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Approximation of common fixed points of a countable family of continuous pseudocontractions in a uniformly smooth Banach space

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ABSTRACT

In this paper, we introduce a new implicit iterative algorithm for finding a common element of a countable family of continuous pseudocontractions in a uniformly smooth Banach space. We obtain some strong convergence theorems under suitable conditions. Our results extend the recent results announced by many others.

1. Introduction

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \left\{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\right\}, x \in E$, where E^* denotes the dual space of E and $\langle ., .\rangle$ denotes the generalized duality pairing. We use $F(T)$ to denote the set of fixed points of the mapping T. It is well known that, if E^* is strictly convex or E is a Banach space with a uniformly Gâteaux differentiable norm, then J is single valued. In what follows, we denote the single-valued normalized duality mapping by j.

Let C be a closed convex subset of E. Recall that a mapping $T : C \to C$ is said to be L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\,\|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

T is said to be non-expansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

T is said to be pseudocontractive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

T is said to be strongly pseudocontractive if there exists a constant $\beta \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le \beta\,\|x - y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

In a Banach space E having a single-valued normalized duality mapping j, we say that an operator A is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, j(x)\rangle \ge \overline{\gamma}\,\|x\|^2, \qquad \|aI - bA\| = \sup_{\|x\|\le 1} |\langle (aI - bA)x, j(x)\rangle| \quad a \in [0, 1],\ b \in [-1, 1], \tag{1.5}$$

where I is the identity mapping.

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Recently, the problems of convergence of an implicit iterative algorithm to a common fixed point for a family of non-expansive mappings and its extensions to Hilbert spaces or Banach spaces have been considered by many authors; see [1–5] for more details.

Yao [2] introduced the following Halpern-type implicit iterative algorithm,

$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n, \quad n \geq 1, \tag{1.6}$$

and proved a strong convergence theorem under suitable conditions.

In this paper, motivated by the above facts, we introduce a new implicit iterative algorithm for a countable family of continuous pseudocontractions in a uniformly smooth Banach space. Then, a strong convergence theorem is established under some suitable conditions. The results presented in this paper improve and extend the corresponding results announced in [2] and many others.

2. Preliminaries

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([6]). *Let E be a Banach space, C a non-empty closed and convex subset of E, and $T : C \to C$ a continuous and strong pseudocontraction. Then T has a unique fixed point in C.*

Lemma 2.2 ([7]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the property $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\beta_n$, $n \geq 0$, where $\{\gamma_n\} \subset (0, 1)$ and $\{\beta_n\} \subset \mathbb{R}$ such that* (i) $\sum_{n=0}^{\infty} \gamma_n = \infty$ *and* (ii) $\limsup_{n\to\infty} \beta_n \leq 0$. *Then $\{a_n\}$ converges to zero as $n \to \infty$.*

Lemma 2.3 ([3]). *Let C be a non-empty closed convex subset of a real Banach space E and $T : C \to C$ be a continuous pseudocontractive map. We denote $B = (2I - T)^{-1}$. Then the following hold.*

(1) *The map B is a non-expansive self-mapping on C.*
(2) *If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $\lim_{n\to\infty} \|x_n - Bx_n\| = 0$.*

Lemma 2.4 ([8]). *Assume that A is a strongly positive linear bounded operator on a smooth Banach space E with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.5. *Let C be a closed convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a continuous pseudocontractive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a fixed Lipschitzian strongly pseudocontractive mapping with pseudocontractive coefficient $\beta \in (0, 1)$ and Lipschitzian constant $L > 0$. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Assume that $C \pm C \subset C$ and $0 < \beta < \bar{\gamma}$. Let $\{x_t\}$ be defined by*

$$x_t = tf(x_t) + (I - tA)Tx_t. \tag{2.1}$$

Then, as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point z of T such that z is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (A - f)z, j(z - p)\rangle \leq 0, \quad \forall p \in F(T). \tag{2.2}$$

Proof. First, we show the uniqueness of the solution of the variational inequality (2.2). Suppose both $z_1 \in F(T)$ and $z_2 \in F(T)$ are solutions to (2.2). We have

$$\langle (A - f)z_1, j(z_1 - z_2)\rangle \leq 0$$

and

$$\langle (A - f)z_2, j(z_2 - z_1)\rangle \leq 0.$$

Adding up the above two inequalities, we obtain

$$\langle (A - f)z_1 - (A - f)z_2, j(z_1 - z_2)\rangle \leq 0.$$

Note that

$$\begin{aligned}\langle (A - f)z_1 - (A - f)z_2, j(z_1 - z_2)\rangle &= \langle A(z_1 - z_2), j(z_1 - z_2)\rangle - \langle f(z_1) - f(z_2), j(z_1 - z_2)\rangle \\ &\geq \bar{\gamma}\, \|z_1 - z_2\|^2 - \beta\, \|z_1 - z_2\|^2 \\ &= (\bar{\gamma} - \beta)\, \|z_1 - z_2\|^2 \geq 0.\end{aligned}$$

Consequently, we have $z_1 = z_2$, and the uniqueness is proved. We use $\tilde{z}$ to denote the unique solution of (2.2).

Next, we prove that $\{x_t\}$ is bounded. Indeed, we may assume, without loss of generality, that $t \leq \|A\|^{-1}$. For $p \in F(T)$, it follows from Lemma 2.4 that

$$\begin{aligned}\|x_t - p\|^2 &= \langle t(f(x_t) - Ap) + (I - tA)(Tx_t - p), j(x_t - p)\rangle \\ &= t(f(x_t) - f(p), j(x_t - p)) + t\,\langle f(p) - Ap, j(x_t - p)\rangle + \langle (I - tA)(Tx_t - p), j(x_t - p)\rangle \\ &\le t\beta\,\|x_t - p\|^2 + (1 - t\bar{\gamma})\,\|x_t - p\|^2 + t\,\|f(p) - Ap\|\,\|x_t - p\|,\end{aligned}$$

which implies that $\|x_t - p\| \le \frac{\|f(p)-Ap\|}{\bar{\gamma}-\beta}$. This shows that $\{x_t\}$ is bounded.

Assume that $t_n \to 0$ as $n \to \infty$. Set $x_n := x_{t_n}$ and define $\mu : C \to \mathbb{R}$ by $\mu(x) = \text{LIM}\,\|x_n - x\|^2$, $x \in C$, where LIM is a Banach limit on l^∞. Let

$$K = \left\{x \in C : \mu(x) = \min_{x\in C} \text{LIM}\,\|x_n - x\|^2\right\}.$$

We see easily that K is a non-empty closed convex subset of E. Note that $\|x_n - Tx_n\| = t_n\,\|f(x_n) - ATx_n\| \to 0$ as $n \to \infty$. From Lemma 2.3, we have that the mapping $B = (2I - T)^{-1} : C \to C$ is non-expansive and $F(T) = F(B)$ and $\lim_{n\to\infty} \|x_n - Bx_n\| = 0$, where I denotes the identity operator. It follows that

$$\mu(Bx) = \text{LIM}\,\|x_n - Bx\|^2 = \text{LIM}\,\|Bx_n - Bx\|^2 \le \text{LIM}\,\|x_n - x\|^2 = \mu(x),$$

which implies that $B(K) \subset K$; that is, K is invariant under B. Since a uniformly smooth space has the fixed point property for non-expansive mapping, B has a fixed point, say $z \in K$. Since z is also a minimizer of μ over C, we have that, for $t \in (0, 1)$ and $x \in C$,

$$\begin{aligned}0 &\le \frac{\mu(z + t(x - Az)) - \mu(z)}{t} \\ &= \text{LIM}\frac{\|x_n - z + t(Az - x)\|^2 - \|x_n - z\|^2}{t} \\ &= \text{LIM}\frac{\langle x_n - z, j(x_n - z + t(Az - x))\rangle + t\,\langle Az - x, j(x_n - z + t(Az - x))\rangle - \|x_n - z\|^2}{t}.\end{aligned}$$

Since E is uniformly smooth, we have that the duality mapping j is norm-to-norm uniformly continuous on a bounded set of E. Letting $t \to 0$, we find that the two limits above can be interchanged, and obtain

$$\text{LIM}\,\langle x - Az, j(x_n - z)\rangle \le 0, \quad x \in C. \tag{2.3}$$

On the other hand, we have $x_n - z = t_n(f(x_n) - Az) + (I - t_nA)(Tx_n - z)$. It follows that

$$\begin{aligned}\|x_n - z\|^2 &= t_n\,\langle f(x_n) - Az, j(x_n - z)\rangle + \langle (I - t_nA)(Tx_n - z), j(x_n - p)\rangle \\ &\le t_n\,\langle f(x_n) - Az, j(x_n - z)\rangle + (1 - t_n\bar{\gamma})\,\|x_n - z\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_n - z\|^2 &\le \frac{1}{\bar{\gamma}}\,\langle f(x_n) - Az, j(x_n - z)\rangle \\ &\le \frac{1}{\bar{\gamma}}\,\langle f(x_n) - x, j(x_n - z)\rangle + \frac{1}{\bar{\gamma}}\,\langle x - Az, j(x_n - z)\rangle.\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\begin{aligned}\text{LIM}\,\|x_n - z\|^2 &\le \frac{1}{\bar{\gamma}}\text{LIM}\,\langle f(x_n) - x, j(x_n - z)\rangle + \frac{1}{\bar{\gamma}}\text{LIM}\,\langle x - Az, j(x_n - z)\rangle \\ &\le \frac{1}{\bar{\gamma}}\text{LIM}\,\langle f(x_n) - x, j(x_n - z)\rangle.\end{aligned}$$

In particular,

$$\bar{\gamma}\text{LIM}\,\|x_n - z\|^2 \le \text{LIM}\,\langle f(x_n) - f(x), j(x_n - z)\rangle \le \beta\text{LIM}\,\|x_n - z\|^2.$$

Hence, $(\bar{\gamma} - \beta)\text{LIM}\,\|x_n - z\|^2 \le 0$. Since $\bar{\gamma} > \beta$, we have $\text{LIM}\,\|x_n - z\|^2 = 0$, and hence there exists a subsequence which is still denoted $\{x_n\}$ such that $x_n \to z$.

Next, we prove that z solves the variational inequality (2.2). Since $x_t = tf(x_t) + (I - tA)Tx_t$, we have

$$(A - f)x_t = -\frac{1}{t}(I - tA)(I - T)x_t.$$

On the other hand, note that, for all $x, y \in C$,

$$\begin{aligned}\langle (I - T)x - (I - T)y, j(x - y)\rangle &= \|x - y\|^2 - \langle Tx - Ty, j(x - y)\rangle \\ &\ge \|x - y\|^2 - \|x - y\|^2 = 0.\end{aligned}$$

For $p \in F(T)$, we have

$$\begin{aligned}\langle (A-f)x_t, j(x_t-p)\rangle &= -\frac{1}{t}\langle (I-tA)(I-T)x_t, j(x_t-p)\rangle \\ &= -\frac{1}{t}\langle (I-T)x_t-(I-T)p, j(x_t-p)\rangle + \langle A(I-T)x_t, j(x_t-p)\rangle \\ &\leq \langle A(I-T)x_t, j(x_t-p)\rangle .\end{aligned}$$

Replacing t with t_n, letting $n \to \infty$, and noting that $(I-T)x_{t_n} \to (I-T)z = 0$, we have that $\langle (A-f)z, j(z-p)\rangle \leq 0$. That is, $z \in F(T)$ is a solution of (2.2). Then $z = \tilde{z}$. In summary, we have that each cluster point of $\{x_n\}$ converges strongly to z as $t_n \to 0$. This completes the proof. □

Lemma 2.6. *Let C be a non-empty closed convex subset of a real Banach space E which has a uniformly Gâteaux norm. Let $T : C \to C$ be a continuous pseudocontractive mapping with $F(T) \neq \emptyset$ and let $f : C \to C$ be a fixed Lipschitzian strongly pseudocontractive mapping with pseudocontractive coefficient $\beta \in (0,1)$ and Lipschitzian constant $L > 0$. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Assume that $C \pm C \subset C$ and that $\{x_t\}$ converges strongly to $z \in F(T)$ as $t \to 0$, where x_t is defined by $x_t = tf(x_t) + (I-tA)Tx_t$, where $\gamma > 0$ is a constant. Suppose that $\{x_n\} \subset C$ is bounded and that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Then $\limsup_{n\to\infty}\langle (f-A)z, j(x_n-z)\rangle \leq 0$.*

Proof. We note that

$$\begin{aligned}x_t - x_n &= tf(x_t) + Tx_t - tATx_t - x_n \\ &= t(f(x_t) - Ax_t) + (Tx_t - x_n) - t(ATx_t - Ax_t) \\ &= t(f(x_t) - Ax_t) + (Tx_t - Tx_n) + (Tx_n - x_n) + t^2A(f(x_t) - ATx_t).\end{aligned}$$

It follows that

$$\begin{aligned}\|x_t - x_n\|^2 &= t\langle f(x_t) - Ax_t, j(x_t-x_n)\rangle + \langle Tx_t - Tx_n, j(x_t-x_n)\rangle + \langle Tx_n - x_n, j(x_t-x_n)\rangle \\ &\quad + t^2\langle A(f(x_t) - ATx_t), j(x_t-x_n)\rangle \\ &\leq t\langle f(x_t) - Ax_t, j(x_t-x_n)\rangle + \|x_t-x_n\|^2 + \|Tx_n - x_n\|\,\|x_t-x_n\| \\ &\quad + t^2\|A(f(x_t) - ATx_t)\|\,\|x_t-x_n\|,\end{aligned}$$

which implies that

$$\langle f(x_t) - Ax_t, j(x_n-x_t)\rangle \leq \frac{\|Tx_n - x_n\|}{t}\|x_t-x_n\| + t\|A(f(x_t) - ATx_t)\|\,\|x_t-x_n\|. \tag{2.5}$$

Since $\{x_t\}$, $\{x_n\}$ and $\{Tx_n\}$ are bounded and $x_n - Tx_n \to 0$, taking the upper limit as $n \to \infty$ in (2.5), we get that

$$\limsup_{n\to\infty}\langle f(x_t) - Ax_t, j(x_n-x_t)\rangle \leq t\|A(f(x_t) - ATx_t)\|\limsup_{n\to\infty}\|x_t-x_n\|. \tag{2.6}$$

Taking the upper limit as $t \to 0$ in (2.6), we obtain

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x_t) - Ax_t, j(x_n-x_t)\rangle \leq 0. \tag{2.7}$$

Since E has a uniformly Gâteaux norm, we obtain that j is single valued and strong–weak* uniformly continuous on a bounded set of E. We get that

$$\begin{aligned}&|\langle f(z) - Az, j(x_n-z)\rangle - \langle f(x_t) - Ax_t, j(x_n-x_t)\rangle| \\ &\quad = |\langle f(z) - Az, j(x_n-z) - j(x_n-x_t)\rangle + \langle f(z) - f(x_t) + Ax_t - Az, j(x_n-x_t)\rangle| \\ &\quad \leq |\langle f(z) - Az, j(x_n-z) - j(x_n-x_t)\rangle| + (\|f(z) - f(x_t)\| + \|Ax_t - Az\|)\,\|x_n-x_t\| \\ &\quad \to 0 \quad \text{as } t \to 0.\end{aligned}$$

Hence, $\forall\,\epsilon > 0$, $\exists\,\delta > 0$ such that $\forall\,t \in (0,\delta)$, for all n, we have

$$\langle f(z) - Az, j(x_n-z)\rangle \leq \langle f(x_t) - Ax_t, j(x_n-x_t)\rangle + \epsilon.$$

By (2.7), we get that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - Az, j(x_n-z)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(z) - Az, j(x_n-z)\rangle \\ &\leq \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x_t) - Ax_t, j(x_n-x_t)\rangle + \epsilon \leq \epsilon.\end{aligned}$$

Since ϵ is arbitrary, we get that $\limsup_{n\to\infty}\langle f(z) - Az, j(x_n-z)\rangle \leq 0$. The proof is complete. □

Lemma 2.7 ([9]). *Let C be a non-empty closed convex subset of a Banach space E. Let $T_1, T_2, \ldots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_n x\| : x \in C\} < \infty$. Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$, for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Tx - T_n x\| : x \in C\} = 0$.*

3. Main results

Theorem 3.1. *Let C be a non-empty closed convex subset of a real uniformly smooth Banach space E such that $C \pm C \subset C$. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of continuous pseudocontractive mappings from C into itself such that $F = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $f : C \to C$ be a fixed Lipschitz strongly pseudocontractive mapping with pseudocontractive coefficient $\beta \in (0, 1)$ and Lipschitz constant $L > 0$. Let $A : C \to C$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $0 < \bar{\gamma} - \beta < 1$. Let $\{x_n\}$ be a sequence generated by the following iterative process:*

$$x_0 \in C, \qquad x_n = \alpha_n f(x_n) + \beta_n x_{n-1} + ((1-\beta_n)I - \alpha_n A)T_n x_n, \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;
(ii) $\sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha_n+\beta_n} = \infty$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C, let T be a mapping of C into itself defined by $Tx = \lim_{n\to\infty} T_n x$, for all $x \in C$, and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then, $\{x_n\}$ converges strongly to a fixed point z of F such that z is a unique solution in F to the following variational inequality:

$$\langle (f - A)z, j(p - z)\rangle \le 0 \quad \text{for all } p \in F. \tag{3.2}$$

Proof. By condition (i), we may assume, without loss of generality, that $\alpha_n \le (1-\beta_n)\|A\|^{-1}$.

Since A is a strongly positive linear bounded operator on C, by (1.5), we have

$$\|A\| = \sup\{|\langle Au, j(u)\rangle| : u \in C, \|u\| = 1\}.$$

Observe that

$$\begin{aligned}\langle ((1-\beta_n)I - \alpha_n A)u, j(u)\rangle &= 1 - \beta_n - \alpha_n \langle Au, j(u)\rangle \\ &\ge 1 - \beta_n - \alpha_n \|A\| \\ &\ge 0.\end{aligned}$$

It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle ((1-\beta_n)I - \alpha_n A)u, j(u)\rangle : u \in C, \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n \langle Au, j(u)\rangle : u \in C, \|u\| = 1\} \\ &\le 1 - \beta_n - \alpha_n \bar{\gamma}.\end{aligned}$$

Next, we show that $\{x_n\}$ is well defined. For each $n \ge 1$, define a mapping $S_n : C \to C$ by

$$S_n x = \alpha_n f(x) + \beta_n x_{n-1} + ((1-\beta_n)I - \alpha_n A)T_n x, \quad \forall x \in C.$$

For every $x, y \in C$, we have

$$\begin{aligned}\langle S_n x - S_n y, j(x-y)\rangle &= \alpha_n \langle f(x) - f(y), j(x-y)\rangle + \langle ((1-\beta_n)I - \alpha_n A)(T_n x - T_n y), j(x-y)\rangle \\ &\le \alpha_n \beta \|x-y\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma})\|x-y\|^2 \\ &= [1 - \beta_n - \alpha_n(\bar{\gamma} - \beta)]\|x-y\|^2.\end{aligned}$$

Therefore, S_n is a continuous strong pseudocontraction for each $n \ge 1$. By Lemma 2.1, we see that there exists a unique fixed point x_n for each $n \ge 1$ such that

$$x_n = \alpha_n f(x_n) + \beta_n x_{n-1} + ((1-\beta_n)I - \alpha_n A)T_n x_n.$$

That is, the sequence $\{x_n\}$ is well defined. Next, we prove that $\{x_n\}$ is bounded. Let $p \in F$. We have

$$\begin{aligned}\|x_n - p\|^2 &= \alpha_n \langle f(x_n) - Ap, j(x_n - p)\rangle + \beta_n \langle x_{n-1} - p, j(x_n - p)\rangle + \langle ((1-\beta_n)I - \alpha_n A)(T_n x_n - p), j(x_n - p)\rangle \\ &\le \alpha_n \langle f(x_n) - f(p), j(x_n - p)\rangle + \alpha_n \langle f(p) - Ap, j(x_n - p)\rangle + \beta_n \|x_{n-1} - p\|\|x_n - p\| \\ &\quad + (1 - \beta_n - \alpha_n \bar{\gamma})\|x_n - p\|^2 \\ &\le \alpha_n \beta \|x_n - p\|^2 + (1 - \beta_n - \alpha_n \bar{\gamma})\|x_n - p\|^2 + \alpha_n \langle f(p) - Ap, j(x_n - p)\rangle + \beta_n \|x_{n-1} - p\|\|x_n - p\| \\ &= (1 - \beta_n - \alpha_n(\bar{\gamma} - \beta))\|x_n - p\|^2 + \alpha_n \|f(p) - Ap\|\|x_n - p\| + \beta_n \|x_{n-1} - p\|\|x_n - p\|,\end{aligned}$$

which implies that

$$\|x_n - p\| \le \frac{\beta_n}{\beta_n + \alpha_n(\bar{\gamma} - \beta)} \|x_{n-1} - p\| + \frac{\alpha_n(\bar{\gamma} - \beta)}{\beta_n + \alpha_n(\bar{\gamma} - \beta)} \frac{\|f(p) - Ap\|}{\bar{\gamma} - \beta}.$$

By induction, we obtain

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|f(p) - Ap\|}{\bar{\gamma} - \beta}\right\}.$$

Therefore, $\{x_n\}$ is bounded. We observe that

$$\begin{aligned} \|x_n - T_n x_n\| &= \|\alpha_n(f(x_n) - AT_n x_n) + \beta_n(x_{n-1} - T_n x_n)\| \\ &\le \alpha_n \|f(x_n) - AT_n x_n\| + \beta_n \|x_{n-1} - T_n x_n\|. \end{aligned} \tag{3.3}$$

It follows from condition (i) and (3.3) that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.4}$$

On the other hand, we have

$$\|x_n - Tx_n\| \le \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\|. \tag{3.5}$$

From Lemma 2.7, (3.4) and (3.5), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.6}$$

Let $x_t = tf(x_t) + (I - tA)Tx_t$. It follows from Lemmas 2.5 and 2.6 that $\{x_t\}$ converges strongly to $z \in F(T) = \cap_{i=1}^{\infty} F(T_i) = F$ and

$$\limsup_{n\to\infty} \langle (f - A)z, j(x_n - z)\rangle \le 0. \tag{3.7}$$

Finally, we show that $x_n \to z$ as $n \to \infty$. We observe that

$$\begin{aligned} \|x_n - z\|^2 &= \alpha_n \langle f(x_n) - Az, j(x_n - z)\rangle + \beta_n \langle x_{n-1} - z, j(x_n - z)\rangle + \langle ((1 - \beta_n)I - \alpha_n A)(T_n x_n - z), j(x_n - z)\rangle \\ &\le (1 - \beta_n - \alpha_n\bar{\gamma}) \|x_n - z\|^2 + \beta_n \|x_{n-1} - z\| \|x_n - z\| + \alpha_n \langle f(x_n) - f(z), j(x_n - z)\rangle \\ &\quad + \alpha_n \langle f(z) - Az, j(x_n - z)\rangle \\ &\le (1 - \beta_n - \alpha_n\bar{\gamma}) \|x_n - z\|^2 + \beta_n \|x_{n-1} - z\| \|x_n - z\| + \alpha_n\beta \|x_n - z\|^2 + \alpha_n \langle f(z) - Az, j(x_n - z)\rangle \\ &\le (1 - \beta_n - \alpha_n(\bar{\gamma} - \beta)) \|x_n - z\|^2 + \frac{\beta_n}{2} \|x_{n-1} - z\|^2 + \frac{\beta_n}{2} \|x_n - z\|^2 + \alpha_n \langle f(z) - Az, j(x_n - z)\rangle \\ &= \left(1 - \frac{\beta_n}{2} - \alpha_n(\bar{\gamma} - \beta)\right) \|x_n - z\|^2 + \frac{\beta_n}{2} \|x_{n-1} - z\|^2 + \alpha_n \langle f(z) - Az, j(x_n - z)\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_n - z\|^2 &\le \frac{\beta_n}{\beta_n + 2\alpha_n(\bar{\gamma} - \beta)} \|x_{n-1} - z\|^2 + \frac{2\alpha_n}{\beta_n + 2\alpha_n(\bar{\gamma} - \beta)} \langle f(z) - Az, j(x_n - z)\rangle \\ &= \left[1 - \frac{2\alpha_n(\bar{\gamma} - \beta)}{\beta_n + 2\alpha_n(\bar{\gamma} - \beta)}\right] \|x_{n-1} - z\|^2 + \frac{2\alpha_n(\bar{\gamma} - \beta)}{\beta_n + 2\alpha_n(\bar{\gamma} - \beta)} \frac{\langle f(z) - Az, j(x_n - z)\rangle}{\bar{\gamma} - \beta}. \end{aligned} \tag{3.8}$$

We note that

$$\frac{2\alpha_n(\bar{\gamma} - \beta)}{2\alpha_n(\bar{\gamma} - \beta) + \beta_n} > \frac{2\alpha_n(\bar{\gamma} - \beta)}{2\alpha_n + 2\beta_n} = (\bar{\gamma} - \beta)\frac{\alpha_n}{\alpha_n + \beta_n}.$$

Therefore, condition (ii) yields $\sum_{n=0}^{\infty} \frac{2\alpha_n(\bar{\gamma}-\beta)}{2(\bar{\gamma}-\beta)\alpha_n+\beta_n} = \infty$. Applying Lemma 2.2 to (3.8), we have that $x_n \to z$ as $n \to \infty$. This completes the proof. □

Remark 3.1. Put $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n^2}$. Then $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (i) and (ii) of Theorem 3.1. But we note that $\frac{\alpha_n}{\beta_n} = n \to \infty$.

Remark 3.2. Theorem 3.1 extends and improves Theorem 3.1 of Yao [2] in the following aspects.

(i) u is replaced by a Lipschitz strongly pseudocontractive mapping.
(ii) One continuous pseudocontractive mapping is replaced by a countable family of continuous pseudocontractive mappings.

(iii) Condition $\frac{\alpha_n}{\beta_n} \to 0$ is weakened to $\alpha_n \to 0$ and $\beta_n \to 0$, as $n \to \infty$.
(iv) We add a strongly positive linear operator A in our iterative algorithm.

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