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Invariants and the Ring of Generic Matrices

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1. INTRODUCTION

Throughout this article K is a field of characteristic zero and $GL(n)$ denotes the general linear group of $n \times n$ invertible matrices over K. There are two major reasons for the restriction to characteristic zero. First, we use the classical description of modules over the general linear group in characteristic zero, which identifies the Grothendieck ring of polynomial $GL(n)$-modules with the ring of symmetric functions in n variables. Second, the "first fundamental theorem" which describes the invariants of m vectors and m covectors, is not yet proved in characteristic $p > 0$.

The two most basic questions one can ask about the polynomial identities of $M_n(K)$ are the qualitative (what are they?) and the quantitative (how many are there?). The first has been answered by a fundamental theorem proved independently by Procesi [10] and Razmyslov [11] (also see Helling [7]). Their theorem gives a complete description of so-called trace identities for $M_n(K)$, and the trace identities contain the ordinary polynomial identities as a proper subset. Unfortunately, it has so far proved difficult to make use of the identification of polynomial identities with trace identities and consequently many basic problems remain unsolved (see [5]).

The quantitative description of identities of $M_n(K)$ is less complete. Here the basic results are due to Regev (e.g. [13]) and for the most part his results apply to T-ideals in general rather than to the T-ideal of identities of $M_n(K)$ in particular. Regev associates with any T-ideal T a cocharacter series which

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describes the finite-dimensional vector space $V_m/(V_m \cap T)$ as a module over S_m, the symmetric group on m letters. Here

$$V_m = \mathrm{span}\{u_{\pi(1)} \cdots u_{\pi(m)} \mid \pi \in S_m\}$$

denotes the space of multilinear polynomials in a set of noncommuting variables $u_1,..., u_m$. More recently, Berele [2] and Drensky [3] have initiated an analogous study of the cocharacter series associated with the action of $GL(m)$ on the m-variable identities in T. They have shown that the multilinear cocharacter series determines the $GL(m)$-cocharacter series, in a sense we will shortly make precise.

There is another series associated with a T-ideal, the Poincare series. Let

$$K\langle U_m\rangle = K\langle u_1, u_2,..., u_m\rangle,$$
$$K\langle U\rangle = K\langle u_1, u_2,...\rangle$$

be free algebras and T a T-ideal in $K\langle U\rangle$. The free algebra has a multigrading relative to which T is homogeneous, so $K\langle U\rangle/T$ and $K\langle U_m\rangle/(T \cap U_m)$ inherit the multigrading. The Poincare series

$$P(K\langle U_m\rangle/(T \cap U_m)) \qquad \text{and} \qquad P(K\langle U\rangle/T)$$

are formal power series whose coefficients count the dimensions of the homogeneous components of $K\langle U_m\rangle/(T \cap U_m)$ and $K\langle U\rangle/T$. The former is a formal power series over

$$\Lambda_m = \mathbf{Z}[x_1,..., x_m]^{S_m},$$

the ring of symmetric functions in m commuting variables. The latter series $P(K\langle U\rangle/T)$ may be regarded as the limit of the $P(K\langle U_m\rangle/(T \cap U_m))$ as $m \to \infty$. It is a formal power series over Λ, "the ring of symmetric functions in infinitely many variables."

Let $\mathrm{Mod}(GL(m))$ be the Grothendieck ring of finite-dimensional $GL(m)$-modules, and let

$$\mathrm{Mod}(S) = \bigoplus_{m \geqslant 0} \mathrm{Mod}(S_m)$$

be the representation ring of the symmetric group, where $\mathrm{Mod}(S_m)$ is the Grothendieck ring of finite-dimensional S_m-modules. Then there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Mod}(S) & \xrightarrow{\mathrm{ch}} & \Lambda \\
\downarrow{\scriptstyle \bar{\rho}(m)} & & \downarrow{\scriptstyle \rho(m)} \\
\mathrm{Mod}(GL(m)) & \xrightarrow{\chi} & \Lambda_m
\end{array}$$

of ring homomorphisms in which the horizontal maps are isomorphisms.

All four rings are graded and the homomorphisms preserve the grading, so they define homomorphisms of the rings of formal power series over these rings. If T is a T-ideal, then the series we have associated with T are formal power series over the above rings, and the Berele–Drensky result becomes (Theorem 7) the fact that

$$\begin{array}{ccc} \text{Multilinear cocharacter series of } T & \overset{\mathrm{ch}}{\longmapsto} & P(K\langle U\rangle/T) \\ \Big\downarrow{\scriptstyle \bar{\rho}(m)} & & \Big\downarrow{\scriptstyle \rho(m)} \\ GL(m)\text{-cocharacter series of } T & \overset{\chi}{\longmapsto} & P(K\langle U_m\rangle/(T\cap U_m)). \end{array}$$

This paper arose as an attempt to compute the Poincare series of $K\langle U\rangle/\mathscr{M}(n)$, where $\mathscr{M}(n)$ is the T-ideal of identities of $M_n(K)$. Aside from the trivial case of 1×1 matrices only the Poincare series for identities of $M_2(K)$ is known [3, 4].

The relatively free algebra $K\langle U\rangle/\mathscr{M}(n)$ has a concrete realization as the ring of generic matrices, which is the subring of

$$M_n(K[u_{ij}(r)]) \qquad (1\leqslant i, j\leqslant n;\ r=1, 2,...)$$

generated by $n\times n$ generic matrices

$$U(r)=(u_{ij}(r)),$$

where $\{u_{ij}(r)\}$ is a set of commuting variables over K. Let

$R=$ ring of generic matrices,

$\bar{C}=$ ring generated by the traces of elements of R,

$\bar{R}=R\bar{C}=$ trace ring.

Procesi [10, Theorems 1.3 and 2.1] has shown that $\bar{C}$ is the fixed ring of an action of $GL(n)$ on $K[U_{ij}(r)]$ and that $\bar{R}$ is the fixed ring of $GL(n)$ acting on $M_n(K[U_{ij}(r)])$ (see Theorem 10). This action extends to $M_n(K(u_{ij}(r))$ in which case the fixed ring is the generic division ring, the quotient ring of R.

Once it is known that $\bar{C}$ and $\bar{R}$ are fixed rings of $GL(n)$, the theory of Schur and Weyl can be applied to obtain formulas for $P(\bar{C})$ and $P(\bar{R})$ (Theorem 12). The form of these Poincare series is (taking $\bar{R}$ for definiteness)

$$P(\bar{R})=\sum_{\lambda}\bar{r}(\lambda)s_\lambda,$$

where λ varies over all partitions of length $\leqslant n^2$ (i.e., with $\leqslant n^2$ parts), $\bar{r}(\lambda)$ is a certain integer valued function of λ, and s_λ is the Schur function associated with λ. Under the homomorphism

$$\mathrm{Mod}(S)\xrightarrow{\mathrm{ch}}\Lambda,$$

irreducible modules are carried to Schur functions. Hence the integer $\bar{r}(\lambda)$ is the multiplicity of an irreducible module.

We can obtain inequalities for the Poincare series of R by choosing an element α of R such that

$$\alpha\bar{R} \subseteq R \subseteq \bar{R}.$$

These inclusions yield more information about the Poincare series of R than could be expected a priori. If we write

$$P(R) = \sum_{\lambda} r(\lambda)\, s_{\lambda},$$

then $r(\lambda) = \bar{r}(\lambda)$ for all sufficiently large λ (Theorem 16). (Too much should not be expected from this theorem; although there are infinitely many partitions which are "sufficiently large", there are also infinitely many which are not.)

The Procesi–Razmyslov theorem also gives a formula for $P(\bar{C})$ and by setting it equal to our formula we get a combinatorial result about the group ring of the symmetric group which has no reference to matrices (Theorem 17).

We conclude by giving explicitly the Poincare series for $P(C)$, $P(R)$, $P(\bar{C})$ and $P(\bar{R})$ in the case of 2×2 generic matrices. Formanek, Halpin and Li [4] have computed all of them for two 2×2 generic matrices and Drensky [3] has determined $P(R)$ for any number of 2×2 matrices. The answers show that $P(R)$ is even closer to $P(\bar{R})$ than Theorem 16 predicts and that the difference depends very strongly on the ideal generated by evaluations of the Capelli polynomial. As an application of the formula for $P(\bar{C})$, we give a presentation of $\bar{C}$ in the case of three 2×2 matrices (Theorem 22).

The rest of the paper is organized as follows. Sections 2 and 3 collect the notation and results on symmetric functions and representations of the symmetric and general linear groups which are used later. Then Sections 4–9 present, in order, the material summarized above.

2. The Ring of Symmetric Functions

This section consists of definitions, notation and a few basic results on the ring of symmetric functions. We follow [9, Chapter 1] although we occasionally modify terminology.

A *degree sequence* (*weight*) of length n is a sequence

$$\alpha = (\alpha_1, ..., \alpha_n)$$

of nonnegative integers. The *total degree* of α is

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

A *partition* (*dominant weight*) of length $\leqslant n$ is a degree sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ with

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n.$$

The *sum* of two degree sequences α and β is

$$\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n).$$

Sometimes we allow degree sequences and partitions to be infinite sequences with only finitely many nonzero terms. Then a finite sequence is identified with the infinite one obtained by adding infinitely many zeros.

Let $x_1, \ldots, x_n$ be commuting indeterminates, and let S_n, the symmetric group on n letters, act on $\mathbf{Z}[x_1, \ldots, x_n]$ and $\mathbf{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by permuting variables. Set

$$\Lambda_n = \Lambda_n(x) = \mathbf{Z}[x_1, \ldots, x_n]^{S_n},$$
$$\bar{\Lambda}_n = \bar{\Lambda}_n(x) = \mathbf{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n},$$

the respective rings of symmetric functions. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a degree sequence, define

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

For any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leqslant n$, the *monomial symmetric function* of type λ is defined by

$$m_\lambda = \sum x^\alpha,$$

where the sum is over all distinct permutations α of $\lambda = (\lambda_1, \ldots, \lambda_n)$.

For each positive integer r, the rth *elementary symmetric function* is

$$\begin{aligned} e_r &= \sum \{x_{i_1} \cdots x_{i_r} \mid i_1 < i_2 \cdots < i_r\} \\ &= m_{(1^r)}, \end{aligned}$$

where $(1^r) = (1, \ldots, 1) = (1, \ldots, 1, 0, \ldots, 0)$ (r ones).

The fundamental theorem of symmetric functions asserts that

$$\Lambda_n = \mathbf{Z}[e_1, \ldots, e_n],$$

a polynomial ring in n independent variables, and it it easy to see that

$$\bar{\Lambda}_n = \mathbf{Z}[e_1, \ldots, e_{n-1}, e_n^{\pm 1}],$$

the localization of Λ_n at the powers of e_n.

The rth *complete symmetric function* is

$$h_r = \sum_{|\lambda| = r} m_\lambda,$$

and the rth *power symmetric function* is

$$p_r = x_1^r + \cdots + x_n^r.$$

The polynomials $e_r(x_1, \ldots, x_n)$ and $h_r(x_1, \ldots, x_n)$ can be defined implicitly in terms of the generating functions $E(t)$ and $H(t)$, where

$$E(t) = \pi(1 + x_i t) = 1 + e_1 t + \cdots + e_n t^n,$$
$$H(t) = \pi(1 - x_i t)^{-1} = 1 + h_1 t + h_2 t^2 + \cdots.$$

For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ *of any length* one defines

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k},$$
$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k},$$
$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}.$$

Define

$$\delta = \delta(n) = (n-1, n-2, \ldots, 1, 0)$$

and for any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leqslant n$, define

$$\begin{aligned} a_\lambda &= a_\lambda(x_1, \ldots, x_n) \\ &= \sum_{\pi \in S_n} (\operatorname{sign} \pi)\, x_{\pi(1)}^{\lambda_1} \cdots x_{\pi(n)}^{\lambda_n}. \end{aligned}$$

Then

$$a_\delta = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j),$$

a_δ divides $a_{\lambda+\delta}$ in $\mathbf{Z}[x_1, \ldots, x_n]$, and the quotient $a_{\lambda+\delta}/a_\delta$ is invariant under S_n.

The *Schur function* $s_\lambda = s_\lambda(x_1, \ldots, x_n)$ is defined by

$$s_\lambda = a_{\lambda+\delta}/a_\delta.$$

The set

$$\{s_\lambda \mid \lambda \text{ a partition of length } \leqslant n\}$$

forms a $\mathbf{Z}$-basis for Λ_n over $\mathbf{Z}$. An inner product $\langle\, ,\, \rangle$ is defined on Λ_n by making the s_λ an orthonormal basis, and $\langle\ ,\ \rangle$ can be extended to $\bar{\Lambda}_n$ by demanding that

$$\langle a, b\rangle = \langle e_n a, e_n b\rangle.$$

The inner product $\langle\ ,\ \rangle$ can also be defined intrinsically, following Weyl [16, pp. 198–200]. Let

$$*: \mathbf{Z}[x_1^{\pm 1},..., x_n^{\pm 1}] \to \mathbf{Z}[x_1^{\pm 1},..., x_n^{\pm 1}]$$

be the involution defined by $x_i^* = x_i^{-1}$, and let

$$\int : \mathbf{Z}[x_1^{\pm 1},..., x_n^{\pm 1}] \to \mathbf{Z}$$

be the linear functional defined by

$$\int 1 = 1$$

$$\int x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0 \qquad \text{if } \alpha_1,..., \alpha_n \text{ are not all zero.}$$

Then for any $a, b \in \bar{\Lambda}_n$,

$$\langle a, b\rangle = \frac{1}{n!}\int ab^* a_\delta (a_\delta)^*.$$

Remark. In the next section the Schur functions s_λ will be identified with irreducible $GL(n)$-modules. The above evaluation of $\langle\ ,\ \rangle$ is a translation of Weyl's method of defining an inner product on modules by integrating over the complex unitary group $U(n, \mathbf{C})$, which has the same module theory as $GL(n)$. After various artifices Weyl [16, pp. 198–200] obtains the inner product of modules as a definite integral of the form

$$\int_0^1 \cdots \int_0^1 \sum_\alpha a_\alpha e^{2\pi i(\alpha_1 z_1 + \cdots + \alpha_n z_n)} dz_1 \cdots dz_n.$$

But of course our definition of "$\int$" satisfies

$$\int x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \int_0^1 \cdots \int_0^1 e^{2\pi i(\alpha_1 z_1 + \cdots + \alpha_n z_n)} dz_1 \cdots dz_n.$$

Finally, we define the "ring of symmetric functions in infinitely many variables." It forms the most natural basis for stating results about T-ideals since they are ideals in a free ring with infinitely many generators. However, actual calculations can always be carried out in the finitely generated rings Λ_n and $\bar{\Lambda}_n$ for some n.

$\mathbf{Z}[x_1,..., x_n]$ is a graded ring and Λ_n inherits the grading. The r-the homogeneous component of Λ_n is denoted Λ_n^r, so that

$$\Lambda_n = \Lambda_n^0 \oplus \Lambda_n^1 \oplus \Lambda_n^2 \oplus \cdots.$$

Note that $e_\lambda, h_\lambda, p_\lambda, s_\lambda$ are homogeneous of degree $|\lambda|$.

Suppose that $n \geqslant m$. Then

$$\rho(n, m)(x_i) = \begin{cases} x_i & (1 \leqslant i \leqslant m) \\ 0 & (m+1 \leqslant i \leqslant n) \end{cases}$$

induces a degree preserving surjection

$$\rho(n, m): \Lambda_n \to \Lambda_m.$$

The $\rho(m, n)$ satisfy the compatibility condition which allows the inverse limit of the Λ_n (in the category of graded rings) to be defined. The *ring of symmetric functions in infinitely many variables* is defined to be this inverse limit, and is denoted $\Lambda = \Lambda(x)$. Its rth homogeneous component is denoted Λ^r, and $\Lambda = \oplus \Lambda^r$. Since

$$\rho(n, m)(m_\lambda(x_1,..., x_n)) = m_\lambda(x_1,..., x_m)$$

and the same is true of the e_λ, h_λ, p_λ, s_λ, these are all well-defined elements of Λ, and the canonical projections

$$\rho(n): \Lambda \to \Lambda_n$$

carry m_λ to $m_\lambda(x_1,..., x_n)$ and likewise for r_λ, h_λ, p_λ, s_λ. However, some of these are zero; for example, if λ has length strictly greater than n, then $m_\lambda(x_1,..., x_n) = 0$; i.e. $\rho(n)(m_\lambda) = 0$.

The set

$$\{s_\lambda \mid \text{length } \lambda \leqslant n, |\lambda| = r\}$$

forms a $\mathbf{Z}$-basis for Λ_n^r. If $n \geqslant r$, all partitions of degree r have length $\leqslant n$. Hence if $n \geqslant m \geqslant r$,

$$\rho(n, m): \Lambda_n^r \to \Lambda_m^r$$

is an isomorphism. It follows that Λ^r is a free $\mathbf{Z}$-module of finite rank with basis

$$\{s_\lambda, |\lambda| = r\}$$

and that

$$\rho(n): \Lambda^r \to \Lambda_n^r$$

is an isomorphism if $n \geqslant r$. On the other hand, if $n < r$, the kernel of $\rho(n): \Lambda^r \to \Lambda_n^r$ is the free $\mathbf{Z}$-module spanned by the set

$$\{s_\lambda \mid \text{length } \lambda > n, |\lambda| = r\}.$$

We will need a basic combinatorial result which expresses the complete symmetric functions of $\{x_i y_j\}$ in terms of symmetric functions of x_i and y_j. If $\lambda = (\lambda_1, ..., \lambda_n)$ is a partition, write

$$\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \cdots)$$

to mean that m_i is the number of λ_j equal to i, and set

$$z_\lambda = \prod i^{m_i} m_i!$$

THEOREM 1 [9, p. 33]. *Let $x = (x_1, x_2, ...)$, $y = (y_1, y_2, ...)$ be two distinct (finite or infinite) sequences of commuting variables over* $\mathbf{Z}$. *Then*

$$\begin{aligned}\prod_{i,j} (1 - x_i y_j)^{-1} &= \sum s_\lambda(x)\, s_\lambda(y) \\ &= \sum m_\lambda(x)\, h_\lambda(y) \\ &= \sum z_\lambda^{-1} p_\lambda(x)\, p_\lambda(y),\end{aligned}$$

where the sum is over all partitions λ.

3. Representations of the General Linear and Symmetric Groups

In this section we summarize the results on the representations of the general linear group $GL(n)$ and the symmetric group S_n which we will use later. They can be found in [6] and [9].

A *polynomial homomorphism*

$$\phi: GL(n) \to GL(m)$$

is a group homomorphism whose coordinate functions are given by polynomials. That is,

$$\phi(a_{ij}) = (f_{pq}(a_{ij})) \qquad (1 \leqslant p, q \leqslant m,\ 1 \leqslant i, j \leqslant n),$$

where each $f_{pq}(x_{ij})$ is a polynomial in n^2 variables. The action of $GL(n)$ on $M = K^m$ is then called a *polynomial representation* of $GL(n)$ and M is called a *polynomial module* over $GL(n)$. If each f_{pq} is a homogeneous polynomial of degree r, the representation is said to be *homogeneous* of degree r.

Rational homomorphisms, representations and modules for $GL(n)$ are defined similarly by letting the coordinate functions be rational functions f_{pq}/g_{pq}. If f_{pq} and g_{pq} are homogeneous of degrees r and s, respectively, the representation is said to be *homogeneous* of degree $r-s$.

If $\mathscr{C}$ is a category of $GL(n)$-modules, the *Grothendieck Ring* of $\mathscr{C}$ is defined to be the additive abelian group generated by all equivalence classes $[M]$, where M is a module in $\mathscr{C}$, modulo the relations

$$[M]+[N]=[Q]$$

if there is an exact sequence

$$0\to M\to Q\to N\to 0.$$

The ring structure is induced by setting

$$[M]+[N]=[M\oplus N],$$
$$[M][N]=[M\otimes N],$$

where $GL(n)$ acts diagonally on $M\otimes N$. Set

$$\begin{aligned}\mathrm{Mod}(GL(n)) &= \text{Grothendieck ring of finite-dimensional}\\ &\qquad \text{polynomial } GL(n)\text{-modules}\\ \overline{\mathrm{Mod}}(GL(n)) &= \text{Grothendieck ring of finite-dimensional}\\ &\qquad \text{rational } GL(n)\text{-modules.}\end{aligned}$$

Let $D(n)$ denote the subgroup of diagonal matrices in $GL(n)$ and let $D(z_1,\ldots,z_n)$ denote the diagonal matrix whose main diagonal is $z_1,\ldots,z_n$. Suppose that $\alpha=(\alpha_1,\ldots,\alpha_n)$ is a degree sequence of length n and that M is a polynomial $GL(n)$-module. Define the *α-homogeneous component* (*α-weight space*) of M to be

$$M^\alpha=\{m\in M\mid D(z_1,\ldots,z_n)m=z_1^{\alpha_1}\cdots z_n^{\alpha_n}m \text{ for all } D\in D(n)\}.$$

Recall that $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ where $x_1,\ldots,x_n$ are commuting indeterminates over $\mathbf{Z}$. The next theorem describes a natural isomorphism between $\mathrm{Mod}(GL(n))$ and Λ_n.

THEOREM 2 (See [6, Chapter 3]). *Let M and N be finite-dimensional polynomial $GL(n)$-modules.*

(a) *M is completely reducible and is a direct sum of homogeneous submodules.*

(b) *As a K-vector space, M is the direct sum of its homogeneous components.*

(c) *Define the character of M to be*

$$\chi(M) = \sum \dim_K(M^\alpha)x^\alpha,$$

where the sum is over the (finitely many) degree sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ *of length n for which* M^α *is nonzero. Then* $\chi(M)$ *is a symmetric polynomial in* $x_1, \ldots, x_n$ *and if M is homogeneous of degree n, so is* $\chi(M)$.

(d) *Suppose* $P \in GL(n)$ *has eigenvalues* $z_1, \ldots, z_n$ *and* $\phi(P) \in GL(M)$ *is a matrix representing the action of P on M. Then the trace of* $\phi(P)$ *is*

$$\operatorname{Tr}(\phi(P)) = \chi(M)(z_1, \ldots, z_n).$$

(e) *Two modules M and N are isomorphic if and only if* $\chi(M) = \chi(N)$. *Furthermore*

$$\chi(M \oplus N) = \chi(M) + \chi(N),$$

$$\chi(M \otimes N) = \chi(M)\,\chi(N).$$

(f) *M is irreducible if and only if* $\chi(M) = s_\lambda(x_1, \ldots, x_n)$, *where* s_λ *is the Schur function associated with a partition* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *of length* $\leqslant n$. *Hence there is a* 1–1 *correspondence between irreducible* $GL(n)$*-modules and partitions of length* $\leqslant n$.

(g) *The induced map*

$$[M] \mapsto \chi(M)$$

is a ring isomorphism between $\operatorname{Mod}(GL(n))$ *and* $\Lambda_n = \mathbf{Z}[x_1, \ldots, x_n]^{S_n}$. *If* $\operatorname{Mod}(GL(n))$ *is graded in the obvious fashion and given an inner product* $\langle\, , \rangle$ *by demanding that the irreducible modules form an orthonormal basis, then* χ *is an isometry of graded rings.*

The corresponding theorem for rational $GL(n)$-modules is only a slight modification of the above.

THEOREM 3. (a) *The only one-dimensional rational* $GL(n)$*-representations are*

$$\phi_r : GL(n) \to K^*,$$

where $r \in \mathbf{Z}$ *and* $\phi_r(P) = (\det P)^r$. *The corresponding character is* $e_n^r = (x_1 \cdots x_n)^r$.

(b) *Let* $(\det)^r$ *denote the above module. Then every finite-dimensional rational* $GL(n)$*-module has the form*

$$(\det)^{-r} \otimes M,$$

where r *is a non-negative integer and* M *is a polynomial module. Moreover,* $(\det)^{-r} \otimes M$ *is irreducible if and only if* M *is irreducible.*

(c) *Any finite-dimensional rational* $GL(n)$*-module is determined up to isomorphism by its character as in Theorem* 2, *but now* $\chi(M)$ *lies in* $\bar{\Lambda}_n = \Lambda_n[e_n^{-1}]$, *the localization of* Λ_n *at the powers of* $e_n = x_1 \cdots x_n$.

(d) *The induced map*

$$[M] \to \chi(M)$$

is a ring isomorphism between $\overline{\mathrm{Mod}}(GL(n))$ *and* $\bar{\Lambda}_n = \mathbf{Z}[{}_1^{\pm 1},\dots, x_n^{\pm 1}]^{S_n}$. *If* $\overline{\mathrm{Mod}}(GL(n))$ *is* $\mathbf{Z}$*-graded and given an inner product by demanding that the irreducible modules form an orthonormal basis, then* χ *is an isometry of graded rings.*

We next define the representation ring of the symmetric group and give a natural isomorphism between it and Λ, the ring of symmetric functions in infinitely many variables. Here we follow [9, pp. 60–68].

For each $n \geqslant 0$, let

$$\sigma\colon \mathrm{Mod}(S_n) \to B_n$$

be the isomorphism between the Grothendieck ring of finite-dimensional S_n-modules and B_n, *the character ring of* S_n (by convention, $\mathrm{Mod}(S_0) = B_0 = \mathbf{Z}$). Both are free $\mathbf{Z}$-modules with one generator for each partition of n, and the elements of B_n are *characters*—certain functions $f\colon S_n \to \mathbf{Z}$ which are constant on conjugacy classes (but not all such functions are characters). Although σ is a ring isomorphism, we ignore the multiplication of $\mathrm{Mod}(S_n)$ and instead define a multiplication

$$\mathrm{Mod}(S_m) \times \mathrm{Mod}(S_n) \to \mathrm{Mod}(S_{m+n}).$$

The *representation ring of the symmetric group* is the graded $\mathbf{Z}$-module

$$\mathrm{Mod}(S) = \mathbf{Z} \oplus \mathrm{Mod}(S_1) \oplus \mathrm{Mod}(S_2) \oplus \cdots$$

with multiplication defined as follows. Let $[M] \in \mathrm{Mod}(S_m)$, $[N] \in \mathrm{Mod}(S_n)$. Then $M \otimes N$ is an $S_m \times S_n$-module with the diagonal action. Identify $S_m \times S_n$ with a subgroup of

$$S_{m+n} = \text{permutations of } \{1,\dots, m+n\}$$

by letting S_m act on $\{1,\ldots,m\}$ and S_n act on $\{m+1,\ldots,m+n\}$. Then induction of modules defines an S_{m+n}-module (denoted $M \oplus N|^{S_{m+n}}$), and the *product* (*outer product*) of $[M]$ and $[N]$ is defined to be

$$[M][N] = [M \otimes N|^{S_{m+n}}],$$

which lies in $\mathrm{Mod}(S_{m+n})$. With this multiplication, $\mathrm{Mod}(S)$ is a graded commutative ring.

If λ is a partition of n (i.e., $|\lambda| = n$), let g_λ be an element of S_n of cycle type λ. The set of g_λ form a set of representatives for the conjugacy classes of S_n. Define the *characteristic map*

$$\mathrm{ch}\colon \mathrm{Mod}(S_n) \to \Lambda^n$$

(where Λ^n is the nth graded part of $\Lambda = \oplus \Lambda^n$) by

$$\mathrm{ch}[M] = \sum_{|\lambda| = n} z_\lambda^{-1} [\sigma(M)(g_\lambda)]\, p_\lambda,$$

where p_λ is the power symmetric function and z_λ is the integer defined before Theorem 1.

THEOREM 4 (Fundamental theorem of the representations of the symmetric group [9, pp. 61–62]). *The characteristic map*

$$\mathrm{ch}\colon \mathrm{Mod}(S) \to \Lambda$$

is an isomorphism of graded rings. An S_n-module M is irreducible if and only if $\mathrm{ch}[M] = s_\lambda$, *the Schur function associated with a partition λ, where $|\lambda| = n$. Hence* ch *is an isometry provided* $\mathrm{Mod}(S)$ *is given an inner product by demanding that the irreducible S_n-modules (for all n) form an orthonormal basis.*

We want to describe the homomorphisms between $\mathrm{Mod}(S)$ and $\mathrm{Mod}(GL(n))$ corresponding to the homomorphisms $\rho(n)$ and $\rho(n, m)$ of Section 1. Suppose $n \geqslant m$, and consider the following diagram:

$$\begin{tikzcd}
\mathrm{Mod}(S) \arrow[rrr, "\mathrm{ch}"] \arrow[dd, "\bar{\rho}(m)"'] \arrow[dr, "\bar{\rho}(n)"] & & & \Lambda \arrow[dl, "\rho(n)"'] \arrow[dd, "\rho(m)"] \\
& \mathrm{Mod}(GL(n)) \arrow[r, "\chi"] \arrow[dl, "\bar{\rho}(n,m)"'] & \Lambda_n \arrow[dr, "\rho(n,m)"] & \\
\mathrm{Mod}(GL(m)) \arrow[rrr, "\chi"] & & & \Lambda_m .
\end{tikzcd}$$

The maps in the right hand triangle are the surjections defined in Section 1 and the horizontal maps are the isomorphisms of Theorem 2 and 4. The left hand maps $\bar{\rho}$ are defined implicitly by requiring that the diagram be commutative. We now descibe them explicitly in terms of modules:

$$\underline{\bar{\rho}(n, m)\colon \mathrm{Mod}(GL(n)) \to \mathrm{Mod}(GL(m)).} \tag{A}$$

Suppose M is a finite-dimensional polynomial $GL(n)$-module, and let

$$M = \oplus M^{\alpha},$$

where α varies over all degree sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ of length n. The set of degree sequences of length m are a subset of these, where $(\alpha_1, \ldots, \alpha_m)$ is identified with $(\alpha_1, \ldots, \alpha_m, 0, \ldots, 0)$. Similarly, the map

$$P \longmapsto \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$$

identifies $GL(m)$ with a subgroup of $GL(n)$ and

$$M(n, m) = \oplus \{M^{\alpha} \mid \text{length } \alpha \leqslant m\}$$

is a $GL(m)$-submodule of M. Then we have

LEMMA 5 [6, pp. 103–104]. $\bar{\rho}(n, m)[M] = [M(n, m)]$:

$$\underline{\bar{\rho}(n)\colon \mathrm{Mod}(S) \to \mathrm{Mod}(GL(n)).} \tag{B}$$

All homomorphisms are degree preserving, and it is enough to define $\bar{\rho}(n)$ on the homogeneous component of degree r, which is $\mathrm{Mod}(S_r)$. Suppose we define

$$\bar{\rho}(r)\colon \mathrm{Mod}(S_r) \to \mathrm{Mod}^r(GL(r)) \tag{$*$}$$

for all r. Then we can define $\bar{\rho}(n)$ and $\bar{\rho}(m)$ for $n \geqslant r \geqslant m$ via the following diagram using the fact that we have already defined $\bar{\rho}(n, r)$ and $\bar{\rho}(r, m)$:

$$\begin{tikzcd}
 & \mathrm{Mod}^r(GL(n)) \arrow[d, "\bar{\rho}(n,r)"] \\
\mathrm{Mod}(S_r) \arrow[ur, "\bar{\rho}(n)"] \arrow[r, "\bar{\rho}(r)"] \arrow[dr, "\bar{\rho}(m)"'] & \mathrm{Mod}^r(GL(r)). \arrow[d, "\bar{\rho}(r,m)"] \\
 & \mathrm{Mod}^r(GL(m))
\end{tikzcd}$$

We need to observe that since $n \geqslant r$, $\rho(n, r)$—and hence also $\bar{\rho}(n, r)$—is an isomorphism, so $\bar{\rho}(n)$ is well-defined by the above diagram provided $\bar{\rho}(r)$ is known. Likewise, (*) is an isomorphism since $\rho(r)\colon \Lambda^r \to \Lambda^r_r$ is an isomorphism, so it is enough to define

$$\rho(r)^{-1}\colon \operatorname{Mod}^r(GL(r)) \to \operatorname{Mod}(S_r).$$

This is accomplished by a construction similar to that used to define $\bar{\rho}(n, m)$.

Suppose M is a homogeneous $GL(r)$-module of degree r, and let (1^r) be the "multilinear" degree sequence of length r. Then $M^{(1^r)}$ is an S_r-module, where S_r is identified with the permutation matrices in $GL(r)$, and we have

LEMMA 6 [6, p. 80]. $\bar{\rho}(r)^{-1}[M] = [M^{(1^r)}]$.

4. T-IDEALS, COCHARACTER SERIES, AND POINCARE SERIES

Let

$$\begin{aligned} K\langle U\rangle &= K\langle u_1, u_2, u_3, \ldots\rangle \\ &= K \oplus U \oplus (U \otimes U) \oplus (U \otimes U \otimes U) \oplus \cdots \end{aligned}$$

be a free associative algebra over K in a countable set of variables $\{u_i\}$ or, equivalently, the tensor algebra of a vector space U of countable dimension with basis $\{u_i\}$. Let U_n denote the subspace of U spanned by $\{u_1, \ldots, u_n\}$, so that

$$K\langle U_n\rangle = K\langle u_1, \ldots, u_n\rangle.$$

The general linear group $GL(n)$ acts on U_n and this action extends to an action as a group of homogeneous automorphisms of $K\langle U_n\rangle$, where $GL(n)$ acts diagonally on $U_n^{\otimes r}$. We can write $U_n^{\otimes r}$ as a K-vector space direct sum of homogeneous components

$$U_n^{\otimes r} = \oplus (U_n^{\otimes r})^{\alpha},$$

where α varies over all degree sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ of length n and total degree $|\alpha| = r$. This is precisely the multigrading on $K\langle u_1, \ldots, u_n\rangle$ obtained by specifying that a monomial $u_{i}^{a_1} \cdots u_{i_k}^{a_k}$ has degree sequence $(\alpha_1, \ldots, \alpha_n)$, where

$$\alpha_j = \sum \{a_{i_l} \mid i_l = j\}.$$

This multigrading can be extended to $K\langle U\rangle$ by letting degree sequences be infinite sequences with only finitely many nonzero coordinates. Even though $K\langle U\rangle$ is infinite dimensional, $K\langle U\rangle^{\alpha}$ is finite dimensional for any degree sequence α.

There is also an action of S_n on $K\langle U_n\rangle$ by permuting variables, and S_n also acts on V_n, where

$$V_n = \operatorname{span}\{u_{\pi(1)} \cdots u_{\pi(n)}\} \mid \pi \in S_n\} = (U^{\otimes n})^{(1^n)},$$

the set of multilinear polynomials in $u_1, \ldots, u_n$.

A *T-ideal* is an ideal of $K\langle U\rangle$ which is invariant under endomorphisms of $K\langle U\rangle$. If T is a T-ideal, then T is invariant under the action of $GL(n)$ for all n and hence is homogeneous with respect to the above multigrading.

A number of invariants have been attached to T and $K\langle U\rangle/T$ (see [5, Section 5]): The S-(co)character sequence (Regev [13]); the GL-(co)character sequence (Berele [2], Drensky [3]); and the Poincare or Hilbert series (Drensky [3], Formanek, Halpin, Li [4]). Berele and Drensky have shown that the S-cocharacter sequence determines the GL-cocharacter sequence. We will restate their result below in the formalism of symmetric functions, at the same time showing that cocharacter series become Poincare series under the character isomorphisms ch and χ defined in Section 2.

To avoid introducing additional notation we adopt the following convention: If

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

is a graded ring (e.g. $\operatorname{Mod}(S)$, $\operatorname{Mod}(GL(n))$, Λ, Λ_n), we allow infinite series

$$a = a_0 + a_1 + a_2 + \cdots,$$

where $a_i \in A_i$.

Let T be a T-ideal in $K\langle U\rangle$.

The *S-module* and *comodule* series of T are, respectively, the following infinite series in $\operatorname{Mod}(S)$:

$$A(T) = [T^{(1^0)}] + [T^{(1^1)}] + [T^{(1^2)}] + \cdots,$$
$$A(K\langle U\rangle/T) = [V_0/T^{(1^0)}] + [V_1/T^{(1^1)}] + \cdots,$$

where $T^{(1^r)} = V_r \cap T$ (and by convention, $V_0 = K$).

The *$GL(n)$-module* and *comodule* series are, respectively, the following infinite series in $\operatorname{Mod}(GL(n))$:

$$B_n(T) = [K \cap T] + [U_n \cap T] + [U_n^{\otimes 2} \cap T] + \cdots,$$
$$B_n(K\langle U\rangle/T) = [K/(K \cap T)] + [U_n/(U_n \cap T)] + [U_n^{\otimes 2}/(U_n^{\otimes 2} \cap T)] + \cdots.$$

The *Poincare Series* of T and $K\langle U\rangle/T$ are the following infinite series in Λ:

$$P(T) = \sum \dim_K(T^\alpha) x^\alpha,$$
$$P(K\langle U\rangle/T) = \sum \dim_K(K\langle U\rangle/T)^\alpha x^\alpha,$$

where α varies over all degree sequences. If we restrict to degree sequences of length n, we get Poincare series for $K\langle U_n\rangle \cap T$ and $K\langle U_n\rangle/(K\langle U_n\rangle \cap T)$.

These definitions are compatible with those of Sections 1 and 2. More precisely, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}(S) & \xrightarrow{\mathrm{ch}} & A \\ \big\downarrow{\scriptstyle \bar{\rho}(n)} & & \big\downarrow{\scriptstyle \rho(n)} \\ \mathrm{Mod}(GL(n)) & \xrightarrow{\chi} & A_n \end{array} \tag{1}$$

$P(K\langle U_n\rangle \cap T)$ is obtained from $P(T)$ by restricting to degree sequences of length n—i.e., to polynomials in $x_1, \dots, x_n$. This is how $\rho(n)$ is defined, so

$$\rho(n)(P(T)) = P(K\langle U_n\rangle \cap T). \tag{2}$$

The map χ is defined for any $GL(n)$-module M by

$$\chi[M] = \sum \dim_K(M^\alpha)\, x^\alpha,$$

where α varies over all degree sequences of length n. Thus

$$\chi(B_n(T)) = P(K\langle U_n\rangle \cap T). \tag{3}$$

The rth term of the series $A[T]$ in $\mathrm{Mod}(S)$ is $[T^{(1^r)}]$ and Lemma 6 asserts that

$$\bar{\rho}(n)[T^{(1^r)}] = [T \cap U_r^{\otimes r}].$$

Thus

$$\bar{\rho}(n)(A(T)) = B_n(T). \tag{4}$$

The commutativity of diagram (1) and the validity of (2), (3) and (4) for all n means that also

$$\mathrm{ch}(A(T)) = P(T), \tag{5}$$

since the top row of (1) is the inverse limit of the bottom row as $n \to \infty$.

Equations (2)–(5) are equally valid for $K\langle U\rangle/T$, so we obtain the following version of Berele [2, Theorem 2.7] and Drensky [3, Lemma 1.1].

THEOREM 7. *Let T be a T-ideal in $K\langle U\rangle$. Then the S-comodule, $GL(n)$-comodule, and Poincare series of $K\langle U\rangle/T$ and $K\langle U_n\rangle/(K\langle U_n\rangle \cap T)$ are related by*

$$\begin{array}{ccc} A(K\langle U\rangle/T) & \xmapsto{\mathrm{ch}} & P(K\langle U\rangle/T) \\ \big\downarrow{\scriptstyle \bar{\rho}(n)} & & \big\downarrow{\scriptstyle \rho(n)} \\ B_n(K\langle U\rangle/T) & \xmapsto{\chi} & P(K\langle U_n\rangle/(K\langle U_n\rangle \cap T)). \end{array}$$

In the light of Theorem 7, it makes sense to work only with Poincare series and we will do this from now on. Whether to choose Λ or Λ_n depends largely on circumstances. When a series is given in terms of the e_λ, h_λ, s_λ, etc., Λ is the natural choice. When series are expressed in terms of $x_1, x_2, x_3, \ldots$, then Λ_n is the natural choice since then the series may be expressible as a rational function of $x_1, \ldots, x_n$. On the other hand, for computations involving *rational* $GL(n)$-modules, $\bar{\Lambda}_n$ is the *only* choice since there is no natural inverse limit of the $\bar{\Lambda}$.

It is clear that Theorem 7 is about GL-grading and multilinearization rather than T-ideals.

5. The Ring of Generic Matrices and the Trace Ring

We now consider the T-ideal $\mathscr{M}(n)$, the ideal of identities satisfied by $M_n(K)$. The relatively free algebra $K\langle U\rangle/\mathscr{M}(n)$ has a concrete model, the ring of generic matrices. For each positive integer r, let

$$U(r) = (u_{ij}(r))$$

be an $n \times n$ *generic matrix* over K—i.e., the $u_{ij}(r)$ $(1 \leqslant i, j \leqslant n,\ r = 1, 2, \ldots)$ are independent *commuting* indeterminates over K. Thus each $U(r)$ is an $n \times n$ matrix over the polynomial ring $K[u_{ij}(r)]$. The K-algebra

$$R = R(n) = K[U(1), U(2), \ldots]$$

they generate is called *the ring of* $n \times n$ *generic matrices* and the map $u_r \mapsto U(r)$ gives rise to an exact sequence

$$0 \to \mathscr{M}(n) \to K\langle U\rangle \to R \to 0.$$

Since $\mathscr{M}(n)$ is a homogeneous ideal of $K\langle U\rangle$, R inherits the multigrading of $K\langle U\rangle$. This of course is the GL-grading of R, where for each m, $GL(m)$ acts linearly on span $\{U(1), \ldots, U(m)\}$. We can extend the multigrading to $K[u_{ij}(r)]$ and $M_n(K[u_{ij}(r)])$ by giving each entry $u_{ij}(r)$ of $U(r)$ the same degree as $U(r)$. Again, this is the GL-grading where for each m and fixed i and j, $GL(m)$ acts linearly on span $\{u_{ij}(1), \ldots, u_{ij}(m)\}$.

The above action of $GL(m)$ $(m = 1, 2, \ldots)$ does no more than define the multigrading of $K[u_{ij}(r)]$. There is another action of $GL(n)$ on $K[u_{ij}(r)]$ which commutes with the action of $GL(m)$ and has greater significance.

If $P \in GL(n)$, let

$$P(u_{ij}(r))P^{-1} = (\bar{u}_{ij}(r)).$$

Then $u_{ij}(r) \mapsto \bar{u}_{ij}(r)$ induces a K-automorphism ϕ^P of $K[u_{ij}(r)]$. The *ring of* *invariants* (more precisely, the ring of *simultaneous polynomial invariants of $n \times n$ matrices*) is by definition the fixed ring of this action. Set

$$\begin{aligned} R &= \textit{ring of } n \times n \textit{ generic matrices}, \\ C &= \textit{center of } R, \\ \bar{C} &= K[u_{ij}(r)]^{GL(n)} = \textit{ring of invariants}, \\ Q(C) &= \textit{quotient field of } C, \\ \bar{R} = \bar{C}R &= \textit{trace ring of } R, \\ Q(R) = Q(C)R &= \textit{classical quotient ring of } R \\ &= \textit{generic division ring}. \end{aligned}$$

We need the following facts (see [5, Section 6]).

THEOREM 8. (a) (First fundamental theorem). *$\bar{C}$ is generated by the traces of elements of R.*

(b) *$\bar{C}$ is the center of $\bar{R} = \bar{C}R$.*

(c) *$C \subseteq \bar{C} \subseteq Q(C)$. Hence $Q(C)$ is the quotient field of $\bar{C}$.*

(d) *If the action of $GL(n)$ on $K[u_{ij}(r)]$ is extended to its quotient field, the fixed field is $Q(C)$. Hence*

$$\bar{C} = K[u_{ij}(r)] \cap Q(C).$$

(e) (Posner's Theorem). *$Q(R)$ is a division ring of dimension n^2 over its center $Q(C)$.*

Remarks. The "first fundamental theorem" is a basic theorem of invariant theory which gives a generating set for the invariants of m vectors and m covectors acted on by $GL(n)$. When translated to a theorem about traces, it becomes (a) (see [10, Theorem 1.3]). Parts (b) and (c) are easy consequences of (a) and standard properties of central simple algebras. To prove (d), one first shows that if f, g are relatively prime polynomials in $K[u_{ij}(r)]$ and f/g is fixed by $GL(n)$, then f and g must be *relative invariants*—that is, Kf and Kg are one-dimensional invariant subspaces of $GL(n)$. But the action of $GL(n)$ on $K[u_{ij}(r)]$ is rational and homogeneous of degree zero, and the only rational one-dimensional representation of $GL(n)$ of degree zero is the trivial representation (Theorem 3(a)). Hence f and g are fixed by $GL(n)$. Part (e) is a standard theorem of PI-theory.

By virtue of Theorem 8 we have the following diagram of inclusions

$$\begin{tikzcd}
& & K[u_{ij}(r)] \arrow[dr] & \\
C \arrow[r] \arrow[d] & \bar{C} \arrow[r] \arrow[d] \arrow[ur] & Q(C) \arrow[r] \arrow[d] & K(u_{ij}(r)) \arrow[d] \\
R \arrow[r] & \bar{R} \arrow[r] \arrow[dr] & Q(R) \arrow[r] & M_n(K(u_{ij}(r))). \\
& & M_n(K[u_{ij}(r)]) \arrow[ur] &
\end{tikzcd}$$

We next show that $Q(R)$ and $\bar{R}$ are also fixed rings of $GL(n)$, a result of Procesi [10, Theorem 2.1]. Procesi considers $\bar{R}$ as the ring of $GL(n)$-invariant polynomial functions $f: W \to M_n(K)$, where $W = M_n(K) \times \cdots \times M_n(K)$ (m times), a formulation equivalent to ours provided $f \in M_n(K[U_{ij}(r)])$ is regarded as defining a polynomial function $W \to M_n(K)$. This action extends to $M_n(K(u_{ij}(r))$, in which case the fixed ring is the generic division ring, the quotient ring of R.

LEMMA 9. $R = Q(R) \cap M_n(K[u_{ij}(r)])$.

Proof. Clearly, $\bar{R} \subseteq Q(R) \cap M_n(K[u_{ij}(r)])$.
Conversely, suppose that

$$f \in Q(R) \cap M_n(K[u_{ij}(r)])$$

and let m be an integer such that f can be expressed in terms of $U(1),..., U(m)$. Let $Z = U(r)$ be an generic matrix with $r > m$—that is, Z is not involved in f. Consider the trace $\mathrm{Tr}(fZ)$. It lies in

$$Q(C) \cap K[u_{ij}(r)],$$

which is $\bar{C}$, by Theorem 8(d). By Theorem 8(a), $\bar{C}$ is generated by the traces of elements of R. Bearing in mind that $\mathrm{Tr}(fZ)$ is homogeneous of degree one in the variable $Z = U(r)$, this means that we can express $\mathrm{Tr}(fZ)$ as a K-linear combination

$$\mathrm{Tr}(fZ) = \sum_g a_g \mathrm{Tr}(g_1) \cdots \mathrm{Tr}(g_l) \mathrm{Tr}(g_{l+1} Z)$$

for some l, where $a_g \in K$ and $g = (g_1,..., g_{l+1})$ is a sequence of monomials (possibly constant) in the generic matrices $U(1),..., U(m)$. But

$$\mathrm{Tr}(g_1) \cdots \mathrm{Tr}(g_l) \mathrm{Tr}(g_{l+1} Z) = \mathrm{Tr}[\mathrm{Tr}(g_1) \cdots \mathrm{Tr}(g_l)\, g_{l+1} Z].$$

Hence $\mathrm{Tr}(fZ) = \mathrm{Tr}(f_1 Z)$, where

$$f_1 = \sum_g a_g \mathrm{Tr}(g_1) \cdots \mathrm{Tr}(g_l)\, g_{l+1},$$

which lies in $\bar{R} = \bar{C}R$. This determination of f_1 is independent of the choice of $Z = U(r)$, provided $r > m$. Hence

$$\mathrm{Tr}(fU(r)) = \mathrm{Tr}(f_1 U(r))$$

for all $r > m$. Regard the above as an equation in $M_n(K(u_{ij}(r)))$. Then the nondegeneracy of the trace together with the fact that the $U(r)$ $(r > m)$ span $M_n(K(u_{ij}(r)))$ over its center implies that $f = f_1$, so $f \in \bar{R}$. ∎

We now define a new action of $GL(n)$ on $M_n(K[u_{ij}(r)])$ and $M_n(K(u_{ij}(r)))$ which reduces to the preceding action on scalar matrices. Let $P \in GL(n)$, $(a_{ij}) \in M_n(K[u_{ij}(r)])$. There is an action by conjugation:

$$(a_{ij}) \to P(a_{ij})P^{-1}.$$

There is an action extending $\phi^P: K[u_{ij}(r)] \to K[u_{ij}(r)]$:

$$(a_{ij}) \mapsto (\phi^P(a_{ij})).$$

If we regard $M_n(K[u_{ij}(r)])$ as

$$M_n(K) \otimes K[u_{ij}(r)],$$

then the first action is on the first factor, fixing the second, and the second action is vice-versa. Thus the two actions commute. Noting that the two actions agree on generic matrices, we define

$$\theta^P: M_n(K[u_{ij}(r)]) \to M_n(K[u_{ij}(r)])$$

by

$$\theta^P(a_{ij}) = P^{-1}(\phi^P(a_{ij}))P.$$

This defines a representation of $GL(n)$ since the two actions used to define θ^P commute. It is a rational representation, homogeneous of degree zero. Of course it extends to $M_n(K(u_{ij}(r)))$.

THEOREM 10 [10, Theorem 2.1]. *Let $GL(n)$ act on $M_n(K(u_{ij}(r)))$ and $M_n(K[u_{ij}(r)])$ by $P \mapsto \theta^P$. Then*

(a) $M_n(K(u_{ij}(r)))^{GL(n)} = Q(R)$, *the generic division ring.*

(b) $M_n(K[u_{ij}(r)])^{GL(n)} = \bar{R}$, *the trace ring.*

Proof. (a) Let $S = M_n(K(u_{ij}(r)))^{GL(n)}$. Each θ^P fixes every generic matrix $U(r)$, so $Q(R) \subseteq S$.

For the reverse inclusion, note that since $Q(R) \subseteq S$, S contains a basis for $M_n(K(u_{ij}(r)))$, so its center will consist of scalar matrices. That is,

$$\text{center}(S) = \{a \in K(u_{ij}(r)) \mid \theta^P(a) = a \text{ for all } P \in GL(n)\}.$$

But the action of θ^P on $K(u_{ij}(r))$ is the same as the action of ϕ^P, so by Theorem 8(d, e),

$$\text{center}(S) = Q(C) = \text{center } Q(R).$$

Since S contains a basis for $M_n(K(u_{ij}(r)))$ it is a prime PI-ring and since its center $Q(C)$ is a field it is central simple of dimension n^2 over $Q(C)$. Since

$$Q(C) \subseteq Q(R) \subseteq S$$

and both $Q(R)$ and S are of dimension n^2 over $Q(C)$,

$$Q(R) = S = M_n(K(u_{ij}(r)))^{GL(n)}.$$

(b) Using (a) and Lemma 9,

$$\begin{aligned} M_n(K[u_{ij}(r)])^{GL(n)} &= M_n(K(u_{ij}(r)))^{GL(n)} \cap M_n(K[u_{ij}(r)]) \\ &= Q(R) \cap M_n(K[u_{ij}(r)]) \\ &= \bar{R}. \quad \blacksquare \end{aligned}$$

Remark. It is clear that the preceding results on the ring of generic matrices remain valid for the ring generated by a finite number $m \geqslant 2$ of $n \times n$ generic matrices.

6. Poincare Series for the Ring of Invariants and the Trace Ring

In the preceding section we showed that the ring of invariants and the trace ring of $n \times n$ generic matrices are fixed rings of $GL(n)$, namely,

$$\bar{C} = K[u_{ij}(r)]^{GL(n)},$$
$$\bar{R} = M_n(K[u_{ij}(r)])^{GL(n)},$$

where the action of $GL(n)$ is given, respectively, by $P \mapsto \phi^P$ and $P \mapsto \theta^P$, which were defined in the last section. We will now use this description to obtain their Poincare series, using the methods of Sections 2 and 3.

For notational simplicity, we assume that we have a finite number m of generic matrices, $U(1),\ldots, U(m)$. The Poincare series we obtain will then be power series over

$$\Lambda_m(x) = \mathbf{Z}[x_1,\ldots, x_m]^{S_m}.$$

However, the series obtained will be independent of m in the sense that letting $m \to \infty$ defines a power series over the inverse limit $\Lambda(x)$, from which the m variable Poincare series can be recovered using the canonical projections $\rho(m)\colon \Lambda(x) \to \Lambda_m(x)$. We use new variables $y_1,\ldots, y_n$ for the characters of the representations $P \mapsto \phi^P$, $P \mapsto \theta^P$ of $GL(n)$. These two representations are rational and homogeneous of degree zero, and their characters lie in

$$\bar{\Lambda}_n(y) = \mathbf{Z}[y_1^{\pm 1},\ldots, y_n^{\pm 1}]^{S_n}.$$

We first handle $\bar{C}$ ($\bar{R}$ is similar). The action of $GL(n)$ on

$$K[u_{ij}(r) \mid 1 \leqslant i, j \leqslant n, 1 \leqslant r \leqslant m]$$

commutes with the action of $GL(m)$, where for fixed i and j, the span of $\{u_{ij}(r) \mid 1 \leqslant r \leqslant m\}$ is the standard $GL(m)$-module. (In other words, the action of $GL(m)$ defines the x-multigrading of $K[u_{ij}(r)]$.) Equivalently,

$$K[u_{ij}(r)] = K[M_n(K) \otimes U_m],$$

the *symmetric algebra* on $M_n(K) \otimes U_m$, where $GL(n)$ acts on $M_n(K)$ by conjugation (for that is how $P \mapsto \phi^P$ is defined) and U_m is the standard $GL(m)$-module.

Since the actions of $GL(n)$ and $GL(m)$ commute, $K[u_{ij}(r)]$ is a $GL(n) \times GL(m)$-module which is rational as a $GL(n)$-module and polynomial as a $GL(m)$-module. The Grothendieck ring of finite-dimensional modules of this type is easily seen to be isomorphic to

$$\mathbf{Z}[y_1^{\pm 1},\ldots, y_n^{\pm 1}, x_1,\ldots, x_m]^{S_n \times S_m} \cong \bar{\Lambda}_n(y) \otimes \Lambda_m(x),$$

and we will denote it by $\mathrm{Mod}(n, m)$. Let

$$\chi_y : \overline{\mathrm{Mod}}(GL(n)) \to \bar{\Lambda}_n(y),$$
$$\chi_x : \mathrm{Mod}(GL(m)) \to \Lambda_m(x)$$

be the isomorphisms of Theorems 2 and 3. Then we have the following analogue (or corollary) of those theorems.

LEMMA 11. *Let W be a finite-dimensional $GL(n) \times GL(m)$-module which is rational as a $GL(n)$-module and polynomial as a $GL(m)$-module.*

(a) *W is completely reducible.*

(b) *W is irreducible if and only if $W = N \otimes M$, where N is an irreducible rational $GL(n)$-module and M is an irreducible polynomial $GL(m)$-module.*

(c) *Defining $\chi(N \otimes M) = \chi_y(N)\,\chi_x(M)$ induces an isomorphisms*

$$\chi\colon \mathrm{Mod}(n, m) \to \mathbf{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}, x_1, \ldots, x_m]^{S_n \times S_m}.$$

(d) *If $z_1, \ldots, z_n, t_1, \ldots, t_m$ are the respective eigenvalues of $P \in GL(n)$ and $Q \in GL(m)$, then the trace of (P, Q) acting on $N \otimes M$ is*

$$\chi(N \otimes M)(z_1, \ldots, z_n, t_1, \ldots, t_m) = \chi_y(N)(z_1, \ldots, z_n)\,\chi_x(M)(t_1, \ldots, t_m).$$

If W is a $GL(n) \times GL(m)$-module, then $W^{GL(n)}$, the set of $GL(n)$ fixed points, is a $GL(m)$-module, and if W has the form $N \otimes M$ as above (but is not necessarily irreducible), then

$$(N \otimes M)^{GL(n)} = N^{GL(n)} \otimes M = \langle [N], 1 \rangle M,$$

where $\langle\, ,\, \rangle$ denotes the inner product on $\overline{\mathrm{Mod}}(GL(n))$ and 1 represents the trivial module.

To analyze $K[u_{ij}(r)]$ as a $GL(n) \times GL(m)$-module, we express it as a graded algebra

$$K[u_{ij}] = K \oplus A_1 \oplus A_2 \oplus \cdots$$

(where each generator $u_{ij}(r)$ has degree one) and define its Poincare series as a $GL(n) \times GL(m)$-module to be

$$P(K[u_{ij}(r)]) = 1 + \chi(A_1) + \chi(A_2) + \ldots,$$

which is a formal power series over $\mathbf{Z}[y_i^{\pm 1}, x_j]^{S_n \times S_m}$.

If $P \in GL(n)$, $Q \in GL(m)$ have eigenvalues $z_1, \ldots, z_n, t_1, \ldots, t_m$, respectively, the eigenvalues of (P, Q) acting on $M_n(K) \otimes U_m$ are

$$\{z_i z_j^{-1} t_k \mid 1 \leqslant i, j \leqslant n,\ 1 \leqslant k \leqslant m\}.$$

Hence, because $K[u_{ij}(r)]$ is the symmetric algebra on $M_n(K) \otimes U_m$, $\chi(A_w)$ is the w-th complete symmetric function of

$$\{y_i y_j^{-1} x_k \mid 1 \leqslant i, j \leqslant n,\ 1 \leqslant k \leqslant m\},$$

that is, the coefficient of t^w in the generating function

$$H(y_i y_j^{-1} x_k; t) = \pi(1 - y_i y_j^{-1} x_k t)^{-1}.$$

Now we invoke Theorem 1, which asserts that

$$\pi(1-y_i y_j^{-1} x_k)^{-1} = \sum_\lambda s_\lambda(y_i y_j^{-1})\, s_\lambda(x_k),$$

where the sum is over all partitions λ. Thus

$$P(K[u_{ij}(r)]) = \sum_\lambda s_\lambda(y_i y_j^{-1})\, s_\lambda(x_k),$$

$$\chi(A_w) = \sum_{|\lambda| = w} s_\lambda(y_i y_j^{-1})\, s_\lambda(x_k).$$

It remains to determine $\chi_x(A_w^{GL(n)})$. Each $s_\lambda(x_k)$ corresponds to an irreducible $GL(m)$-module (or zero), but $s_\lambda(y_i y_j^{-1})$ in general corresponds to a sum of irreducibles. It is the image of $s_\lambda(z_{ij})$ $(1 \leqslant i, j \leqslant n)$ under the homomorphism

$$\mathbf{Z}[z_{ij}]^{S_n^2} \to \mathbf{Z}[y_1^{\pm 1}, ..., y_n^{\pm 1}]^{S_n}$$

induced by $z_{ij} \mapsto y_i y_j^{-1}$. In any case, $s_\lambda(y_i y_j^{-1}) = \chi_y(N)$ for some rational $GL(n)$-module N, and to obtain the fixed points we replace N by

$$N^{GL(n)} = \langle [N], 1 \rangle \cdot 1.$$

Since χ_y is an isometry,

$$\chi_y(N^{GL(n)}) = \langle \chi(N), 1 \rangle = \langle s_\lambda(y_i y_j^{-1}), 1 \rangle.$$

Thus the Poincare series of $\bar{C}$ as a $GL(m)$-module is

$$\begin{aligned} P(\bar{C}) &= P(K[u_{ij}(r)]^{GL(n)}) \\ &= 1 + \chi_y(A^{GL(n)}) + \chi_y(A_2^{GL(n)}) + \cdots \\ &= \sum_\lambda \langle s_\lambda(y_i y_j^{-1}), 1 \rangle\, s_\lambda(x_k). \end{aligned}$$

To obtain $P(\bar{R})$, we need only take account of the difference between ϕ^P and θ^P. As graded rings

$$K[u_{ij}(r)] = K \oplus A_1 \oplus A_2 \oplus \cdots,$$

$$M_n(K[u_{ij}(r)]) = M_n(K) \oplus (M_n(K) \otimes A_1) \oplus (M_n(K) \otimes A_2) \oplus \cdots.$$

If $P \in GL(n)$, P acts on A_w by

$$a \mapsto \phi^P(a)$$

and on $M_n(K) \otimes A_w$ by

$$b \otimes a \mapsto \theta^P(b \otimes a) = (P^{-1} b P) \otimes \phi^P(a),$$

where $a \in A_w$, $b \in M_n(K)$. Hence

$$\chi(M_n(K) \otimes A_w) = \left(\sum y_i y_j^{-1}\right) \chi(A_w)$$
$$= s_{(1)}(y_i y_j^{-1}) \chi(A_w),$$

and the Poincare series of $\bar{R}$ as a $GL(m)$-module is obtained from that of $\bar{C}$ by replacing $\langle s_\lambda(y_i y_j^{-1}), 1\rangle$ by

$$\langle s_{(1)}(y_i y_j^{-1}) s_\lambda(y_i y_j^{-1}), 1\rangle = \langle s_\lambda(y_i y_j^{-1}), s_{(1)}(y_i y_j^{-1})\rangle.$$

(This last equality uses the fact that $\langle ab, c\rangle = \langle a, b^*c\rangle$, where $*: \bar{\Lambda}(y) \to \bar{\Lambda}(y)$ is the involution defined by $y_i^* = y_i^{-1}$.)

Although $P(\bar{C})$ and $P(\bar{R})$ are summations over all partitions λ, $s_\lambda(y_i y_j^{-1})$ is zero unless λ has length $\leqslant n^2$, so it is enough to sum over partitions of length $\leqslant n^2$. And while the series are over $\Lambda_m(x)$, they clearly converge to series over $\Lambda(x)$ as $m \to \infty$. Hence they can be expressed without reference to m.

THEOREM 12. *Let $R = K[U(1), U(2),...]$ be the ring of $n \times n$ generic matrices and let*

$$\bar{C} = K[u_{ij}(r)]^{GL(n)},$$
$$\bar{R} = M_n(K[u_{ij}(r)])^{GL(n)}$$

be the corresponding ring of invariants and trace ring. Then the Poincare series of $\bar{C}$ and $\bar{R}$ as infinite series over $\Lambda(x)$ are

$$P(\bar{C}) = \sum_\lambda \langle s_\lambda(y_i y_j^{-1}), 1\rangle s_\lambda(x),$$

$$P(\bar{R}) = \sum_\lambda \langle s_\lambda(y_i y_j^{-1}), s_{(1)}(y_i y_j^{-1})\rangle s_\lambda(x),$$

where λ varies over all partitions $\lambda = (\lambda_1,..., \lambda_{n^2})$ of length $\leqslant n^2$, $s_\lambda(y_i y_j^{-1})$ denotes the evaluation of $s_\lambda(z_{ij})$ $(1 \leqslant i, j \leqslant n)$ at $z_{ij} = y_i y_j^{-1}$, and the inner product is evaluated on $\bar{\Lambda}_n(y) = \mathbf{Z}[y_1^{\pm 1},..., y_n^{\pm 1}]^{S_n}$.

Remarks. (1) Other expressions for $P(\bar{C})$ and $P(\bar{R})$ can be obtained by using the other expansions of $\pi(1 - x_i y_j)^{-1}$ in Theorem 1. For example,

$$P(\bar{C}) = \sum_\lambda z_\lambda^{-1} \langle p_\lambda(y_i y_j^{-1}), 1\rangle p_\lambda(x),$$

but the sum must be taken over all partitions λ since $p_\lambda(y_i y_j^{-1})$ is nonzero for all partitions. The formulation in terms of the s_λ is the primary one in the sense that the s_λ's correspond to irreducible modules. The coefficient of s_λ in

$P(\bar{R})$ is what Regev calls the multiplicity of the irreducible character of $S_{|\lambda|}$ associated with λ in the cocharacter sequence of trace identities of $n \times n$ matrices [13, p. 1418].

(2) If the number of generic matrices is finite, say m, then Hilbert's theory asserts that $\bar{C}$ is finitely generated by homogeneous elements and that $P(\bar{C})$ is a rational function of $x_1, ..., x_m$. Since $\bar{R}$ is a finite module over $\bar{C}$, the same is true for $P(\bar{R})$. I thank Pat Halpin for pointing this out.

(3) Since the series involve only partitions of length $\leqslant n^2$, the Poincare series in n^2 variables determine the Poincare series in any number of variables.

(4) Our determination of $P(\bar{C})$ and $P(\bar{R})$ is via standard methods of invariant theory, which stem from Schur and Weyl. Schur solved the classical problem of decomposing the symmetric algebra $K[U_n \otimes U_m]$ as a $GL(n) \times GL(m)$ module, where U_n and U_m are the standard modules of $GL(n)$, $GL(m)$, respectively. A survey article of Verma [15] discusses Schur's solution from many points of view: Invariant theory, algebraic groups, coalgebras, Hopf algebras, Lie algebras.

(5) When the number of generic matrices is finite, the Molien–Weyl theorem [17, pp. 5–6] gives equivalent formulas for $P(\bar{C})$ and $P(\bar{R})$ as integrals over the complex unitary group $U(n, \mathbf{C})$ with normalized Haar measure μ. For example, for m generic $n \times n$ matrices

$$P(\bar{C}) = \int_{U(n,\mathbf{C})} \left(\prod_{i=1}^{m} \det(1 - x_i \phi^P)^{-1} \right) d\mu(P),$$

$$P(\bar{R}) = \int_{U(n,\mathbf{C})} \operatorname{Tr}(\phi^P) \left(\prod_{i=1}^{m} \det(1 - x_i \phi^P)^{-1} \right) d\mu(P),$$

where for each $P \in U(n, \mathbf{C})$, $\phi^P \in U(n^2, \mathbf{C})$ is a matrix giving the action of P by conjugation on $M_n(\mathbf{C})$.

(6) The Poincare series of $\bar{R}$ can be obtained from that of $\bar{C}$ by the following device. Let $\bar{C}_m$, $\bar{R}_m$ denote the rings obtained by taking a finite number m of generic $n \times n$ matrices. Their Poincare series lie in

$$\Lambda_m(x) = \mathbf{Z}[x_1, ..., x_m]^{S_m} \subseteq \mathbf{Z}[x_1, ..., x_m].$$

The map

$$f \mapsto \operatorname{Tr}(f \cdot U(m+1))$$

defines a monomorphism from $\bar{R}_m$ onto the subspace of $\bar{C}_{m+1}$ consisting of elements of degree one in $U(m+1)$—i.e., the sum of all homogeneous components with degree sequence $(\alpha_1, \ldots, \alpha_m, 1)$. Translating to Poincare series, this means that $P(\bar{R}_m)$ is the coefficient of x_{m+1} in $P(\bar{C}_{m+1})$, or

$$P(\bar{R}_m) = \frac{\partial P(\bar{C}_{m+1})}{\partial x_{m+1}} \bigg|_{x_{m+1}=0}.$$

The map

$$g(x_1, \ldots, x_{m+1}) \mapsto \frac{\partial g}{\partial x_{m+1}} \bigg|_{x_{m+1}=0}$$

induces a map

$$\partial : \Lambda_{m+1}(x) \to \Lambda_m(x)$$

characterized by the properties

$$\partial(h_r) = h_{r-1},$$
$$\partial(gh) = \partial(g)\,\rho(h) + \rho(g)\,\partial(h),$$

where $\rho = \rho(m+1, m) : \Lambda_{m+1}(x) \to \Lambda_m(x)$ is the standard projection. A derivation $\partial : \Lambda(x) \to \Lambda(x)$ is induced by letting $m \to \infty$ and for this derivation

$$\partial(P(\bar{C})) = P(\bar{R}).$$

7. The Discriminant and the Conductor

The ring of $n \times n$ generic matrices R and its center C are subrings, respectively, of the trace ring $\bar{R}$ and the ring of invariants $\bar{C}$.

$$\begin{array}{ccc} C & \longrightarrow & \bar{C} \\ \downarrow & & \downarrow \\ R & \longrightarrow & \bar{R}. \end{array}$$

We introduce the following notation for the Poincare series of these rings

$$P(C) = \sum c(\lambda)\, s_\lambda, \qquad P(\bar{C}) = \sum \bar{c}(\lambda)\, s_\lambda,$$
$$P(R) = \sum r(\lambda)\, s_\lambda, \qquad P(\bar{R}) = \sum \bar{r}(\lambda)\, s_\lambda,$$

where λ varies over all partitions of length $\leqslant n^2$ and the coefficients are nonnegative integers.

The coefficients $\bar{c}(\lambda)$ and $\bar{r}(\lambda)$ are given by Theorem 12. The main object of this section is to give estimates for $c(\lambda)$ and $r(\lambda)$ in terms of $\bar{c}(\lambda)$ and $\bar{r}(\lambda)$.

Of course $c(\lambda) \leqslant \bar{c}(\lambda)$ and $r(\lambda) \leqslant \bar{r}(\lambda)$, so the problem is to find lower bounds. The idea is to take the number of generic matrices to be n^2—which does not affect the coefficients—and then find an element α in C which satisfies:

(a) The one-dimensional subspace $K\alpha$ is invariant under $GL(n^2)$. I.e., α is a *relative invariant.*

(b) $\alpha \bar{R} \subseteq R$.

These two properties imply that

$$\alpha \bar{R} \subseteq R \subseteq \bar{R}$$

is a series of inclusions of $GL(n^2)$-modules and hence that

$$P(\alpha \bar{R}) \leqslant P(R) \leqslant P(\bar{R}), \tag{$*$}$$

where we say that

$$\sum a(\lambda)\, s_\lambda \leqslant \sum b(\lambda)\, s_\lambda$$

if $a(\lambda) \leqslant b(\lambda)$ for all partitions λ. (This condition is strictly stronger than demanding that

$$f(x_1,..., x_{n^2}) = \sum (a(\lambda) - b(\lambda)) s_\lambda$$

has nonnegative coefficients as an element of $\mathbf{Z}[x_1,..., x_{n^2}]$. For example, $s_{(2)}(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$, $s_{(1,1)}(x_1, x_2) = x_1 x_2$.)

Although a priori ($*$) only gives bounds for the coefficients of $P(R)$, a fortunate event occurs: For all sufficiently large partitions λ, the multiplicity of s_λ in $P(\alpha \bar{R})$ equals $\bar{r}(\lambda)$, its multiplicity in $P(\bar{R})$. Hence $r(\lambda) = \bar{r}(\lambda)$ for these partitions.

For the rest of this section we assume that

$$R = K[U(1),..., U(n^2)]$$

is the generic matrix ring generated by n^2 *generic* $n \times n$ *matrices.* Since the Poincare series of $\bar{R}$ (when expressed as a formal power series in the Schur functions s_λ) only involves partitions of length $\leqslant n^2$, the results we obtain on Poincare series will be valid for any number of $n \times n$ generic matrices.

The Poincare series

$$P(\bar{R}) = \sum \bar{r}(\lambda)\, s_\lambda(x_1,..., x_{n^2})$$

gives the structure of $\bar{R}$ as a $GL(n^2)$-module, where $\theta = (\theta_{rs}) \in GL(n^2)$ acts on

$$M_n(K[u_{ij}(r)]) = M_n(K) \otimes K[u_{ij}(r)]$$

by acting trivially on $M_n(K)$ and acting on $K[u_{ij}(r)]$ by

$$\theta(u_{ij}(r)) = \sum_s \theta_{rs} u_{ij}(s).$$

In other words, for fixed i and j, the K-vector space spanned by

$$\{u_{ij}(r) \mid 1 \leqslant r \leqslant n^2\}$$

is the standard $GL(n^2)$-module and (θ_{rs}) is the $n^2 \times n^2$ matrix giving the action of θ (acting from the left on column vectors) relative to the ordered basis

$$(u_{ij}(1),..., u_{ij}(n^2)).$$

Similarly, θ acts on R as a linear transformation of the K-vector space spanned by $U(1),..., U(n^2)$:

$$\theta(U(r)) = \sum_s \theta_{rs} U(s).$$

The action of $GL(n^2)$ is a polynomial action and there are no *absolute* $GL(n^2)$-invariants (fixed points) of positive degree. But there are *relative invariants*—i.e., one-dimensional $GL(n^2)$-modules. These are all powers of the determinant (Theorem 3(a))—more precisely, let $(\det)^r$ denote K as a $GL(n^2)$-module with action

$$\theta(\alpha) = (\det \theta)^r \alpha.$$

This module is homogeneous of degree rn^2 and its character is

$$(x_1 \cdots x_{n^2})^r = (e_{n^2})^r = s_{(r^{n^2})},$$

where $(r^{n^2}) = (r,..., r)$ (n^2 times).

Let

$$\mu = (1^{n^2}), \qquad \delta = (n^2 - 1, n^2 - 2,..., 1, 0)$$

and recall (Section 2) that the Schur function $s_\lambda = s_\lambda(x_1,..., x_{n^2})$ is defined, for any partition $\lambda = (\lambda_1,..., \lambda_{n^2})$ of length $\leqslant n^2$, by

$$s_\lambda = a_{\lambda+\delta}/a_\delta,$$

where

$$a_\lambda(x_1,..., x_{n^2}) = \sum_{\pi \in S_{n^2}} (\text{sign } \pi)\, x_{\pi(1)}^{\lambda_1} \cdots x_{\pi(n^2)}^{\lambda_{n^2}}.$$

Since $a_{\mu+\lambda} = x_1 \cdots x_{n^2} a_\lambda$,

$$\begin{aligned} s_\mu s_\lambda &= x_1 \cdots x_{n^2} a_{\lambda+\delta}/a_\delta \\ &= a_{\mu+\lambda+\delta}/a_\delta \\ &= s_{\mu+\lambda}. \end{aligned}$$

We can use this to derive information about the coefficients $\bar{c}(\lambda)$, $\bar{r}(\lambda)$. By Theorem 12,

$$\bar{c}(\lambda) = \langle s_\lambda(y_i y_j^{-1}), 1\rangle,$$

where $s_\lambda(y_i y_j^{-1})$ denotes the image of $s_\lambda(x_1,..., x_{n^2})$ under the homomorphism

$$\mathbf{Z}[x_1,..., x_{n^2}]^{S_{n^2}} \to \mathbf{Z}[y_1^{\pm 1},..., y_n^{\pm 1}]^{S_n}$$

obtained from the specialization

$$(x_1,..., x_{n^2}) \mapsto (y_i y_j^{-1} \mid 1 \leqslant i, j \leqslant n).$$

(The statement of Theorem 12 used variables $\{z_{ij} \mid 1 \leqslant i, j \leqslant n\}$ instead of $\{x_1,..., x_{n^2}\}$.) Since $s_\mu(y_i y_j^{-1}) = 1$,

$$\begin{aligned} \bar{c}(\mu+\lambda) &= \langle s_{\mu+\lambda}(y_i y_j^{-1}), 1\rangle \\ &= \langle s_\mu(y_i y_j^{-1})\, s_\lambda(y_i y_j^{-1}), 1\rangle \\ &= \langle s_\lambda(y_i y_j^{-1}), 1\rangle \\ &= \bar{c}(\lambda), \end{aligned}$$

and similarly $\bar{r}(\mu+\lambda) = \bar{r}(\lambda)$. Moreover, for the empty partition 0,

$$\begin{aligned} \bar{c}(0) &= \langle 1, 1\rangle = 1, \\ \bar{r}(0) &= \langle 1, s_{(1)}(y_i y_j^{-1})\rangle \\ &= \langle 1, s_{(1)}(y_i)\, s_{(1)}(y_i^{-1})\rangle \\ &= \langle s_{(1)}(y_i), s_{(1)}(y_i)\rangle = 1. \end{aligned}$$

Hence we have

LEMMA 13. *Let* $P(\bar{C}) = \sum \bar{c}(\lambda)\, s_\lambda$, $P(\bar{R}) = \sum \bar{r}(\lambda) s_\lambda$, *and let* $\mu = (1^{n^2})$. *Then*

(a) *For all partitions* $\lambda = (\lambda_1, \ldots, \lambda_{n^2})$ *of length* $\leqslant n^2$,

$$\bar{c}(\mu + \lambda) = \bar{c}(\lambda), \qquad \bar{r}(\mu + \lambda) = \bar{r}(\lambda).$$

(b) *For all* $m \geqslant 0$, $\bar{c}(m\mu) = \bar{r}(m\mu) = 1$.

Remark. Any partition $\lambda = (\lambda_1, \ldots, \lambda_{n^2})$ of length $\leqslant n^2$ can be written

$$\lambda = (\lambda_{n^2})\,\mu + \lambda',$$

where $\lambda' = (\lambda_1 - \lambda_{n^2}, \ldots, \lambda_{n^2-1} - \lambda_{n^2}, 0)$ has length $\leqslant n^2 - 1$. Hence, modulo Lemma 13, the Poincare series for $\bar{C}$ or $\bar{R}$ in any number of variables is determined by the series in $n^2 - 1$ variables.

Part (b) of the lemma asserts that in $\bar{C}$ (or $\bar{R}$) there is a unique relative invariant of degree mn^2 for each $m \geqslant 0$. This implies that all the relative invariants are powers of a single relative invariant of degree n^2, which we now describe. Set

$$\begin{aligned}
\Delta &= \Delta(U(1), \ldots, U(n^2)) \\
&= \det \begin{pmatrix} u_{11}(1)\, u_{12}(1) & \cdots & u_{nn}(1) \\ \vdots & & \vdots \\ u_{11}(n^2)\, u_{12}(n^2) & \cdots & u_{nn}(n^2) \end{pmatrix} \\
&= \det \mathscr{U},
\end{aligned}$$

the determinant of the $n^2 \times n^2$ matrix $\mathscr{U}$ whose rows are $U(1), \ldots, U(n^2)$ written as $1 \times n^2$ row vectors. The generic matrices $U(r)$ $(1 \leqslant r \leqslant n^2)$ form a basis for $M_n(K(u_{ij}(r)))$ over its center and Δ is the discriminant of this basis.

Suppose $P \in GL(n)$ and $\theta = (\theta_{rs}) \in GL(n^2)$. Let ϕ^P denote the linear transformation of $M_n(K)$ defined by

$$\phi^P(A) = PAP^{-1},$$

and let (ϕ^P_{rs}) be the matrix of ϕ^P relative to the ordered basis $e_{11}, e_{12}, \ldots, e_{nn}$ (acting on row vectors from the right, while (θ_{rs}) acts on column vectors from the left). As usual, ϕ^P and θ also act as automorphisms of $K[u_{ij}(r)]$. Then

$$\begin{aligned}
\phi^P(\Delta) &= \det \begin{pmatrix} \phi^P(u_{11}(1)) & \cdots & \phi^P(u_{nn}(1)) \\ \vdots & & \vdots \\ \phi^P(u_{11}(n^2)) & \cdots & \phi^P(u_{nn}(n^2)) \end{pmatrix} \\
&= \det(\mathscr{U}(\phi^P_{rs})) \\
&= (\det \mathscr{U})(\det \phi^P) \\
&= \Delta.
\end{aligned}$$

$$\theta(\Delta) = \det \begin{pmatrix} \theta(u_{11}(1)) & \cdots & \theta(u_{nn}(1)) \\ \vdots & & \vdots \\ \theta(u_{11}(n^2)) & \cdots & \theta(u_{nn}(n^2)) \end{pmatrix}$$
$$= \det((\theta_{rs})\mathscr{U})$$
$$= (\det\theta)(\det\mathscr{U})$$
$$= (\det\theta)\Delta.$$

The first equation says that

$$\Delta \in K[u_{ij}(r)]^{GL(n)} = \bar{C}$$

and the second says that it is a relative invariant of degree n^2 for $GL(n^2)$. Hence Δ is the unique relative invariant in $\bar{C}$ of lowest degree, and all other relative invariants in $\bar{C}$ or $\bar{R}$ are powers of Δ.

Since $\Delta \in \bar{C}$, the first fundamental theorem (Theorem 8(a)) asserts that it can be expressed in terms of traces. The following expression for Δ was found by Procesi and me in 1975.

Suppose that $\lambda = (\lambda_1, ..., \lambda_k)$ is a partition with

$$|\lambda| = \lambda_1 + \cdots + \lambda_k = n^2.$$

For each such λ, define $A_\lambda = A_\lambda(U(1), ..., U(n^2))$ by

$$A_\lambda = \sum_{\pi \in S_{n^2}} (\operatorname{sign}\pi)\operatorname{Tr}[U(\pi(1)) \cdots U(\pi(\lambda_1))]$$
$$\operatorname{Tr}[U(\pi(\lambda_1+1)) \cdots U(\pi(\lambda_1+\lambda_2))]$$
$$\ddots$$
$$\operatorname{Tr}[U(\pi(\lambda_1+\cdots+\lambda_{k-1}+1)) \cdots U(\pi(n^2))].$$

It is easy to see that since Δ can be expressed in terms of traces of monomials in $U(1), ..., U(n^2)$ and is multilinear and alternating as a function of $U(1), ..., U(n^2)$, it can be written as a K-linear combination

$$\Delta = \sum a(\lambda) A_\lambda.$$

(Given any expression $\Delta = f(U(r))$, where f is a polynomial in traces of monomials in $U(1), ..., U(n^2)$, throw away the terms which are not multilinear and then replace f by the formally alternating function

$$f'(U(r)) = \frac{1}{(n^2)!} \sum_{\pi \in S_{n^2}} (\operatorname{sign}\pi) f(U(\pi(r))).$$

Then $f' = \Delta$ and $f' = \sum a(\lambda) A_\lambda$.)

Conversely, each A_λ is a multilinear alternating function of $U(1),\ldots, U(n^2)$—thus a relative invariant of $GL(n^2)$ of degree n^2—thus a scalar multiple of Δ. The real problem is to discover which A_λ are nonzero. The following observations show that all A_λ except one are zero and hence that Δ is a scalar multiple of the remaining one.

Let $\lambda = (\lambda_1,\ldots, \lambda_k)$, $|\lambda| = n^2$.

(a) If any λ_i is even, then $A_\lambda = 0$, since the trace of any standard identity of *even* degree is zero, and A_λ can be expressed as a sum of products, each having the trace of the standard identity of degree λ_i as a factor.

(b) If two λ_i's are odd and equal, then $A_\lambda = 0$ since it is alternating.

(c) If $\lambda_1 \geqslant 2n$, then $A_\lambda = 0$ since the ring of $n \times n$ generic matrices satisfies the standard identity of degree $2n$ (Amitsur–Levitzki theorem).

This shows that $A_\lambda = 0$ uness the parts λ_i of λ are distinct, odd, and less than $2n$. But there is only one way to express n^2 as a sum of distinct odd integers less than $2n$, namely,

$$n^2 = 1 + 3 + 5 + \cdots + 2n - 1.$$

Hence A_λ is nonzero only for $\lambda_0 = (2n - 1, 2n - 3,\ldots, 3, 1)$.

We summarize the above remarks as

LEMMA 14. *Suppose $U(r)$ $(1 \leqslant r \leqslant n^2)$ are $n \times n$ generic matrices, and $\bar{C}$ and $\bar{R}$ are the associated ring of invariants and trace ring. Let $GL(n^2)$ act on $\bar{C}$ and $\bar{R}$ as above, and let Δ and A_{λ_0} be defined as above, where $\lambda_0 = (2n - 1, 2n - 3,\ldots, 3, 1)$. Then*

(a) *$K\Delta$ is a $GL(n^2)$-invariant subspace of $\bar{C}$.*

(b) *The only one-dimensional $GL(n^2)$-invariant subspaces of $\bar{C}$ and $\bar{R}$ are the subspaces $K\Delta^m (m = 0, 1, 2,\ldots)$.*

(c) *$\Delta = cA_{\lambda_0}$, where c is a nonzero scalar depending only on n.*

The *conductor* from $\bar{R}$ to R is the largest ideal of R which is also an ideal of $\bar{R}$, namely,

$$(\bar{R} : R) = \{\alpha \in R \mid \bar{R}\alpha\bar{R} \subseteq R\} = \{\alpha \in R \mid \alpha\bar{C} \subseteq R\}.$$

The *Capelli polynomial* was introduced and exploited by Razmyslov [12], who showed that all evaluations of the n^2th Capelli polynomial lie in the conductor. The mth Capelli polynomial is the polynomial in noncommuting variables u_i, v_i defined by

$$C_m(u_1,\ldots, u_m, v_1,\ldots, v_m) = \sum_{\pi \in S_m} (\operatorname{sign} \pi)\, u_{\pi(1)} v_1 \cdots u_{\pi(m)} v_m.$$

Remarks. (a) Let $V(1),\ldots, V(n^2)$ be a new set of $n \times n$ generic matrices and let $\bar{R}(V)$ denote the trace ring associated with them. Consider

$$Z = C_{n^2}(U(1),\ldots, U(n^2), V(1),\ldots, V(n^2)),$$

which is known to be nonzero [12]. Since Z is alternating and multilinear in $U(1),\ldots, U(n^2)$, Z can be factored

$$Z = \Delta(U(r))\, f(V(r)),$$

where $f(V(r))$ lies in $\bar{R}(V)$ and is multilinear in $V(1),\ldots, V(n^2)$ (see [1, 3.2]). It would be useful to known that $f(V(r))$ is, particularly in connection with the conjecture of Regev given following Theorem 16.

(b) The Procesi–Razmyslov theorem on trace identities implies that $\bar{C}$ is generated as a K-vector space by all products

$$\operatorname{Tr}(\mu_1) \cdots \operatorname{Tr}(\mu_n)$$

of n traces, where the μ_i are monomials in the generic matrices. The point is that the number of factors needed is uniformly bounded. If we take R to be generated by an infinite number of generic matrices (so that for any element f of R, there are "other" generic matrices not involved in f), it follows that $f = f(U(1),\ldots, U(k))$ lies in the conductor if and only if

$$\operatorname{Tr}(Z_1) \cdots \operatorname{Tr}(Z_n)\, f \in R,$$

where $Z_1,\ldots, Z_n$ are distinct generic matrices different from $U(1),\ldots, U(k)$. This formulation shows that if we let

$$(\bar{R} : R) = J/\mathscr{M}(n) \subseteq K\langle U\rangle/\mathscr{M}(n) = R,$$

then J is a T-ideal in $K\langle U\rangle$ and again it would be useful to know what it is. The next lemma suggests that J may be generated by all evaluations of the n^2th Capelli polynomial, at least modulo $\mathscr{M}(n)$. The lemma is valid without any hypothesis on the number of generic matrices, except that there must be at least two.

LEMMA 15. (a) *The Capelli polynomial C_{n^2} is not a polynomial identity for $M_n(K)$ but it is a polynomial identity for $M_{n-1}(K)$.*

(b) *The K-vector space D generated by all evaluations of C_{n^2} on R is an ideal of $\bar{R}$. In particular, D is contained in the conductor from $\bar{R}$ to R.*

(c) *Let M be the ideal of R generated by the identities of $M_{n-1}(K)$. Then the discriminant $\Delta = \Delta(U(r))$ lies in $\bar{R}M$.*

(d) *There is an integer k such that Δ^k lies in D. Hence $\Delta^k\bar{R} \subseteq R$, $\Delta^k\bar{C} \subseteq C$.*

Proof. (a) and (b) are due to Razmyslov [12] (see also [1, Theorem 6 and Corollary 8]).

(c) Set

$$\begin{aligned}\Delta_0 = \sum_{\pi \in S_{n^2}} &(\operatorname{sign} \pi) \operatorname{Tr}[U(\pi(1))] \operatorname{Tr}[U(\pi(2))\, U(\pi(3))\, U(\pi(4))] \\ &\cdots \operatorname{Tr}[U(\pi(n-2)^2+1) \cdots U(\pi(n-1)^2)] \\ &U(\pi(n-1)^2+1) \cdots U(\pi(n^2)).\end{aligned}$$

Since Δ_0 is alternating and multilinear in $U(1),..., U(n^2)$, it is a $GL(n^2)$ relative invariant and hence a scalar multiple of Δ, by the uniqueness of Δ. But

$$\operatorname{Tr}(\Delta_0) = A_{\lambda_0} = c\Delta,$$

where A_{λ_0} and $c \neq 0$ are as in Lemma 14. Hence $\Delta_0 = c\Delta/n$.

By suitably collecting terms, Δ_0 can be expressed as a sum of products of traces and evaluations of the standard identity of degree $2n-1$, which lies in M. Thus Δ_0—and also Δ—lies in $\bar{R}M$.

(d) Consider the finitely generated PI-algebra $\bar{R}/D$. If P is a prime ideal of $\bar{R}$, then (a) implies that $P \supseteq M$ (i.e. $\bar{R}/P$ has PI-degree $<n$) if and only if $P \supseteq D$. Hence

$$\text{Prime radical } (D) \supseteq \bar{R}M,$$

so Δ lies in the prime radical of D. The prime radical of any ring is a nil ideal, so $\Delta^k \in D$ for some k. ∎

Using Lemma 15, we can now show that for sufficiently large partitions λ, the multiplicity of s_λ is the same in $P(C)$ as in $P(\bar{C})$, and the same in $P(R)$ as in $P(\bar{R})$.

THEOREM 16. *Let k be an integer such that $\Delta^k\bar{R} \subseteq R$, $\Delta^k\bar{C} \subseteq C$ (k exists by Lemma 15), and let*

$$P(C) = \sum c(\lambda)\, s_\lambda, \qquad P(\bar{C}) = \sum \bar{c}(\lambda)\, s_\lambda,$$
$$P(R) = \sum r(\lambda)\, s_\lambda, \qquad P(\bar{R}) = \sum \bar{r}(\lambda)\, s_\lambda.$$

Suppose that λ is a partition such that $\lambda \geqslant (k^{n^2}) = k\mu$, where $\mu = (1^{n^2}) = (1,..., 1)$. Then $c(\lambda) = \bar{c}(\lambda)$, $r(\lambda) = \bar{r}(\lambda)$.

Proof. We show that $c(\lambda) = \bar{c}(\lambda)$; the proof that $r(\lambda) = \bar{r}(\lambda)$ is the same. Note that since Δ^k is a $GL(n^2)$ relative invariant

$$\Delta^k\bar{C} \subseteq C \subseteq \bar{C} \tag{1}$$

is a series of inclusions of $GL(n^2)$-modules. Let

$$P(\Delta^k \bar{C}) = \sum c'(\lambda)\, s_\lambda.$$

Then (1) implies that

$$c'(\lambda) \leqslant c(\lambda) \leqslant \bar{c}(\lambda) \tag{2}$$

for all partitions λ.

Since $K\Delta^k$ has character $s_{k\mu} = (x_1 \cdots x_{n^2})^k$ and $\bar{c}(k\mu + \lambda) = \bar{c}(\lambda)$ for all partitions λ (Lemma 13(a)),

$$\begin{aligned} P(\Delta^k \bar{C}) &= \chi(K\Delta^k)\, P(\bar{C}) \\ &= s_{k\mu} \sum \bar{c}(\lambda)\, s_\lambda \\ &= \sum \bar{c}(k\mu + \lambda)\, s_{k\mu+\lambda} \\ &= \sum \{\bar{c}(\lambda)\, s_\lambda \mid \lambda \geqslant k\mu\}. \end{aligned}$$

Hence $c'(\lambda) = \bar{c}(\lambda)$ for any $\lambda \geqslant k\mu$, and then (2) implies that $c(\lambda) = \bar{c}(\lambda)$. ∎

Remark. Regev [13, Conjecture 2.4] has conjectured that the polynomial

$$\begin{aligned} f(u_1,\ldots,u_{n^2}, v_1,\ldots,v_{n^2}) &= \sum_{\pi,\sigma \in S_{n^2}} \operatorname{sign}(\pi\sigma)\, u_{\pi(1)} v_{\sigma(1)} u_{\pi(2)} u_{\pi(3)} u_{\pi(4)} v_{\sigma(2)} v_{\sigma(3)} v_{\sigma(4)} u_{\pi(5)} \\ &\quad \cdots u_{\pi((n-1)^2+1)} \cdots u_{\pi(n^2)} v_{\sigma((n-1)^2+1)} \cdots v_{\sigma(n^2)} \end{aligned}$$

is not a polynomial identity for $M_n(K)$ and verified his conjecture for $n = 2, 3$. Since f is multilinear and alternating, its value under the substitution of $n \times n$ generic matrices $u_r = U(r)$, $v_r = V(r)$ must be a scalar multiple of $\Delta(U(r))\,\Delta(V(r))$. If Regev's conjecture is true, the scalar is nonzero and we can take $k = 2$ in Theorem 16.

On the other hand, for $n > 1$, Δ does not lie in R since it would necessarily be an evaluation of the standard identity of degree n^2, which vanishes on R.

8. Comparison with the Procesi–Razmyslov Theorem

A fundamental theorem proved independently by Procesi [10] and Razmyslov [11] gives an explicit description of all multilinear "trace identities" for $n \times n$ matrices. Their result leads to an expression for $P(\bar{C})$ which is different from that of Theorem 12, and setting the two equal yields a result about the representations of S_m which has no reference to matrices or identities (Theorem 17). To give complete details of the derivation of this alternate expression for $P(\bar{C})$ requires a considerable amount of new definitions and notation, so we will only sketch it. For more details the reader is referred to [5, Sections 4–5 and especially Theorem 23 and the subsequent discussion].

Recall that the usual tensor product of modules with diagonal group action defines a multiplication

$$\mathrm{Mod}(S_m) \times \mathrm{Mod}(S_m) \to \mathrm{Mod}(S_m),$$

the *inner product*, which we ignored in Section 2. Using the characteristic map ch: $\mathrm{Mod}(S_m) \to \Lambda_m(x)$ we define the inner product on $\Lambda^m(x)$ and denote by

$$[M] * [N] \text{ and } a * b$$

the two inner products on $\mathrm{Mod}(S_m)$, $\Lambda_m(x)$, respectively. The inner product on $\Lambda_m(x)$ is not easy to describe in terms of the s_λ, but in terms of the p_λ it has the following simple form [8, p. 145] (z_λ is the integer defined before Theorem 1).

$$p_\lambda * p_\lambda = z_\lambda p_\lambda,$$
$$p_\lambda * p_\mu = 0 \qquad (\lambda \neq \mu).$$

The group algebra KS_m is a direct product of simple factors J_λ, one for each partition λ such that $|\lambda| = m$. For J_λ as an S_m-module by left multiplication, denoted $l(J_\lambda)$,

$$\mathrm{ch}[l(J_\lambda)] = a_\lambda s_\lambda,$$

where $a_\lambda = (\dim_K(J_\lambda))^{1/2}$. But as an S_m-module where S_m acts by conjugation, denoted simply J_λ,

$$\mathrm{ch}[J_\lambda] = s_\lambda * s_\lambda.$$

The Procesi–Razmyslov theorem implies that if $\bar{C}$ is expressed as a graded ring

$$\bar{C} = K \oplus \bar{C}_1 \oplus \bar{C}_2 \oplus \cdots,$$

then $(\bar{C}_m)^{(1^m)}$, the multilinear part of $\bar{C}_m$, is isomorphic as an S_m-module to

$$\sum \{J_\lambda \mid |\lambda| = m, \text{ length } \lambda \leqslant n\},$$

where S_m acts by conjugation. In analogy with Theorem 7 the Poincare series of $\bar{C}$ can be defined using the multilinear parts of $\bar{C}_m$ and the characteristic map. Thus we obtain

THEOREM 17. *Let $\bar{C}$ be the ring of invariants of $n \times n$ matrices. Then*

$$P(\bar{C}) = \sum s_\lambda(x) * s_\lambda(x),$$

where λ varies over all partitions of length $\leqslant n$. Hence for each m,

$$\sum s_\lambda(x) * s_\lambda(x) = \sum \langle s_\lambda(y_i y_j^{-1}), 1\rangle s_\lambda(x),$$

where the left sum is over partitions λ such that $|\lambda| = m$ and length $\lambda \leqslant n$, and the right sum is over partitions λ such that $|\lambda| = m$ and length $\lambda \leqslant n^2$, and the coefficients are those of Theorem 12.

It is also possible to derive a formula for $P(\bar{R})$ in a similar way, but it does not yield anything more about the inner product on $\Lambda_m(x)$.

9. 2×2 GENERIC MATRICES

In this section we give two expressions for $P(C)$, $P(R)$, $P(\bar{C})$ and $P(\bar{R})$ in the case of 2×2 generic matrices: As formal power series of the Schur functions s_λ, with the coefficient of s_λ given as a simple function of λ; and as rational functions in four variables x_1, x_2, x_3, x_4 when the number of generic matrices is assumed to be four.

The determination of $P(\bar{C})$ and $P(\bar{R})$ is formal modulo Theorem 12, and in principle can be carried out for matrices of any size. In practice the computations already appear formidable for 3×3 matrices. In contrast, the determination of $P(C)$ and $P(R)$ relies on an analysis of how R sits inside $\bar{R}$, and it is not clear how to carry out such an analysis for larger matrices. This was the method of [4] for two 2×2 matrices. Drensky [3, Theorem 2.1(iii)] has given a formula equivalent to expressing $P(R)$ as a formal power series of Schur functions. As an application of the formula for $P(\bar{C})$, we give a presentation for $\bar{C}$ as a ring when the number of matrices is three (Theorem 22). It is a polynomial ring in ten variables modulo a principal ideal.

As in Section 7, let

$$P(C) = \sum c(\lambda) s_\lambda, \qquad P(\bar{C}) = \sum \bar{c}(\lambda) s_\lambda,$$
$$P(R) = \sum r(\lambda) s_\lambda, \qquad P(\bar{R}) = \sum \bar{r}(\lambda) s_\lambda.$$

The coefficients in $P(\bar{C})$ and $P(\bar{R})$ can be computed using Theorem 12. The process is lengthy but routine and the conclusion will be given later (Theorem 21). For the present, we examine the difference between $P(R)$ and $P(\bar{R})$. Set

$$R_m = K[U(1), \ldots, U(m)],$$

where the $U(r)$ are 2×2 generic matrices, and let $C_m, \bar{C}_m, \bar{R}_m$ denote the corresponding center, ring of invariants, and trace ring. By analyzing $R_3 \subseteq \bar{R}_3$ and $R_4 \subseteq \bar{R}_4$ we will obtain the difference $\bar{r}(\lambda) - r(\lambda)$, first for partitions of length $\leqslant 3$, then for partitions of length 4. The next lemma reduces the problem to commutator ideals.

LEMMA 18 (cf. [4, Theorem 6]).

(a) $\bar{R}_m = R_m[\mathrm{Tr}(U(1)),..., \mathrm{Tr}(U(m))] = K[U(r), \mathrm{Tr}(U(r)) \mid 1 \leqslant r \leqslant m]$.

(b) $\bar{R}_m/[\bar{R}_m, \bar{R}_m]$ *is a polynomial ring over K in 2m independent variables, the images of* $U(1),..., U(m)$ *and* $\mathrm{Tr}(U(1)),..., \mathrm{Tr}(U(m))$.

(c) $P(R_m/[R_m, R_m]) = \prod_{i=1}^{m}(1 - x_i)^{-1} = \sum_{m_1 \geqslant 0} s_{(m_1)}$.

(d) $P(\bar{R}_m/[\bar{R}_m, \bar{R}_m]) = \prod_{i=1}^{m}(1 - x_i)^{-2} = \sum_{m_1, m_2 \geqslant 0}(m_1 + 1) s_{(m_1 + m_2, m_2)}$.

Proof. (a) By definition, $\bar{R}_m = R_m \bar{C}_m$ is generated over R_m by all $\mathrm{Tr}(X)$, for X a monomial in R_m. The multilinear Cayley–Hamilton theorem for 2×2 matrices says that

$$\mathrm{Tr}(XY) = \mathrm{Tr}(X)\,\mathrm{Tr}(Y) - \mathrm{Tr}(Y)\,X - \mathrm{Tr}(X)\,Y + XY + YX.$$

A simple induction on the degree of X shows that $\mathrm{Tr}(U(1)),..., \mathrm{Tr}(U(m))$ generate $\bar{R}_m$ over R_m.

(b) By (a), it suffices to find a commutative homomorphic image of $\bar{R}_m$ in which the images of $U(r)$, $\mathrm{Tr}(U(r))$ $(1 \leqslant r \leqslant m)$ are algebraically independent. Specializing the off-diagonal entries of the $U(r)$ to zero gives such a homomorphic image.

(c) is clear, and (d) follows from (b). ∎

Set

$$\Delta = \Delta(U(1), U(2), U(3), U(4)),$$
$$\Delta_0 = \Delta(U(1), U(2), U(3), I),$$

the discriminants of the indicated bases for $M_2(K(u_{ij}(r)))$. Note that Δ is a relative invariant for $GL(4)$ acting on $\bar{R}_4$ and Δ_0 is a relative invariant for $GL(3)$ acting on $\bar{R}_3$. Define

$$\bar{A}_m = \bar{R}_m(\mathrm{Tr}(Y) - 2Y,\ Y \in \bar{R}_m)\bar{R}_m,$$

the ideal of $\bar{R}_m$ generated by all $\mathrm{Tr}(Y) - 2Y$. The motivation for introducing A_m is the fact that the fourth Capelli polynomial has the evaluations

$$
\begin{aligned}
&C_4(U(1), U(2), U(3), U(4), Y_1, Y_2, Y_3, Y_4) \\
&\quad = \Delta\{[\mathrm{Tr}(Y_2)\,\mathrm{Tr}(Y_3) - \mathrm{Tr}(Y_2 Y_3)][\mathrm{Tr}(Y_1) - Y_1] \\
&\qquad - [\mathrm{Tr}(Y_1)\,\mathrm{Tr}(T_2) - \mathrm{Tr}(Y_1 Y_2)][\mathrm{Tr}(Y_3) - Y_3]\}\, Y_4, \\
&C_4(U(1), U(2), U(3), U(4), Y, I, I, I) \\
&\quad = \Delta(\mathrm{Tr}(Y) - 2Y),
\end{aligned}
$$

which lie in, and generate, $\Delta\bar{A}_m$ as $Y, Y_1, \ldots, Y_4$ vary over R_m. It follows, using standard properties of the Capelli polynomial (e.g. [1, Corollary 8]) that $\Delta\bar{A}_4$ is an ideal of R_4 and is contained in $[R_4, R_4]$. Similarly, $\Delta_0\bar{A}_3$ is an ideal of R_3 contained in $[R_3, R_3]$.

Using Lemma 18(b), it is easy to see that

LEMMA 19. (a) $\bar{R}_m/\bar{A}_m \cong K[\mathrm{Tr}\, U(1)), \ldots, \mathrm{Tr}(U(m))]$, *a polynomial ring over K in m idependent variables.*

(b) $P(\bar{R}_m/\bar{A}_m) = \prod_{i=1}^{m} (1 - x_i)^{-1} = \sum_{m_1 \geqslant 0} s_{(m_1)}$.

The next lemma is the crucial step in obtaining $P(\bar{R}_m) - P(R_m)$ in terms of known Poincare series. Its proof involves lengthy computations which we omit.

LEMMA 20. (a) $[R_3, R_3] + \Delta_0 R_3 = [\bar{R}_3, \bar{R}_3]$.

(b) $[R_3, R_3] \cap \Delta_0\bar{R}_3 = \Delta_0\bar{A}_3$.

(c) $R_4 \cap \Delta\bar{R}_4 = \Delta\bar{A}_4$.

Consider the following diagram of inclusions of $GL(3)$-modules

$$
\begin{tikzcd}
 & {[R_3, R_3]} \arrow[rr] \arrow[dr] & & R_3 \arrow[dr] & \\
\Delta_0\bar{A}_3 \arrow[ur] \arrow[dr] & & {[\bar{R}_3, \bar{R}_3]} \arrow[r] & \bar{A}_3 \arrow[r] & \bar{R}_3. \\
 & \Delta_0\bar{R}_3 \arrow[ur] & & &
\end{tikzcd}
$$

By Lemma 20(a, b)

$$P([\bar{R}_3, \bar{R}_3]) - P([R_3, R_3]) = P(\Delta_0\bar{R}_3/\Delta_0\bar{A}_3).$$

Using this equation and the Poincare series of Lemma 18(c, d) and Lemma 19(b) as well as the fact that Δ_0 us a relative invariant for $GL(3)$ with character $x_1x_2x_3 = s_{(1,1,1)}$, we have

$$P(\bar{R}_3)-P(R_3)=P(\bar{R}_3/[\bar{R}_3,\bar{R}_3])-P(R_3/[R_3,R_3])+P(\Delta_0\bar{R}_3/\Delta_0\bar{A}_3)$$
$$=\sum_{m_1,m_2\geqslant 0}(m_1+1)\,s_{(m_2+m_1,m_2)}-\sum_{m_1\geqslant 0}s_{(m_1)}+\sum_{m_1\geqslant 0}s_{(m_1+1,1,1)}. \tag{1}$$

Equation (1) tells us $\bar{r}(\lambda)-r(\lambda)$ for partitions of length $\leqslant 3$. Since $P(\bar{R})$ only involves partitons of length $\leqslant 4$, we need only find $\bar{r}(\lambda)-r(\lambda)$ for partitions whose length is exactly four. Here we are aided by the fortunate circumstance that Δ is the unique relative invariant of lowest degree in $\bar{R}_4$ (c.f. Lemma 13). This implies that $\Delta\bar{R}_4$ is the $GL(4)$-submodule of $\bar{R}_4$ generated by irreducible modules corresponding to partitions of length exactly four, or

$$P(\Delta\bar{R}_4)=\sum\{\bar{r}(\lambda)\,s_\lambda \mid \text{length }\lambda=4\}.$$

Using Lemmas 19(b) and 20(c) and the fact that Δ is a $GL(4)$ relative invariant with character $x_1x_2x_3x_4=s_{(1,1,1,1)}$, we have

$$\begin{aligned}\sum\{(\bar{r}(\lambda)-r(\lambda))\,s_\lambda \mid \text{length }\lambda=4\}&=P(\Delta\bar{R}_4/(R_4\cap\Delta\bar{R}_4)\\ &=P(\Delta\bar{R}_4/\Delta\bar{A}_4)\\ &=\sum_{m_1\geqslant 1}s_{(m_1+1,1,1,1)}.\end{aligned} \tag{2}$$

We now turn to $P(C)$. For the coefficients of the partitions of length $\leqslant 2$, we use the determination of $P(C_2)$ in [4, Theorem 9]:

$$\begin{aligned}P(C_2)&=1+x_1^2x_2^2P(\bar{C}_2)\\ &=1+s_{(2,2)}\sum\bar{c}(\lambda)\,s_\lambda\\ &=1+\sum_{\lambda=(\lambda_1,\lambda_2)}\bar{c}(\lambda)\,s_{\lambda+(2,2)}.\end{aligned} \tag{3}$$

Note that since $C=R\cap\bar{C}$, $\bar{C}/C$ is a GL-submodule of $\bar{R}/R$. Hence

$$0\leqslant\bar{c}(\lambda)-c(\lambda)\leqslant\bar{r}(\lambda)-r(\lambda)$$

for all partitions. Since $\bar{r}(\lambda)-r(\lambda)=0$ for most partitions of lengths 3 and 4 and $\bar{r}(\lambda)-r(\lambda)=1$ for the remaining ones, all that remains to be decided is whether $\bar{c}(\lambda)-c(\lambda)$ is 0 or 1 for the partitions

$$\lambda=(m_1+1,1,1)\qquad(m_1\geqslant 0),$$
$$\lambda=(m_1+1,1,1,1)\qquad(m_1\geqslant 0).$$

In case $\lambda=(m_1+1,1,1)$, Eq. (1) shows that the multiplicity 1 of s_λ in

$\bar{R}_3/R_3$ is represented by the module $\Delta_0\bar{R}_3/\Delta_0\bar{A}_3$. Hence to show that s_λ also has multiplicity 1 in $\bar{C}_3/C_3$ is suffices to show that

$$\bar{C}_3 + \Delta_0\bar{A}_2 \supseteq \Delta_0\bar{R}_3.$$

But by Lemma 19(a),

$$\bar{A}_3 + \bar{C}_3 = \bar{R}_3,$$

so

$$\Delta_0\bar{A}_3 + \Delta_0\bar{C}_3 = \Delta_0\bar{R}_3.$$

A similar argument works for $\lambda = (m_1 + 1, 1, 1, 1)$. The net result is

$$\text{If } \lambda \text{ is a partition of length 3 or 4,} \qquad \bar{c}(\lambda) - c(\lambda) = \bar{r}(\lambda) - r(\lambda). \tag{4}$$

Using Eqs. (1)–(4), $P(C)$ and $P(R)$ can be computed from $P(\bar{C})$ and $P(\bar{R})$. The next theorem gives the coefficients of these Poincare series. As noted earlier, the details of the computation of $P(\bar{C})$ and $P(\bar{R})$ are not included, and the details of the translation to a rational function of four variables are also omitted.

THEOREM 21. *For 2×2 generic matrices, the coefficients of*

$$P(C) = \sum c(\lambda)\, s_\lambda, \qquad P(\bar{C}) = \sum \bar{c}(\lambda)\, s_\lambda,$$
$$P(R) = \sum r(\lambda)\, s_\lambda, \qquad P(\bar{R}) = \sum \bar{r}(\lambda)\, s_\lambda$$

are given by the following formulas. In all cases the summations are over all partitions λ of length $\leqslant 4$, where

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$
$$= (m_1 + m_2 + m_3 + m_4, m_2 + m_3 + m_4, m_3 + m_4).$$

(a) $\bar{r}(\lambda) = (m_1 + 1)(m_2 + 1)(m_3 + 1)$

(b) $\bar{c}(\lambda) = \frac{1}{4}(\bar{r}(\lambda) + \bar{d}(\lambda))$, *where $\bar{d}(\lambda)$ depends on the parity of m_1, m_2, m_3 according to the following table*:

(m_1, m_2, m_3) (mod 2)	$\bar{d}(\lambda)$
(0, 0, 0)	$m + m_2 + m_3 + 3$
(1, 0, 0)	$m_1 + 1$
(0, 1, 0)	$-(m_2 + 1)$
(0, 0, 1)	$m_3 + 1$
all others	0

(c) *$r(\lambda)$ and $c(\lambda)$ are given in terms of $\bar{r}(\lambda)$ and $\bar{c}(\lambda)$ by the following table*:

λ	$r(\lambda)$	$c(\lambda)$
$(0, 0, 0, 0)$	1	1
$(m_1, 0, 0, 0)$ $(m_1 \geqslant 1)$	1	0
$(m_1 + 1, 1, 0, 0)$	$m_1 + 1$	0
$(m_1 + m_2, m_2, 0, 0)$ $(m_2 \geqslant 2)$	$(m_1 + 1)m_2$	$\bar{c}(m_1 + m_2 - 2, m_2 - 2, 0, 0)$
$(m_1 + 1, 1, 1, 0)$	$\bar{r}(\lambda) - 1$	$\bar{c}(\lambda) - 1$
$(m_1 + 1, 1, 1, 1)$	$\bar{r}(\lambda) - 1$	$\bar{c}(\lambda) - 1$
all others	$\bar{r}(\lambda)$	$\bar{c}(\lambda)$

(d) *When the number of 2×2 generic matrices is four, the Poincare series are given by the following rational functions*

$$P(\bar{R}_4) = \frac{1 - x_1 x_2 x_3 x_4}{\pi(1 - x_i)^2 \pi(1 - x_i x_j)},$$

$$P(\bar{C}_4) = \frac{1 + \sum x_i x_j x_k - x_1 x_2 x_3 x_4 \sum x_i - (x_1 x_2 x_3 x_4)^2}{\pi(1 - x_i)(1 - x_i^2)\, \pi(1 - x_i x_j)},$$

$$P(R_4) = P(\bar{R}_4) - \frac{1}{\pi(1 - x_i)^2} + \frac{1 - \sum x_i x_j x_k}{\pi(1 - x_i)},$$

$$P(C_4) = P(\bar{C}_4) - \frac{1 + \sum x_i x_j + x_1 x_2 x_3 x_4}{\pi(1 - x_i)(1 - x_i^2)} - \frac{\sum x_i x_j x_k}{\pi(1 - x_i)} + 1.$$

(The sums and products are over $1 \leqslant i \leqslant 4$, $1 \leqslant i < j \leqslant 4$, $1 \leqslant i < j < k \leqslant 4$, respectively.)

Finally, we use the Poincare series for $\bar{C}_3$ in conjunction with a result of Siberskii [14] to gve a presentation for $\bar{C}_3$. Siberskii showed that $\bar{C}_3$ is generated by the following 10 elements

$$\begin{aligned} &\operatorname{Tr}(U(i)), \det(U(i)) && (1 \leqslant i \leqslant 3), \\ &\operatorname{Tr}(U(i)\, U(j)) && (1 \leqslant i < j \leqslant 3), \qquad (*) \\ &\operatorname{Tr}(U(1)\, U(2)\, U(3)). && \end{aligned}$$

These elements are homogeneous of degrees

$$(1, 0, 0)(0, 1, 0), (0, 0, 1), (2, 0, 0), (0, 2, 0), (0, 0, 2),$$
$$(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1).$$

The Poincare series of a commutative polynomial ring generated by independent elements of the above degrees is

$$Q = \frac{1}{(1 - x_1 x_2 x_3)\, \pi(1 - x_i)(1 - x_i^2)\, \pi(1 - x_i x_j)}.$$

Specializing $x_4 = 0$ in Theorem 21(d) shows that

$$P(\bar{C}_3) = \frac{1 + x_1 x_2 x_3}{\pi(1 - x_i)(1 - x_i^2)\, \pi(1 - x_i x_j)} = (1 - (x_1 x_2 x_3)^2)Q.$$

Hence if we show that the ten elements (*) satisfy a homogeneous relation of degree (2, 2, 2), it will follow easily that $\bar{C}_3$ is isomorphic to a polynomial ring in ten variables modulo this relation. The desired relation is

$$\begin{aligned} &E^2 - (A_1 C_1 + A_2 C_2 + A_3 C_3 - A_1 A_2 A_3)E, \\ &+ B_1 C_1^2 + B_2 C_2^2 + B_3 C_3^2 - A_1 A_2 B_3 C_3 - A_1 A_3 B_2 C_2 - A_2 A_3 B_1 C_1, \quad (**) \\ &+ A_1^2 B_2 B_3 + A_1^2 B_3 B_1 + A_3^2 B_1 B_2 - 4B_1 B_2 B_3 + C_1 C_2 C_3 = 0, \end{aligned}$$

where $A_i = \mathrm{Tr}(U(i))$, $B_i = \det(U(i))$, $C_i = \mathrm{Tr}(U(j)\, U(k))$ $(i \neq j, k;\ j \neq k)$, $E = \mathrm{Tr}(U(1)\, U(2)\, U(3))$.

THEOREM 22. *Let $\bar{C}_3$ denote the ring of invariants of three 2×2 matrices. Then $\bar{C}_3$ is generated by the ten elements in* (*) *modulo the principal ideal generated by the relation* (**). *Alternatively, $\bar{C}_3$ is a free module of rank* 2 (*with basis* 1, $\mathrm{Tr}(U(1)\, U(2)\, U(3))$) *over the polynomial ring freely generated by the first* 9 *of the elements in* (*).

REFERENCES

1. S. A. AMITSUR, Alternating identities, *in* "Ring Theory," Proceedings, Ohio University Conference (S. K. Jain, Ed.), pp. 1–14, Lecture Notes in Mathematics No. 25, Dekker, New York, 1977.
2. A. BERELE, Homogeneous polynomial identities, *Israel J. Math.* **42** (1982), 258–272.
3. V. S. DRENSKY, Codimensions of T-ideals and Hilbert series of relatively free algebras, *C. R. Acad. Bulgare Sci.* **34** (1981), 1201–1204.
4. E. FORMANEK, P. HALPIN, AND W-C. W. LI, The Poincare series of the ring of 2×2 generic matrices, *J. Algebra* **69** (1981), 105–112.
5. E. FORMANEK, The polynomial identities of matrices, *in* "Algebraists' Homage" (S. A. Amitsur *et al.*, Eds.), Contemporary Mathematics, pp. 41–79, Amer. Math. Soc., Providence, 1982.
6. J. A. GREEN, "Polynomial Representations of GL_n," Lecture Notes in Mathematics No. 830, Springer-Verlag, Berlin/Heidelberg/New York, 1980.
7. H. HELLING, Eine Kennzeichnung von Charakteren auf Gruppen und Assoziativen Algebren, *Comm. Algebra* **1** (1974), 491–501.

8. D. KNUTSON, "λ-rings and the Representation Theory of the Symmetric Group," Lecture Notes in Mathematics No. 308, Springer-Verlag, Berlin/Heidelberg/New York, 1973.
9. I. G. MACDONALD, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press (Clarendon), Oxford, 1979.
10. C. PROCESI, The invariant theory of $n \times n$ matrices, *Adv. in Math.* **19** (1976), 306–381.
11. Y. P. RAZMYSLOV, Trace identities of full matrix algebras over a field of characteristic zero, *Izv. Akad. Nauk. SSSR Ser. Mat.* **38** (1974), 723–756 [Russian]; *Math. USSR Izv.* **8** (1974), 727–760 [Trans.].
12. Y. P. RAZMYSLOV, The Jacobson radical in *PI*-algebras, *Algebra i Logika* **13** (1974), 337–360 [Russian]; *Algebra and Logic* **13** (1974), 192–204 [Trans.].
13. A. REGEV, The polynomial identities of matrices in characteristic zero, *Comm. Algebra* **8** (1980), 1417–1467.
14. K. S. SIBERSKII, Algebraic invariants for a set of matrices, *Sib. Mat. Zh.* **9** (1) (1968), 152–164 [Russian]; *Sib. Math. J.* **9** (1968), 115–124 [Trans.].
15. D-N. VERMA, On a classical stability result on invariants of isotypical modules, *J. Algebra* **63** (1980), 15–40.
16. H. WEYL, "The Classical Groups," Princeton Univ. Press, Princeton, N. J., 1946.
17. H. WEYL, Zur Darstellungstheorie und Invariantenabzählung der projectiven, der Komplex und der Drehungsgruppe, Ges. Abh. Bd. III, pp. 1–25, Springer-Verlag, Berlin/Heidelberg/New York, 1968.