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Perturbations of the Spectrum of Nonlinear Eigenvalue Problems

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1. INTRODUCTION

In the application of nonlinear eigenvalue problems, we are frequently faced with a degree of uncertainty concerning the description of the region or the values of the coefficients in both the differential equation and the boundary conditions. As with linear eigenvalue problems, it is therefore of interest to ascertain whether the "spectrum" of such problems is continuous with respect to small perturbations of the operators and of the region. The problems considered here arise in the determination of the critical conditions for thermal ignition (using the steady-state model). The equations are generalizations of the steady-state heat-conduction with a nonlinear source term depending on the temperature (but not on its gradient). This leads to problems of the form

$$-\nabla \cdot (k(x)\, \nabla T) = \lambda G(x, T) \qquad x \in \Omega, \tag{1.1}$$

subject to the appropriate boundary conditions on the boundary $\partial\Omega$ of region Ω, where $k(x)$ is the thermal conductivity of the inhomogeneous region Ω, and T is the local temperature in Ω. The form of the nonlinearity G is not known precisely in many practical situations, such as in an exothermic chemical reaction. Commonly, it is taken as varying with temperature T as in the Arrhenius law

$$G(x, T) \propto \exp(-E/RT), \tag{1.2}$$

where E is the activation energy (determined empirically) and R is the gas constant. Even this form is likely to be approximated over a large range of temperatures.

For the present paper, it is of no additional difficulty to study equations of the form

$$L(u) = \lambda f(x, u) \qquad x \in \Omega, \tag{1.3}$$

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where $x = (x_1, \ldots, x_n)$, Ω is an open bounded subset of $\mathbb{R}^n$ with closure $\bar{\Omega}$ and boundary $\partial\Omega \in C^{2+\alpha}$, L is a real, linear, second-order, uniformly elliptic, self-adjoint operator given by

$$L(u) \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)\,u,$$

and the matrix $(a_{ij}(x))$ is symmetric and uniformly positive definite on Ω. We will assume that $a_0 \geqslant 0$ in Ω, a_{ij}, a_0, $\partial a_{ij}/\partial x_i \in C^\alpha(\bar{\Omega})$. On the boundary $\partial\Omega$ of Ω, we will consider a linearized radiation condition given by

$$B_\epsilon(u) \equiv au + \epsilon\frac{\partial u}{\partial \nu} = 0 \qquad x \in \partial\Omega, \tag{1.4}$$

where $\partial u/\partial v$ is the conormal dervative given by

$$\frac{\partial u}{\partial \nu} \equiv \sum_{i,j=1}^{n} m_i a_{ij}\frac{\partial u}{\partial x_j},$$

and $m = (m_1, \ldots, m_n)$ is an outward normal unit vector field on $\partial\Omega$. Also, ϵ is 0 or 1, and a satisfies the following conditions:

$$a > 0,\ \epsilon = 0 \qquad \text{when} \quad x \in \partial\Omega_1,$$
$$a \geqslant 0,\ \epsilon = 1 \qquad \text{when} \quad x \in \partial\Omega_2 = \partial\Omega - \partial\Omega_1.$$

Moreover, we require that a and a_0 are not both identically zero, $m_i \in C^{1+\alpha}(\partial\Omega)$, $a \in C^{2-\epsilon+\alpha}(\partial\Omega)$. (For the necessity of these conditions, see Amann [1], Ladyzenskaja and Ural'tseva [8].)

In this case we can apply the strong maximum principle to problem (1.3, 1.4). Further, the pair of operators (L, B_ϵ) is invertible on its domain, so we can consider the problem in the form

$$u = \lambda KFu, \tag{1.5}$$

where $(Ku)(x) = \int_\Omega K(x, y)\,u(y)\,dy$, $u \in C(\bar{\Omega})$; $K(x, y)$ is the Green's function of operator pair (L, B_ϵ), and F is the corresponding Nemytskii operator to f. More precisely, if S is an arbitrary bounded subset of $\mathbb{R}$, and if $f: \bar{\Omega} \times S \to \mathbb{R}$, then for every $u: \bar{\Omega} \to S$, we define $F[u](x) \equiv f(x, u(x))$.

In what follows we will assume that f satisfies all of the following conditions:

H1: $f \in C^\alpha(\bar{\Omega} \times S)$, for all bounded $S \subseteq \mathbb{R}^+ \equiv (0, \infty)$;

H2: $f(x, 0) \geqslant 0$, for all $x \in \bar{\Omega}$, but $f(x, 0) \not\equiv 0$;

H3: $f_\xi(x, \xi), f_{\xi\xi}(x, \xi) \in C^\alpha(\bar{\Omega} \times S)$, for all bounded $S \subseteq \mathbb{R}^+$;

H4: $\inf_{(x,\xi)\in\Omega\times\mathbb{R}^+} f_\xi(x, \xi) > 0$.

(It is true that the function in Eq. (1.2) does not satisfy H4, but the common approximations to it that are used in ignition theory do satisfy this condition, for example, $f \propto \exp. u$, where $u = E(T - T_a)/RT_a^2$ and T_a is the ambient temperature. See Fradkin and Wake [4].) The conditions H1–H3 imply that F is twice Frechét differentiable. In particular,

$$F_u(u)[v](x) = f_\xi(x, u(x))\, v(x), \qquad F_{uu}(u)[v](x) = f_{\xi\xi}(x, u(x))(v(x))^2.$$

In what follows we will use the linearized problem corresponding to problem (1.3, 1.4), that is,

$$L(v) = \mu r(x) v \qquad x \in \Omega, \tag{1.6}$$

$$B_\epsilon(v) = 0 \qquad x \in \partial\Omega, \tag{1.7}$$

where $r(x) = f_\xi(x, u(x))$ and u is a solution of (1.3, 1.4).

We are interested in the positive classical solutions of problem (1.3, 1.4) only, that is, in those from the space $C^2(\Omega) \cap C^\alpha(\bar{\Omega})$. We shall be concerned with the "spectrum" of the problem, which is the set Λ of λ for which there exists a positive classical solution of (1.3, 1.4). For any $\lambda \in \Lambda$, we denote by $\underline{u}(\lambda, x)$ the corresponding minimal solution, that is, for any other solution $u(\lambda, x)$, we have that $u(\lambda, x) \geqslant \underline{u}(\lambda, x)$ for all $x \in \bar{\Omega}$. The minimal solution $\underline{u}(\lambda, x)$ can be constructed by means of a standard iteration scheme (as in Amann [1] and Keller and Cohen [7]). Also, it is well-known that, if $\lambda' > 0$ belongs to Λ, then $(0, \lambda'] \subseteq \Lambda$, that is Λ is an interval. (See Keller and Cohen [7]). The fact that $\lambda^* \equiv \sup \Lambda$ is positive is an immediate corollary of the fact that the solution set $\{(\lambda, u(\lambda, x)) : \lambda \in \Lambda\}$ of problem (1.3, 1.4) contains an unbounded component emanating from point $(0, 0)$ (see Amann [2]). Often λ^* is taken as the critical value for ignition of the corresponding thermal regime. We have shown previously (Fradkin [3]) that under H1–H4, $\lambda^* < \infty$. In this paper, we discuss how small perturbations of operators F and (L, B_ϵ) and region Ω affect λ^*, that is, the continuity properties of λ^*.

For a given function f, satisfying H1–H4, we shall denote by $\Lambda(f)$ and $\lambda^*(f)$ the spectrum and supremum of the spectrum, respectively. Further, we shall need to use the first eigenvalue μ_1 of problem (1.6, 1.7), when

$$r(x) \equiv f_\xi(x, \underline{u}(\lambda, x)), \tag{1.8}$$

in Eq. (1.6). Since this implies that r depends on λ, so will $\mu_1 \equiv \mu_1(\lambda)$ depend on λ.

2. Continuity Properties of λ^*

Our first result is obtained for the case in which the operator (L, B_ϵ) remain unchanged while operator F is varied in a certain way. This leads to bounds on $\lambda^*(f)$.

THEOREM 2.1. *If f, g satisfy* H1–H3, $\lambda^*(f)$ *is finite, and there exists* p: $\bar{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$g = (1 + p)f \qquad (x, \xi) \in \bar{\Omega} \times \mathbb{R}^+,$$
$$|p| \leqslant C < \infty \qquad (x, \xi) \in \bar{\Omega} \times \mathbb{R}^+,$$

then $\lambda^*(f)/(1 + a) \leqslant \lambda^*(g) \leqslant \lambda^*(f)/(1 + b)$, *where* $a = \sup_{(x,\xi)\in\bar{\Omega}\times\mathbb{R}^+} p$ *and* $b = \inf_{(x,\xi)\in\bar{\Omega}\times\mathbb{R}^+} p \neq -1$.

Proof. By the requirements of the theorem, a, b are finite and $a, b > -1$ (otherwise H2 would not be satisfied for g), and

$$(1 + b)f \leqslant g \leqslant (1 + a)f. \tag{2.1}$$

Keller and Cohen [7] showed that $\lambda \in \Lambda(z)$ if $\lambda \in \Lambda(q)$, when $q(x, u) \geqslant z(x, u)$ and q, z satisfy H1–H3. So we have

$$\lambda^*(q) \leqslant \lambda^*(z) \qquad \text{if} \quad q \geqslant z. \tag{2.2}$$

As a, b are constants, we obtain

$$\lambda^*((1 + b)f) = \lambda^*(f)/(1 + b), \tag{2.3}$$

and a similar equation for a replacing b. The theorem then follows directly from Eqs. (2.1–2.3). Hence $\lambda^*(g)$ is continuous at $g = f$ under the continuous perturbations of f. Q.E.D.

Remark 2.1. We can consider g in the form $g = f + s$, under the condition that s/f is uniformly bounded on its domain (see Fradkin [3]).

Considering perturbations of the operators (L, B_ϵ), we are able to prove that λ^* is continuous from below with respect to continuous perturbations of (L, B_ϵ). This means that if λ^* decreases as a result of such perturbations, then it decreases continuously. In practice, this means that small experimental errors in the functions of the operators (L, B_ϵ) cause small perturbations in λ^* if these lead to under-estimation of λ^*. Small errors in the operators, which cause over-estimates of λ^*, *may* yield discontinuous changes in λ^*.

We give some preliminary results first.

LEMMA 2.1. *Let* Φ: $\mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ *with the following properties*:

(i) $\Phi \in C^1(\mathbb{R}) \times \{C(\bar{\Omega}) \cap C^1(\Omega)\}$. *Then, by the first mean-value theorem, for every* $\tau \in \mathbb{R}$, *there exists a* τ', *such that* $\tau' \in [0, \tau]$ *if* $\tau \geqslant 0$ *and* $\tau' \in [\tau, 0]$ *if* $\tau \leqslant 0$, *and*

$$\Phi(\tau, x) = \Phi(0, x) + \Phi_\tau(\tau', x)\tau, \qquad \text{in } \bar{\Omega};$$

(ii) $\Phi(0, x) > 0$ *in* Ω;

(iii) *$\forall \tau_1$, such that $\tau_1 \in (0, \infty)$ there exists an $N(\tau_1) < \infty$, such that for each τ with $|\tau| \leqslant \tau_1$*

$$|\Phi_\tau(\tau', x)/\Phi(0, x)| \leqslant N(\tau_1), \quad \textit{in } \Omega \qquad \forall\, |\tau| \leqslant \tau_1;$$

where τ' is defined for every τ with $|\tau| \leqslant \tau_1$ by (i). *Then there exists a $\tau_0 > 0$, such that*

$$\Phi(\tau, x) = \Phi(0, x) + \Phi_\tau(\tau', x)\tau > 0, \quad \textit{in } \Omega \qquad \forall\, |\tau| \leqslant \tau_0. \tag{2.4}$$

Proof. One can see that even without condition (iii) the statement of Lemma 2.1 is obvious—under the additional requirement that there do not exist points x_0 on boundary $\partial\Omega$ with $\Phi(0, x_0) = 0$. Without this requirement, condition (iii) becomes necessary. Indeed, it implies that $\Phi_\tau(\tau', x_0) = 0$, too, and, as $x \to x_0$, $\Phi_\tau(\tau', x)$ has zero of the same order as $\Phi(0, x)$.

Let us now fix an arbitrary $\tau_1 > 0$ and introduce a τ_2, such that $N(\tau_1) \equiv 1/\tau_2$ (and, hence, $\tau_2 > 0$, too). We can now rewrite condition (iii) as follows:

$$-N(\tau_1) \leqslant -\Phi_\tau(\tau', x)/\Phi(0, x) \leqslant N(\tau_1) \equiv 1/\tau_2, \quad \text{in } \Omega \qquad \forall\, |\tau| \leqslant \tau_1. \tag{2.5}$$

It is obvious that either $1/\tau_2 \leqslant 1/\tau_1$ or $1/\tau_2 > 1/\tau_1$. Let us consider these two cases separately.

(a) If $1/\tau_2 < 1/\tau_1$, then for any τ with $|\tau| \leqslant \tau_1$ it follows that $1/\tau_2 \leqslant 1/|\tau|$. Therefore, $1/\tau_2 < 1/\tau$ if $\tau \in (0, \tau_1)$, while $1/\tau < -1/\tau_2$ if $\tau \in (\tau_1, 0)$.

It follows from (2.5) that

$$-\Phi_\tau(\tau', x)/\Phi(0, x) < 1/\tau, \quad \text{in } \Omega \qquad \forall \tau \in (0, \tau_1) \tag{2.6}$$

and

$$1/\tau < -\Phi_\tau(\tau', x)/\Phi(0, x), \quad \text{in } \Omega \qquad \forall \tau \in (-\tau_1, 0). \tag{2.7}$$

Since $\Phi(0, x) > 0$ in Ω, $\tau > 0$ if $\tau \in (0, \tau_1)$, and $\tau < 0$ if $\tau \in (-\tau_1, 0)$, it follows immediately from (2.6) and (2.7) that

$$\Phi(\tau, x) = \Phi(0, x) + \Phi_\tau(\tau', x)\tau > 0, \quad \text{in } \Omega \qquad \forall\, |\tau| < \tau_1.$$

(b) If $1/\tau_2 > 1/\tau_1$ we introduce a $\tau_3 \equiv 1/N(\tau_2)$ (and, hence, $\tau_3 > 0$). Then we rewrite condition (iii) as follows:

$$-1/\tau_3 \equiv -N(\tau_2) < -\Phi_\tau(\tau', x)/\Phi(0, x) < N(\tau_2) \equiv 1/\tau_3, \text{ in } \Omega \qquad \forall\, |\tau| < \tau_2.$$

We notice that $1/\tau_3 \equiv N(\tau_2) \leqslant N(\tau_1) \equiv 1/\tau_2$, since $\tau_2 < \tau_1$. And therefore,

$$-\Phi_\tau(\tau', x)/\Phi(0, x) < 1/\tau, \text{ in } \Omega \qquad \forall \tau \in (0, \tau_2) \tag{2.8}$$

and

$$1/\tau < -\Phi_\tau(\tau', x)/\Phi(0, x), \text{ in } \Omega \qquad \forall \tau \in (-\tau_2, 0). \tag{2.9}$$

Then, as before, the result follows from (2.8) and (2.9), namely:

$$\Phi(\tau, x) = \Phi(0, x) + \Phi_\tau(\tau', x)\tau > 0, \text{ in } \Omega \qquad \forall\, |\tau| < \tau_2 .$$

This means that whatever $\tau_1 > 0$, there always exist a τ_0 (namely, $\tau_0 \leqslant \min\{\tau_1, \tau_2\}$), such that (2.4) holds. Q.E.D.

In the following we shall make use of the class M_1 of functions, where $M_1 \equiv \{u: u \in C^2(\bar{\Omega}) \cap C^\alpha(\bar{\Omega}); B_\epsilon u = 0 \text{ on } \partial\Omega\}$.

LEMMA 2.2. *If $f(x, u(x))$ satisfies* H1–H4, *then for every $\lambda' \in (0, \lambda^*)$ with the property $\lambda' < \mu_1(\lambda')$ there exists a $\tilde{u} \in M_1$, such that $\underline{u}(\lambda', x) \leqslant \tilde{u}(\lambda', x)$ on $\bar{\Omega}$, where the equality sign holds on $\partial\Omega_1$ only, and*

$$\tilde{u}(\lambda', x) - \lambda' KF[\tilde{u}(\lambda', \cdot)](x) > 0, \qquad in\ \Omega. \tag{2.10}$$

Proof. We notice that λ^* is finite.

We fix an arbitrary $\lambda' \in (0, \lambda^*)$ and denote by $\mu_1' \equiv \mu_1(f_\epsilon(\cdot, \underline{u}(\lambda', \cdot))$ and $\underline{v}' \equiv \underline{v}(\lambda', x)$, the principal eigenvalue and the principal eigenfunction of problem (1.6, 1.7). We notice that under all conditions laid upon (L, B_ϵ), $\underline{v}' \in M_1$ (as does $\underline{u}$).

To prove the existence of $\tilde{u}$ we consider the class of positive real functions $\tilde{u}(\tau) \equiv \tilde{u}(\tau, \lambda', x)$, where $\tau \in \mathbb{R}$, defined by $\tilde{u}(\tau) \equiv \underline{u}' + \tau\underline{v}'$. It is clear that for each $\tau \in R^+$, $\tilde{u}(\tau) \in M_1$ ($\underline{u}' \equiv \underline{u}(\lambda', x)$).

Let us now apply Taylor's theorem (Schwartz [9, p. 29]) to the operator $(I - \lambda KF)$ at $u = \tilde{u}(\tau)$, $\exists\, \tau' \in [0, \tau]$ such that:

$$\begin{aligned} &\tilde{u}(\tau) - \lambda' KF[\tilde{u}(\tau)] \\ &\quad = \{\tilde{u}(\tau) - \lambda' KF[\tilde{u}(\tau)]\} - \{\underline{u}' - \lambda' KF[\underline{u}']\} \\ &\quad = \tau\{\underline{v}' - \lambda' KF_u(\underline{u}')\,[\underline{v}']\} - \frac{\tau^2}{2!} KF_{uu}(\underline{u}' + \tau'\underline{v}')\,[\underline{v}']^2, \quad \text{on } \bar{\Omega}. \end{aligned} \tag{2.11}$$

If we introduce the notation $\Phi(\tau, x) \equiv \frac{1}{2}\{\underline{v}' - \lambda' KF_u(\underline{u}')[\underline{v}']\} - \tau/2F_{uu}(\underline{u}' + \tau'\underline{v}')[\underline{v}']^2$, then

$$\begin{aligned} \Phi(0, x) &= \tfrac{1}{2}\{\underline{v}'(x) - \mu_1' KF_u(\underline{u}')\,[\underline{v}']\,(x) + (\mu_1' - \lambda')\,KF_u(\underline{u}')\,[\underline{v}']\,(x)\} \\ &= \tfrac{1}{2}\{(\mu_1' - \lambda')\,KF_u(\underline{u}')\,[\underline{v}']\,(x)\} > 0, \qquad \text{in} \quad \Omega \end{aligned} \tag{2.12}$$

since $\lambda' < \mu_1'$; $\underline{v}'(x) > 0$ in Ω, and H4 holds (i.e. $KF_u(\underline{u}')[\underline{v}'] > 0$ in Ω). We can also see that

$$\Phi_\tau(\tau', x) = -\tfrac{1}{2}KF_{uu}(\underline{u}' + \tau'\underline{v}')[\underline{v}']^2. \tag{2.13}$$

As both $\Phi_\tau(\tau', x)$ and $\Phi(0, x) \in M_1$, and $\Phi(0, x) > 0$ in Ω, we have $\Phi_\tau(\tau', x)/\Phi(0, x) \in C(\bar{\Omega})$ (see Keener and Keller [6, Theorem 3.2]), and so the conditions of Lemma 2.1 are satisfied. Using this lemma, we conclude that there exists a

$\tau_0 > 0$, such that $\Phi(\tau, x) > 0$ in Ω, and, hence, (2.10) holds (via (2.11) and (2.12)) for any $\tilde{u} \equiv \tilde{u}(\tau, \lambda', x)$, $\forall \tau \in (0, \tau_0)$. We notice that if $\partial\Omega = \phi$ both Lemma 2.1 and this conclusion are trivial. Q.E.D.

We consider the family of operators $(L(\tau), B_\epsilon(\tau))$, $\forall \tau \in \mathbb{R}$, such that for every fixed τ all the conditions laid upon the operator pair $(L(0), B_\epsilon(0)) \equiv (L, B_\epsilon)$ are satisfied and all functions $a_{ij}(\tau, x)$, $a_0(\tau, x)$, and $a(\tau, x)$ depend on τ holomorphically. Neither Ω nor $\partial\Omega$ change. It is known that in this case $K(\tau) \equiv K(\tau, x, y)$ depends on τ holomorphically as well (see Kato [5, pp. 365, 426]). The corresponding spectra we denote by $\Lambda(\tau)$. We proceed now to the last lemma needed.

LEMMA 2.3. *If λ', with $\lambda' < \mu_1(\lambda')$, belongs to the spectrum $\Lambda(0)$, then there exists a $\tau_0 > 0$, such that $\lambda' \in \Lambda(\tau)$, $\forall \tau \in [-\tau_0, \tau_0]$.*

Proof. Let us construct $\tilde{u}$, such that (2.10) is satisfied for the operator pair $(L(0), B_\epsilon(0))$ (see Lemma 2.2). Let us prove that there exists a $\tau_0 > 0$, such that condition (2.10) is satisfied for a function $\tilde{u}(\tau, \lambda', x) \in C^1(\mathbb{R}) \times M_1{}^+(\tau)$ ($\tilde{u}(0, \lambda', x) \equiv \tilde{u}$), and the operator pair $(L(\tau), B_\epsilon(\tau))$, $\forall \tau \in [-\tau_0, \tau_0]$, too, where $M_1{}^+(\tau) \equiv \{u: u \in C^2(\bar{\Omega}) \cap C^\alpha(\bar{\Omega}); u > 0 \text{ in } \Omega; B_\epsilon(\tau) = 0 \text{ on } \partial\Omega\}$ (we recall that λ' is a fixed point of $\Lambda(0)$). We consider

$$\begin{aligned}\Phi(\tau, x) &\equiv \tilde{u}(\tau, \lambda', x) - \lambda' K(\tau) F[\tilde{u}(\tau, \lambda', \cdot)](x)\\ &= \{\tilde{u}(\tau, \lambda', x) - \lambda' K(0) F[\tilde{u}](x)\} - \tau\{\lambda' K(\tau') F_u(\tilde{u}(\tau', \lambda', \cdot)[\tilde{u}_\tau(\tau', \lambda', \cdot)]\\ &\quad + \lambda' K_\tau(\tau) F[\tilde{u}(\tau', \lambda', \cdot)]\}(x) \equiv \Phi(0, x) + \Phi_\tau(\tau', x)\tau, \text{ in } \Omega \quad \tau' \in [0, \tau].\end{aligned} \tag{2.14}$$

We notice that $\Phi(0, x) > 0$ in Ω (see (2.10)), and $\Phi(0, x) \in M_1{}^+(\tau)$, $\Phi_\tau(\tau', x) \in M_1(\tau)$. The desired result follows, therefore, as above, by Lemma 2.1.

We have proved, in other words, that $\tilde{u}(\tau, \lambda', x)$ is an "upper solution" for operator K and eigenvalue $\lambda' \in \Lambda(\tau)$. As K is an increasing operator, this means that there exists a function $\underline{u}(\tau, \lambda', x)$, the minimal solution of the corresponding problem. (We do not reproduce the proof of this statement here.) Therefore, $\lambda' \in \Lambda(\tau)$, $\forall \tau \in [-\tau_0, \tau_0]$. Q.E.D.

Remark 2.2. We have applied the method of upper solutions to K and not (L, B_ϵ) by necessity, not by choice. Indeed, only in this case, Theorem 3.2 [6] of Keener and Keller and, hence, our Lemma 2.1, may be used.

We are able to prove our main result now.

THEOREM 2.2. *If $\lambda^*(\tau)$ decreases in a vicinity of $\lambda^*(0)$ and there are λ', such that $\lambda' < \mu_1(\lambda')$ in every neighborhood of $\lambda^*(0)$, then $\lambda^*(\tau)$ decreases continuously.*

Proof. In Lemma 2.3 we have proved that if $\lambda' < \mu_1(\lambda')$ and $\lambda' \in \Lambda(0)$, then also $\lambda' \in \Lambda(\tau)$, $\forall \tau \in [-\tau_0, \tau_0]$, for some $\tau_0 > 0$. The statement of the theorem implies that λ', with $\lambda' < \mu_1(\lambda')$ exists in any vicinity of $\lambda^*(0)$. In view of this fact, if $\lambda^*(\tau)$ decreases, Lemma 2.3 can be restated as follows: $\forall\lambda'$, with $\lambda' < \mu_1(\lambda')$, there exists a τ_0 such that $\lambda' \leqslant \lambda^*(\tau) \leqslant \lambda^*(0)$ if $|\tau| \leqslant \tau_0$. In other words, $\forall\delta$ there exists a $\tau_0 > 0$, such that $\lambda^*(0) - \delta \leqslant \lambda' \leqslant \lambda^*(\tau) \leqslant \lambda^*(0)$ if $|\tau| \leqslant \tau_0$. But this means that $\lambda^*(\tau)$ is continuous in τ in a vicinity of $\tau = 0$. Q.E.D.

Remark 2.3. It is known that for convex nonlinearities, the second requirement of the theorem is not necessary, as $\lambda < \mu_1(\lambda)$ for all $\lambda \in [0, \lambda^*)$ in this case (see Keener and Keller [6, Theorem 3.2(i)]).

Remark 2.4. Let region Ω also change continuously in such a way that $\Omega(\tau_2) \subset \Omega(\tau_1)$ for $|\tau_2| > |\tau_1|$, sign τ_2 = sign τ_1. That is, let $\partial\Omega(\tau)$ change in such a way that $\Omega(\tau) \subset \Omega(0)$ where $\partial\Omega(0)$ corresponds, say, to the real boundary and $\partial\Omega(\tau)$ to the experimentally found one. Then the expression for $\Phi(\tau, x)$ is still defined by (2.14), but in $\Omega(\tau)$. We now have $\Phi(0, x) > 0$ on $\bar{\Omega}(\tau)$, $\forall\tau \in \mathbb{R}^+$ and, therefore, both Lemma 2.3 and (hence) Theorem 2.2 are valid. In practice, this means that only in this case can we be sure that experimental errors cause small errors in λ^*.

Let us consider some examples.

EXAMPLE 2.1. If the functions $a(x)$ and $a_0(x)$ change in such a way that

(i) $a_0(\tau_1, x) \geqslant a_0(\tau_2, x)$ in Ω $\quad 0 \leqslant \tau_1 \leqslant \tau_2$,

(ii) $a(\tau_1, x) \geqslant a(\tau_2, x)$ on $\partial\Omega$ $\quad 0 \leqslant \tau_1 \leqslant \tau_2$,

then $\lambda^*(\tau_1) \geqslant \lambda^*(\tau_2)$, that is, $\lambda^*(\tau)$ decreases with τ.

Proof. To show that this is true, we need to prove that if $\lambda \in \Lambda(\tau_2)$, then necessarily $\lambda \in \Lambda(\tau_1)$. By using the iteration scheme for construction of minimal solutions, the first step gives

$$\begin{aligned} L(\tau_1)\, u_0(\tau_1, x) &= \lambda f(x, 0), \qquad \text{in} \quad \Omega, \\ B_\epsilon(\tau_1)\, u_0(\tau_1, x) &= 0, \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} L(\tau_2)\, u_0(\tau_2, x) &= \lambda f(x, 0), \qquad \text{in} \quad \Omega, \\ B_\epsilon(\tau_2)\, u_0(\tau_2, x) &= 0, \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.16}$$

By using the conditions (i) and (ii) on $a(x)$ and $a_0(x)$, we obtain

$$\begin{aligned} &L(\tau_2)[u_0(\tau_2, x) - u_0(\tau_1, x)] \\ &\quad = L(\tau_2)\, u_0(\tau_2, x) - L(\tau_2)\, u_0(\tau_1, x) + L(\tau_1)\, u_0(\tau_1, x) - L(\tau_1)\, u_0(\tau_1, x) \\ &\quad = L(\tau_1)\, u_0(\tau_1, x) - L(\tau_2)\, u_0(\tau_1, x) \\ &\quad = (a_0(\tau_1, x) - a_0(\tau_2, x))\, u_0(\tau_1, x) \geqslant 0, \quad \text{in } \Omega, \end{aligned} \tag{2.17}$$

$$
\begin{aligned}
&B_\epsilon(\tau_2)[u_0(\tau_2, x) - u_0(\tau_1, x)] \\
&\quad = B_\epsilon(\tau_2)\, u_0(\tau_2, x) - B_\epsilon(\tau_2)\, u_0(\tau_1, x) + B_\epsilon(\tau_1)\, u_0(\tau_1, x) - B_\epsilon(\tau_1)\, u_0(\tau_1, x) \\
&\quad = B_\epsilon(\tau_1)\, u_0(\tau_1, x) - B_\epsilon(\tau_2)\, u_0(\tau_1, x) \\
&\quad = (a(\tau_1, x) - a(\tau_2, x))\, u_0(\tau_1, x) \geqslant 0, \qquad \text{on } \partial\Omega.
\end{aligned}
$$

The maximum principle (Keller and Cohen [7]) yields accordingly that

$$u_0(\tau_2, x) \geqslant u_0(\tau_1, x), \quad \text{in } \bar{\Omega} \qquad 0 \leqslant \tau_1 \leqslant \tau_2 .$$

Similarly, we obtain

$$u_n(\tau_2, x) \geqslant u_n(\tau_1, x), \quad \text{in } \bar{\Omega} \qquad 0 \leqslant \tau_1 \leqslant \tau_2, \quad n \in Z. \tag{2.18}$$

By our assumption that $\lambda \in \Lambda(\tau_2)$, we see that $\underline{u}(\tau_2, \lambda, x)$ exists and by (2.18) so does $\underline{u}(\tau_1, \lambda, x)$. It follows that $\lambda \in \Lambda(\tau_1)$. Q.E.D.

EXAMPLE 2.3. If all the functions of the operator pair (L, B_ϵ) change in such a way that $K(\tau_2)[u(\cdot)] \geqslant K(\tau_1)[u(\cdot)]$, $\forall u \in C(\bar{\Omega})$, $0 \leqslant \tau_1 \leqslant \tau_2$, then $\lambda^*(\tau_1) \geqslant \lambda^*(\tau_2)$, $0 \leqslant \tau_1 \leqslant \tau_2$ as well. This statement can be proved as above, but using the iteration scheme for the operator K instead.

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