Multiple Eisenstein series and multiple cotangent functions

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Abstract

We construct the multiple Eisenstein series and we show a relation to the multiple cotangent function. We calculate a limit value of the multiple Eisenstein series.

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1. Introduction

For integers \( r \geq 1 \) and \( k \geq r + 2 \), we define the multiple Eisenstein series as

\[
F_k(\tau_1, \ldots, \tau_r) = \sum_{n=1}^{\infty} \frac{n^{k-1}q_1^n \cdots q_r^n}{(1 - q_1^n) \cdots (1 - q_r^n)}
\]

with \( q_j = e^{2\pi i \tau_j} \) for \( \text{Im}(\tau_j) > 0 \). We remark that the series converges for \((\tau_1, \ldots, \tau_r) \in (\mathbb{C} - \mathbb{R}_{\leq 0})^r \) if \( \text{Im}(\tau_j) > 0 \) for at least one \( j \). This function is considered to be the multiple \( q \) (quantum) polylogarithm since

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\[ F_k(\tau_1, \ldots, \tau_r) = \frac{1}{(1-q_1) \cdots (1-q_r)} \sum_{n=1}^{\infty} \frac{q_1^n \cdots q_r^n}{n^{1-k}[n]_{q_1} \cdots [n]_{q_r}}, \]

where

\[ [x]_q = \frac{1 - q^x}{1 - q}. \]

We recall the one-variable case

\[ F_k(\tau) = \sum_{n=1}^{\infty} \frac{n^{k-1}q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \]

with

\[ \sigma_{k-1}(n) = \sum_{d|n} d^{k-1} \]

and

\[ q = e^{2\pi i \tau} \]

occurs frequently in the form

\[ E_k(\tau) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \]

with the constant term \( \zeta(1-k)/2 \).

In a previous paper [K2] we proved that

\[ \lim_{\tau \to 1} \left( E_k\left(-\frac{1}{\tau}\right) - \tau^k E_k(\tau) \right) = \frac{(-1)^kB_{k-1}}{2\pi i} \]

for all positive integers \( k \). This is checked for even integers \( k \) by the modularity of \( E_k(\tau) \) for \( SL_2(\mathbb{Z}) \), but the above limit value seems to be curious especially for odd integers \( k \). We remark that the above result is written also as

\[ \lim_{\tau \to 1} \left( F_k\left(-\frac{1}{\tau}\right) - \tau^k F_k(\tau) \right) = \frac{(-1)^kB_{k-1}}{2\pi i}. \]

In this paper we generalize this result to the case of several variables.

To describe our results we introduce multiple sine functions and multiple cotangent functions. The multiple sine function \( S_r(x, (\omega_1, \ldots, \omega_r)) \) is constructed as
\[ S_r(x, (\omega_1, \ldots, \omega_r)) = \prod_{n_1, \ldots, n_r \geq 0} (n_1 \omega_1 + \cdots + n_r \omega_r + x) \left( \prod_{m_1, \ldots, m_r \geq 1} (m_1 \omega_1 + \cdots + m_r \omega_r - x) \right)^{(-1)^{r-1}}, \]

where \( \prod \) denotes the regularized product of Deninger [D]:

\[ \prod_{\lambda} = \exp \left( -d \frac{d}{ds} \sum_{\lambda} \lambda^{-s} \bigg|_{s=0} \right). \]

We refer to [K1, KK] for details of the theory of multiple sine functions; see the survey of Manin [M]. The multiple cotangent function \( \text{Cot}_r(x, (\omega_1, \ldots, \omega_r)) \) is defined as the logarithmic derivative of the multiple sine function:

\[ \text{Cot}_r(x, (\omega_1, \ldots, \omega_r)) = \frac{S'_r(x, (\omega_1, \ldots, \omega_r))}{S_r(x, (\omega_1, \ldots, \omega_r))}. \]

Our first result relates \( F_k(\tau_1, \ldots, \tau_r) \) to \( \text{Cot}_{r+1}^{(k-1)}(\tau_1 + \cdots + \tau_r, (\tau_1, \ldots, \tau_r, 1)) \).

**Theorem 1.** Let \( k \geq r + 2 \) and \( 0 < \arg(\tau_1) < \cdots < \arg(\tau_r) < \pi \). Then

\[ \text{Cot}_{r+1}^{(k-1)}(\tau_1 + \cdots + \tau_r, (\tau_1, \ldots, \tau_r, 1)) = \frac{\left( F_k(\tau_1, \ldots, \tau_r) - \frac{1}{\tau_1} F_k \left( -\frac{1}{\tau_1}, \frac{\tau_2}{\tau_1}, \ldots, \frac{\tau_r}{\tau_1} \right) \right)}{\left( 1 - \frac{1}{\tau_1} \right) \left( 1 - \frac{1}{\tau_2} \right) \cdots \left( 1 - \frac{1}{\tau_r} \right)}. \]

**Examples.**

1. \( \text{Cot}_2^{(k-1)}(\tau, (\tau, 1)) = -(2\pi i)^k \left( F_k(\tau) - \frac{1}{\tau} F_k \left( -\frac{1}{\tau} \right) \right) \).
2. \( \text{Cot}_3^{(k-1)}(\tau + \sigma, (\tau, \sigma, 1)) = -(2\pi i)^k \left( F_k(\tau, \sigma) - \frac{1}{\tau} F_k \left( -\frac{1}{\tau}, \frac{\sigma}{\tau} \right) - \frac{1}{\sigma} F_k \left( \frac{\tau}{\sigma}, -\frac{1}{\sigma} \right) \right) \).

**Theorem 2.** Let \( k \geq r + 2 \). Then

\[ \lim_{\tau_1, \ldots, \tau_r \to 1} \left( F_k(\tau_1, \ldots, \tau_r) - \frac{1}{\tau_1} F_k \left( -\frac{1}{\tau_1}, \frac{\tau_2}{\tau_1}, \ldots, \frac{\tau_r}{\tau_1} \right) \right) \cdots \frac{1}{\tau_r} F_k \left( \frac{\tau_1}{\tau_r}, \ldots, \frac{\tau_r-1}{\tau_r}, -\frac{1}{\tau_r} \right) = \frac{(-1)^{r-1}(k-1)!}{r!(2\pi i)^k} \sum_{m=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-1)^m \pi^{2m} \alpha(r, k-2m)}{(2m)!} B_{2m}. \]
where \( a(r, j) \in \mathbb{Z} \) is a Stirling like number defined by

\[
x(x + 1) \cdots (x + r - 1) = \sum_{j=1}^{n} a(r, j)x^j.
\]

Examples.

1. \[
\lim_{\tau \to 1} \left( F_k(\tau) - \frac{1}{\tau^k} F_k \left( -\frac{1}{\tau} \right) \right) = \frac{(-1)^{k-1} B_{k-1}}{2\pi i},
\]

2. \[
\lim_{0 < \arg(\tau) < \pi} \left( F_k(\tau, \sigma) - \frac{1}{\tau^k} F_k \left( -\frac{1}{\tau} \right) - \frac{1}{\sigma^k} F_k \left( \frac{\tau}{\sigma}, -\frac{1}{\sigma} \right) \right) = \begin{cases} 
-\frac{B_{k-1}}{4\pi i} & \text{if } k \geq 1 \text{ is odd}, \\
\frac{(k-1) B_{k-2}}{8\pi^2} & \text{if } k \geq 2 \text{ is even}.
\end{cases}
\]

Our functions can be investigated from the viewpoint of the “Stirling modular form” which generalizes a notion used by Barnes [B]. Instead of going into the detailed theory, here we only indicate that “Stirling modular forms” mean some suitable functions of “semi-lattices” similar to the situation of usual modular forms which are functions of “lattices.” We notice one example which shows that \( F_k(\tau_1, \ldots, \tau_r) \) is a function of the “semi-lattice” \( \mathbb{Z}_{\geq 1} \tau_1 + \cdots + \mathbb{Z}_{\geq 1} \tau_r + \mathbb{Z} \cdot 1. \)

**Theorem 3.** Let \( k \geq r + 2 \) and \( \text{Im}(\tau_1), \ldots, \text{Im}(\tau_r) > 0. \) Then

\[
F_k(\tau_1, \ldots, \tau_r) = \frac{(-1)^k (k - 1)!}{(2\pi i)^k} \sum_{m_1, \ldots, m_r \geq 1} \sum_{n=-\infty}^{\infty} \frac{1}{(m_1 \tau_1 + \cdots + m_r \tau_r + n)^k}.
\]

**2. Proof of Theorem 1**

As proved in [KW] we have

\[
\log S_r(x, (\omega_1, \ldots, \omega_r)) = -\sum_{l=1}^{r} \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n} \frac{e^{-2\pi i n \omega_l}}{\prod_{j \neq l} (1 - e^{-2\pi i n \omega_j})} + Q(x),
\]

where \( Q(x) \) is a polynomial of \( x \) with \( \deg Q \leq r. \) Hence we have

\[
\text{Cot}_r(x, (\omega_1, \ldots, \omega_r)) = -2\pi i \sum_{l=1}^{r} \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{\omega_l} \frac{e^{-2\pi i n \omega_l}}{\prod_{j \neq l} (1 - e^{-2\pi i n \omega_j})} + Q'(x)
\]

and

\[
\text{Cot}_r^{(k-1)}(x, (\omega_1, \ldots, \omega_r)) = -(2\pi i)^k \sum_{l=1}^{r} \sum_{n=1}^{\infty} \frac{n^{k-1} e^{2\pi i n x}}{\omega_l^k} \frac{e^{-2\pi i n \omega_l}}{\prod_{j \neq l} (1 - e^{-2\pi i n \omega_j})} + Q^{(k)}(x)
\]
for \( k \geq 1 \). In particular

\[
\cot_r^{(k-1)}(x, (\omega_1, \ldots, \omega_r)) = -(2\pi i)^k \sum_{l=1}^{r} \sum_{n=1}^{\infty} \frac{\omega_l^{-k} e^{\frac{2\pi inx}{\omega_l}}}{\omega_l \prod_{j \neq l} (1 - e^{\frac{2\pi in\omega_j}{\omega_l}})}
\]

for \( k \geq r + 1 \). Thus, changing \( r \) to \( r + 1 \) and taking \((\omega_1, \ldots, \omega_{r+1}) = (\tau_1, \ldots, \tau_r, 1)\) with \( x = \tau_1 + \cdots + \tau_r \), we obtain Theorem 1.

3. Proof of Theorem 2

Theorem 1 shows that the limit value is given by

\[
-\frac{1}{(2\pi i)^k} \cot_r^{(k-1)}(r, (1, \ldots, 1)).
\]

We calculate it by using the formula

\[
\cot_{r+1}(x, (1, \ldots, 1)) = (-1)^r \begin{pmatrix} x - 1 \\ r \end{pmatrix} \pi \cot(\pi x)
\]

proved in [KK]. We look at

\[
\cot_{r+1}(r + x, (1, \ldots, 1)) = (-1)^r \begin{pmatrix} x + r - 1 \\ r \end{pmatrix} \pi \cot(\pi x)
\]

and its Taylor expansion around \( x = 0 \). Since

\[
\begin{pmatrix} x + r - 1 \\ r \end{pmatrix} = \frac{x(x + 1) \cdots (x + r - 1)}{r!} = \frac{1}{r!} \sum_{j=1}^{r} a(r, j) x^j
\]

and

\[
\cot(\pi x) = \sum_{m=0}^{\infty} \frac{(-1)^m \pi 2m-1 B_{2m}}{(2m)!} x^{2m-1}
\]

we have

\[
\cot_r^{(k-1)}(r, (1, \ldots, 1)) = (k - 1)! \begin{pmatrix} k-2 \\ r \end{pmatrix} \pi \sum_{j=1}^{r} \sum_{m=0}^{\infty} \frac{(-1)^m \pi 2m-1 B_{2m} a(r, j)}{(2m)!}
\]

Thus we have Theorem 2.

We notice that examples are easily checked by Theorem 1.
4. Proof of Theorem 3

The well-known Lipschitz formula says that

$$
\sum_{n=-\infty}^{\infty} \frac{(\tau + n)^{-k}}{n!} = (-1)^k \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n
$$

for $\text{Im}(\tau) > 0$ with $q = e^{2\pi i \tau}$. Especially, for $\tau = m_1 \tau_1 + \cdots + m_r \tau_r$ we have

$$
\sum_{n=-\infty}^{\infty} \frac{(m_1 \tau_1 + \cdots + m_r \tau_r + n)^{-k}}{n!} = (-1)^k \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} (q_1^{m_1} \cdots q_r^{m_r})^n
$$

with $q_j = e^{2\pi i \tau_j}$. Hence we obtain

$$
\sum_{m_1,\ldots,m_r \geq 1} \sum_{n=-\infty}^{\infty} \frac{(m_1 \tau_1 + \cdots + m_r \tau_r + n)^{-k}}{n!} = (-1)^k \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \frac{n^{k-1} q_1^n \cdots q_r^n}{(1 - q_1^n) \cdots (1 - q_r^n)}.
$$

This gives Theorem 3.

References