Journal of Mathematical Analysis and Applications **249,** 652–667 (2000)
doi:10.1006/jmaa.2000.6967, available online at http://www.idealibrary.com on IDEAL®

Spectral Synthesis of Jordan Operators

Steven M. Seubert

Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, Ohio 43403-0221
E-mail: sseuber@math.bgsu.edu

Submitted by Paul S. Muhly

Received November 22, 1999

The purpose of this paper is to extend results of J. Wermer (1952, *Proc. Amer. Math. Soc.* **3**, 270–277), L. Brown et al. (1960, *Trans. Amer. Math. Soc.* **96**, 162–183), and D. Sarason (1972, *J. Reine Agnew. Math.* **252**, 1–15; 1966, *Pacific J. Math.* **17**, 511–517) on spectral subspaces of diagonalizable operators on separable complex Hilbert space to the class of so-called Jordan operators or infinite direct sums of Jordan cells. © 2000 Academic Press

1. INTRODUCTION

In this paper, we are concerned with a special case of the following general problem: Let $\mathscr{X}$ be a Banach space and let $T\colon \mathscr{X} \to \mathscr{X}$ be a bounded linear operator which is complete, that is, whose root vectors have dense linear span in $\mathscr{X}$. Under what conditions will every subspace invariant for T be the closed linear span of the root vectors for T that it contains? (Recall that a closed subspace $\mathscr{M}$ of a Banach space $\mathscr{X}$ is *invariant* for a bounded linear operator $T\colon \mathscr{X} \to \mathscr{X}$ if $T\mathscr{M} \subseteq \mathscr{M}$ and that a vector x in $\mathscr{X}$ is a *root vector* for T if there exists a complex number λ and a positive integer n such that $(T - \lambda I)^n x$ is the zero vector.) Invariant subspaces for a complete operator which are the closed linear span of the root vectors for the operator they contain are called *spectral* subspaces for the operator and complete operators all of whose invariant subspaces are spectral are said to admit *spectral synthesis*.

In particular, we take $\mathscr{X}$ to be a separable complex Hilbert space and T to be the infinite direct sum of Jordan cells or a so-called Jordan operator. Recall that a bounded linear operator $T\colon \mathscr{H} \to \mathscr{H}$ on a finite-dimensional Hilbert space $\mathscr{H}$ is a *Jordan cell* if there exists a complex number λ and an

0022-247X/00 $35.00

orthonormal basis $\{e_i : 1 \le i \le m\}$ for $\mathscr{H}$ such that the matrix representation for T with respect to $\{e_i : 1 \le i \le m\}$ is

$$J(\lambda, m) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ & \lambda & 1 & \cdots & 0 & 0 \\ & & \lambda & \cdots & 0 & 0 \\ & & & \ddots & & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{pmatrix}.$$

That is, a bounded linear operator $J: \mathscr{H} \to \mathscr{H}$ on a separable complex Hilbert space $\mathscr{H}$ is a *Jordan operator* if there exists a bounded sequence $\{\lambda_n\}$ of complex numbers, a sequence $\{m_n\}$ of positive integers, and a sequence $\{\mathscr{H}_n\}$ of Hilbert spaces such that $\mathscr{H} = \oplus \mathscr{H}_n$ and for each positive integer n, the restriction $J \,|\, \mathscr{H}_n$ of J to $\mathscr{H}_n$ is the Jordan cell $J(\lambda_n, m_n)$.

In this paper, we seek necessary and sufficient conditions for a Jordan operator to admit spectral synthesis.

The special case of diagonalizable Jordan operators $\oplus J(\lambda_n, 1)$ (that is, of complete normal operators) was studied by Wermer [14] and Brown et al. [2], and was solved by Sarason [10, 11] in 1972. The following is a synopsis of the results relevant to our study. The most important feature of the result is the connection between condition (v) and the rest. Also see Nikolskii [7, Wermer–Sarason theorem, p. 99]) for additional conditions.

THEOREM 1. *Let $\{\lambda_n\}$ be a sequence of distinct points in the open unit disc $\mathbf{D} \equiv \{z \in \mathbf{C} : |z| < 1\}$ and let $D = \oplus J(\lambda_n, 1)$ be a diagonalizable operator acting on a Hilbert space $\mathscr{H} =$ span$\{e_n\}$. The following are equivalent*:

(i) *D admits spectral synthesis*,

(ii) *a vector x in $\mathscr{H}$ is cyclic for D if and only if $\langle x, e_n \rangle \neq 0$ for all positive integers n*,

(iii) *there does not exist a sequence $\{w_n\}$ of complex numbers for which $0 < \Sigma |w_n| < \infty$ and $\Sigma w_n \lambda_n^i = 0$ for all nonnegative integers i*,

(iv) *the weakly closed algebra generated by D and the identity coincides with the commutant of D, and*

(v) *there does not exist a bounded complex domain Ω such that* $\sup\{|f(z)| : z \in \Omega\} = \sup\{|f(z)| : z \in \Omega \cap \{\lambda_n\}\}$ *for all functions f bounded and analytic on Ω.*

If, in addition, the points λ_n accumulate only on the boundary of $\mathbf{D}$, *then these conditions are equivalent to*:

(vi) *there does not exist a sequence $\{w_n\}$ of complex numbers for which $0 < \Sigma |w_n| < \infty$ and $\Sigma w_n e^{\lambda_n z} = 0$ for all complex numbers z*,

(vii) *the map* $T\colon H^\infty \to l_\infty(\mu)$ *from the space* H^∞ *of functions bounded and analytic on the unit disc to* $l_\infty(\mu)$ *where* $\mu \equiv \sum \delta_{\lambda_n}$ *is the measure consisting of point masses at the* λ_n *defined by* $T\colon f \to \{f(\lambda_n)\}$ *is not an isometry, and*

(viii) *not almost every point of the unit circle is in the nontangential cluster set of* $\{\lambda_n\}$.

Throughout this paper, we let $\{\lambda_n\}$ denote a bounded sequence of distinct complex numbers, we let $\{m_n\}$ denote a bounded sequence of positive integers, and we let $J = \oplus J(\lambda_n, m_n)$ denote a Jordan operator acting on a Hilbert space $\mathscr{H} = \oplus_{n=1}^{\infty} \mathscr{H}_n = \oplus_{n=1}^{\infty} \operatorname{span}\{e_{n,i} : 1 \le i \le m_n\}$. It is understood that for each positive integer $n, \{e_{n,i} : 1 \le i \le m_n\}$ denotes the unique orthonormal basis for $\mathscr{H}_n$ for which the matrix representation for $J \,|\, \mathscr{H}_n$ with respect to $\{e_{n,i} : 1 \le i \le m_n\}$ is the Jordan cell $J(\lambda_n, m_n)$. For distinct points λ_n, the Jordan operator $J \equiv \oplus J(\lambda_n, m_n)$ has a dense set of cyclic vectors (see [4]) and moreover, the spectral subspaces of J are precisely those subspaces of the form $\oplus \operatorname{span}\{e_{i,j} : i \le d_j\}$ where $\{d_n\}$ is any sequence of nonnegative integers with $d_n \le m_n$. (If $d_n = 0$, then we interpret $\operatorname{span}\{e_{i,j} : i \le d_j\}$ as the zero subspace.) We have restricted our attention to the case of distinct points λ_n in order to avoid uninteresting complications. The techniques of the paper apply in the more general case and yield similar results when suitably modified. The restriction that the block sizes $\{m_n\}$ be bounded, however, is imposed only so our proofs are valid.

For a survey of the present state of the spectral synthesis problem, see Nikolskii [6]. Also see Markus [5].

2. EQUIVALENT CONDITIONS FOR SPECTRAL SYNTHESIS

In this section, we give several equivalent conditions for a Jordan operator to admit spectral synthesis. These conditions are analogues of conditions (i), (ii), (iii), and (vi) of Theorem 1 on diagonalizable operators.

THEOREM 2. *Let* $\{\lambda_n\}$ *be a bounded sequence of distinct complex numbers, let* $\{m_n\}$ *be a bounded sequence of positive integers, and let* $J = \oplus J(\lambda_n, m_n)$ *be a Jordan operator acting on a Hilbert space* $\mathscr{H} = \oplus_{n=1}^{\infty} \mathscr{H}_n = \oplus_{n=1}^{\infty} \operatorname{span}\{e_{n,i} : 1 \le i \le m_n\}$. *The following are equivalent*:

(i) *J admits spectral synthesis*,

(ii) *a vector* $x = \oplus x_n$ *in* $\mathscr{H}$ *is cyclic for J if and only if for each positive integer* n, x_n *is a cyclic vector for* $J(\lambda_n, m_n)$ *on* $\mathscr{H}_n$,

(iii) *there does not exist a set* $\{w_{n,j}\}$ *of complex numbers such that* $0 < \sum_{n=1}^{\infty}\sum_{j=1}^{m_n}|w_{n,j}| < \infty$ *and* $\sum_{n=1}^{\infty}\sum_{j=1}^{\min(i+1,m_n)}\binom{i}{j-1}\lambda_n^{i-j+1}w_{n,j} = 0$ *for all nonnegative integers* i, *and*

(iv) *there does not exist a set* $\{w_{n,j}\}$ *of complex numbers such that* $0 < \sum_{n=1}^{\infty}\sum_{j=1}^{m_n}|w_{n,j}| < \infty$ *and* $0 \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{m_n}(w_{n,j}/(j-1)!)z^{j-1}e^{\lambda_n z}$ *for all complex numbers* z.

Proof. (i) $\Leftrightarrow$ (ii). Let $x = \oplus x_n$ be an arbitrary vector in $\mathscr{H} = \oplus\mathscr{H}_n$ and denote by $\mathscr{M}_x \equiv \operatorname{span}\{J^k x : k \geq 0\}$ the smallest subspace invariant for J containing x. Since J admits spectral synthesis, it follows that $\mathscr{M}_x = \oplus\operatorname{span}\{J^k(\lambda_n, m_n)x_n : k \geq 0\}$. Since $\operatorname{span}\{J^k(\lambda_n, m_n)x_n : k \geq 0\}$ is a subspace of H_n for each n, x is cyclic for J if and only if each x_n is cyclic for $J(\lambda_n, m_n)$ on $\mathscr{H}_n$.

(iii) $\Leftrightarrow$ (iv). If $\{w_{n,j}\}$ is any collection of complex numbers for which $\sum_{n=1}^{\infty}\sum_{j=1}^{m_n}|w_{n,j}| < \infty$, then $F(z) \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{m_n}(w_{n,j}/(j-1)!)z^{j-1}e^{\lambda_n z}$ is an entire function. Moreover, $F(z) \equiv 0$ if and only if for all nonnegative integers i, $0 = F^{(i)}(0) = \sum_{n=1}^{\infty}\sum_{j=1}^{\min(i+1,m_n)}\binom{i}{j-1}\lambda_n^{i-j+1}w_{n,j}$.

(iii) $\Rightarrow$ (ii). Let $\{x_n\}$ be any collection of vectors for which x_n is a cyclic vector for $J(\lambda_n, m_n)$ on $\mathscr{H}_n$ for each positive integer n. We show that $x \equiv \oplus x_n$ is cyclic for J on $\mathscr{H}$. By means of contradiction, assume not. So there exists a nonzero vector $y \equiv \oplus y_n$ in $\mathscr{H}$ such that for all nonnegative integers i,

$$\begin{aligned} 0 &= \langle J^i x, y\rangle \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{\min(i+1,m_n)}\binom{i}{j-1}\lambda_n^{i-j+1}\sum_{k=1}^{m_n-j+1}\langle x_n, e_{n,k+j-1}\rangle\overline{\langle y_n, e_{n,k}\rangle}. \end{aligned} \tag{1}$$

Define $w_{n,j} \equiv \langle x_n, e_{n,k+j-1}\rangle\overline{\langle y_n, e_{n,k}\rangle}$ for all positive integers n and all j in $\{1, 2, \ldots, m_n\}$. Since x and y are in $\mathscr{H}$, it follows that $\sum_{n=1}^{\infty}\sum_{j=1}^{m_n}|w_{n,j}| < \infty$. By (ii), $0 = w_{n,j}$ for all positive integers n and all j in $\{1, 2, \ldots, m_n\}$. For all positive integers n, x_n is a cyclic vector for $J(\lambda_n, m_n)$ and so $\langle x_n, e_{n,m_n}\rangle$ is nonzero. Letting $j = m_n$ in Eq. (1) yields $\langle y_n, e_{n,1}\rangle = 0$. Induction on $j = m_n, m_n - 1, \ldots, 1$ yields $\langle y_n, e_{n,j}\rangle = 0$ for all n and all j in $\{1, 2, \ldots, m_n\}$. Hence $y = \oplus y_n$ is the zero vector, a contradiction.

(ii) $\Rightarrow$ (iii). Let $\{w_{n,j}\}$ be any collection of complex numbers for which $\sum_{n=1}^{\infty}\sum_{j=1}^{m_n}|w_{n,j}| < \infty$ and $\sum_{n=1}^{\infty}\sum_{j=1}^{\min(i+1,m_n)}\binom{i}{j-1}\lambda_n^{i-j+1}w_{n,j} = 0$ for all nonnegative integers i. Let n be a positive integer. If $w_{n,j} = 0$ for all $j = 1, 2, \ldots, m_n$, define $x_n = 2^{-n}(0, \ldots, 0, 1)$ and $y_n = 0 \in \mathscr{H}_n$. Otherwise, let j_n denote the largest integer in $\{1, 2, \ldots, m_n\}$ for which $w_{n,j}$ is nonzero and let $\gamma_n \equiv \max\{|w_{n,j}|^{1/2} : 1 \leq j \leq m_n\}$. Define $x_n = \gamma_n^{-1}(0, \ldots, 0, w_{n,1}, \ldots, w_{n,j_n})$ and define $y_n = (0, \ldots, 0, \gamma_n, 0, \ldots, 0)$ where the term γ_n occurs in the $(m_n - j_n + 1)$st coordinate. Since $|w_{n,j}|^{1/2} \leq \gamma_n$ and $\sup m_n < \infty$, $x \equiv \oplus x_n$ and $y \equiv \oplus y_n$ are norm convergent. Moreover, the vectors x_n and y_n are chosen so that $w_{n,j} = \sum_{k=1}^{m_n - j+1} \langle x_n, e_{n,k+j-1}\rangle \overline{\langle y_n, e_{n,k}\rangle}$ for all positive integers n and all $j = 1, 2, \ldots, m_n$. Hence for all nonnegative integers i, we have that $\langle J^i x, y\rangle = \sum_{n=1}^{\infty} \sum_{j=1}^{\min(i+1, m_n)} \binom{i}{j-1} \lambda_n^{i-j+1} w_{n,j} = 0$.

For each positive integer n, $\langle x_n, e_n\rangle$ is nonzero. Hence x_n is cyclic for $J(\lambda_n, m_n)$ on $\mathscr{H}_n$ and so, by hypothesis, $x = \oplus x_n$ is cyclic for J on $\mathscr{H}$. Since $\langle J^i x, y\rangle = \sum_{n=1}^{\infty} \sum_{j=1}^{\min(i+1, m_n)} \binom{i}{j-1} \lambda_n^{i-j+1} w_{n,j} = 0$ for all nonnegative integers i and x is cyclic for J, we have that $y = \oplus y_n$ is the zero vector. By definition of y_n, it follows that $0 = \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} |w_{n,j}|$.

The result follows.

3. SUFFICIENT CONDITIONS

In this section, we show that a Jordan operator $J = \oplus J(\lambda_n, m_n)$ admits spectral synthesis whenever for each positive integer i, λ_i is in the unbounded component of $(\overline{\{\lambda_k : k \neq i\}})^c$.

THEOREM 3. *Let $\{\lambda_n\}$ be a bounded sequence of distinct complex numbers, let $\{m_n\}$ be a bounded sequence of positive integers, and let $J = \oplus J(\lambda_n, m_n)$ be a Jordan operator acting on a Hilbert space $\mathscr{H} = \oplus_{n=1}^{\infty} \mathscr{H}_n$. If for each positive integer i, the orthogonal projection $P_{\mathscr{H}_i}: \mathscr{H} \to \mathscr{H}_i$ is in the weakly closed algebra generated by J and the identity, then the Jordan operator $J = \oplus J(\lambda_n, m_n)$ admits spectral synthesis.*

Proof. Let $\mathscr{M}$ be an arbitrary invariant subspace for J and let x be an arbitrary element of $\mathscr{M}$. Since $P_{\mathscr{H}_i}$ is in the weakly closed algebra generated by J and the identity, $\mathscr{M}$ is invariant for J and the identity, and x is in $\mathscr{M}$, it follows that $P_{\mathscr{H}_i} x$ is in $\mathscr{M}$.

For each positive integer n, let d_n denote the largest integer $j = 1, 2, \ldots, m_n$ for which there exists a vector x in $\mathscr{M}$ with $\langle x, e_{n,j}\rangle$ nonzero (if no such vector exists, take d_n to be zero). Clearly $\mathscr{M} \subseteq \oplus \operatorname{span}\{e_{n,j} : j \leq d_n\}$ (if $d_n = 0$, we interpret $\operatorname{span}\{e_{n,j} : j \leq d_n\}$ as $\{0\}$). We show equality. Let n be any positive integer for which d_n is nonzero and let x_n be any vector in $\mathscr{M}$ for which $\langle x_n, e_{n,d_n}\rangle$ is nonzero. By the first part of the proof, $P_{\mathscr{H}_n} x_n$ is in $\mathscr{M}$. Moreover, $\langle P_{\mathscr{H}_n} x_n, e_{n,d_n}\rangle = \langle x_n, e_{n,d_n}\rangle$ is nonzero and so $P_{\mathscr{H}_n} x_n$ is a cyclic vector for $J(\lambda_n, m_n)$ restricted to $\operatorname{span}\{e_{n,j} : j \leq d_n\}$. Hence $\mathscr{M} \supseteq \{p(J) P_{\mathscr{H}_n} x_n : p \text{ is a polynomial}\} = \{p(J(\lambda_n, m_n)) P_{\mathscr{H}_n} x_n : p \text{ is a polynomial}\} = \operatorname{span}\{e_{n,j} : j \leq d_n\}$. So $\mathscr{M} \supseteq \oplus \operatorname{span}\{e_{n,j} : j \leq d_n\}$. That is, $\mathscr{M} = \oplus \operatorname{span}\{e_{n,j} : j \leq d_n\}$ is a spectral subspace of J.

THEOREM 4. *Let $\{\lambda_n\}$ be a bounded sequence of distinct complex numbers, let $\{m_n\}$ be a bounded sequence of positive integers, and let $J = \oplus J(\lambda_n, m_n)$ be a Jordan operator acting on a Hilbert space $\mathscr{H} = \oplus_{n=1}^{\infty}\mathscr{H}_n$. Let i be any positive integer and let $\{p_\alpha\}$ be a set of polynomials. Then $\{p_\alpha(J)\}$ converges in the weak operator topology to the projection operator $P_{\mathscr{H}_i}$ if and only if*

$$\text{(i)}\quad \lim_\alpha p_\alpha(\lambda_k) = \begin{cases} 0 & \text{if } k \neq i, \\ 1 & \text{if } k = i, \end{cases}$$

$$\text{(ii)}\quad \lim_\alpha \hat{p}_\alpha^{(j)}(\lambda_k) = 0 \qquad \text{for all } j, k \geq 1, \text{ and}$$

$$\text{(iii)}\quad \sup_{\alpha, k}|\hat{p}_\alpha^{(j)}(\lambda_k)| < \infty \qquad \text{for all } j \geq 0$$

where

$$\hat{p}_\alpha^{(j)}(\lambda_k) \equiv \begin{cases} 0 & \text{if } j \geq m_k, \\ p_\alpha^{(j)}(\lambda_k) & \text{if } j < m_k. \end{cases}$$

Proof. Suppose that the polynomials $\{p_\alpha\}$ satisfy properties (i), (ii), and (iii). We show that $\{p_\alpha(J)\}$ converges in the weak operator topology to the projection operator $P_{\mathscr{H}i}$. Let $x = \oplus(x_{n,1}, x_{n,2}, \ldots, x_{n,m_n})$ and $y = \oplus(y_{n,1}, y_{n,2}, \ldots, y_{n,m_n})$ denote arbitrary elements of $\mathscr{H} = \oplus\mathscr{H}_n$. We have that

$$\begin{aligned} &\langle p_\alpha(J)x, y\rangle \\ &\quad = \sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\bar{y}_{k,j}\left(\sum_{r=1}^{m_k-j+1}\frac{1}{(r-1)!}p_\alpha^{(r-1)}(\lambda_k)x_{k,r+j-1}\right) \\ &\quad = \sum_{j=1}^{m_i}\bar{y}_{i,j}\left(\sum_{r=1}^{m_i-j+1}\frac{1}{(r-1)!}p_\alpha^{(r-1)}(\lambda_i)x_{i,r+j-1}\right) \\ &\qquad + \sum_{k\neq i}\bar{y}_{k,j}\left(\sum_{r=1}^{m_k-j+1}\frac{1}{(r-1)!}p_\alpha^{(r-1)}(\lambda_k)x_{k,r+j-1}\right). \end{aligned} \tag{2}$$

By (i) and (ii),

$$\lim_\alpha \sum_{j=1}^{m_i}\bar{y}_{i,j}\left(\sum_{r=1}^{m_i-j+1}\frac{1}{(r-1)!}p_\alpha^{(r-1)}(\lambda_i)x_{i,r+j-1}\right) = \langle P_{\mathscr{H}_i}x, y\rangle. \tag{3}$$

By (iii), $M \equiv \sup_{\alpha,k}\{|\hat{p}_\alpha^{(j)}(\lambda_k)| : j = 0, 1, \ldots, m_k - 1\} < \infty$ from which it follows that

$$\sum_{k\neq i}\bar{y}_{k,j}\left(\sum_{r=1}^{m_k-j+1}\frac{1}{(r-1)!}p_\alpha^{(r-1)}(\lambda_k)x_{k,r+j-1}\right) \leq M(\sup m_n)\|x\|\,\|y\| < \infty. \tag{4}$$

By means of contradiction, suppose that $\{\langle p_\alpha(J)x, y\rangle\}$ does not converge to $\langle P_{\mathscr{H}_i}x, y\rangle$. Then there exists a sequence $\{\alpha_n\}$ such that $\{\langle p_{\alpha_n}(J)x, y\rangle\}$ does not converge to $\langle P_{\mathscr{H}_i}x, y\rangle$. Hence by the dominated convergence theorem, Eq. (4), and (iii),

$$\begin{aligned}\lim_{n\to\infty} \sum_{k\neq i} \bar{y}_{k,j}\left(\sum_{r=1}^{m_k-j+1} \frac{1}{(r-1)!} p_{\alpha_n}^{(r-1)}(\lambda_k) x_{k,r+j-1}\right)\\ = \sum_{k\neq i} \bar{y}_{k,j}\left(\sum_{r=1}^{m_k-j+1} \frac{1}{(r-1)!} \lim_{n\to\infty} p_{\alpha_n}^{(r-1)}(\lambda_k) x_{k,r+j-1}\right)\\ = 0,\end{aligned}$$

a contradiction as $\{\langle p_\alpha(J)x, y\rangle\}$ does not converge to $\langle P_{\mathscr{H}_i}x, y\rangle$. Hence

$$\begin{aligned}\lim_{\alpha} \sum_{k\neq i} \bar{y}_{k,j}\left(\sum_{r=1}^{m_k-j+1} \frac{1}{(r-1)!} p_{\alpha}^{(r-1)}(\lambda_k) x_{k,r+j-1}\right)\\ = \sum_{k\neq i} \bar{y}_{k,j}\left(\sum_{r=1}^{m_k-j+1} \frac{1}{(r-1)!} \lim_{\alpha} p_{\alpha}^{(r-1)}(\lambda_k) x_{k,r+j-1}\right)\\ = 0. \qquad (5)\end{aligned}$$

So by Eqs. (2)–(5), $\langle p_\alpha(J)x, y\rangle \to \langle P_{\mathscr{H}_i}x, y\rangle$ for arbitrary elements x and y in $\mathscr{H}$. That is, $\{p_\alpha(J)\}$ converges in the weak operator topology to the projection operator $P_{\mathscr{H}_i}$.

Conversely, suppose that $\{p_\alpha(J)\}$ converges in the weak operator topology to the projection operator $P_{\mathscr{H}_i}$. We show that the polynomials $\{p_\alpha\}$ satisfy properties (i), (ii), and (iii). Let $x = \oplus x_n = \oplus(x_{n,1}, x_{n,2}, \dots, x_{n,m_n})$ and $y = \oplus y_n = \oplus(y_{n,1}, y_{n,2}, \dots, y_{n,m_n})$ denote arbitrary elements of $\mathscr{H} = \oplus\mathscr{H}_n$. Since $\{p_\alpha(J)\}$ converges in the weak operator topology to $P_{\mathscr{H}_i}$, we have that

$$\begin{aligned}\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \bar{y}_{k,j}\left(\sum_{r=1}^{m_k-j+1} \frac{1}{(r-1)!} p_{\alpha}^{(r-1)}(\lambda_k) x_{k,r+j-1}\right)\\ = \langle p_\alpha(J)x, y\rangle\\ \to \langle P_{\mathscr{H}_i}x, y\rangle = \langle x_i, y_i\rangle = \sum_{j=1}^{m_i} x_{i,j}\bar{y}_{i,j}. \qquad (6)\end{aligned}$$

Judicious choices of coordinates for the vectors x and y in Eq. (6) show that the polynomials $\{p_\alpha\}$ satisfy properties (i) and (ii). Hence we need only show that the polynomials $\{p_\alpha\}$ satisfy property (iii). To this end, fix $j \geq 0$

and let $\{w_n\}$ be an arbitrary sequence in l_1. Define $x = \oplus x_k$ where $x_k = (0, \dots, 0, \sqrt{|w_k|}, 0, \dots, 0)$ with the term $\sqrt{|w_k|}$ occuring in the jth coordinate (take $x_k = 0 \in \mathscr{H}_k$ if $j > m_k$). Similarly, define $y_k = ((j-1)! e^{i \arg w_k} \sqrt{|w_k|}, 0, \dots, 0)$. We have that $\langle p_\alpha(J)x, y\rangle = \sum_{k=1}^\infty \hat{p}_\alpha^{(j)}(\lambda_k) w_k$. By hypothesis, $\{\langle p_\alpha(J)x, y\rangle\}$ converges to $\langle P_{\mathscr{H}_i} x, y\rangle$. Hence $\sum_{k=1}^\infty \hat{p}_\alpha^{(j)}(\lambda_k) w_k$ converges for each $\{w_k\}$ in l_1. Moreover,

$$\lim_\alpha \sum_{k=1}^{\infty} \hat{p}_\alpha^{(j)}(\lambda_k) w_k = \lim_\alpha \langle p_\alpha(J)x, y\rangle = \langle P_{\mathscr{H}_i} x, y\rangle. \tag{7}$$

For each $\{w_k\}$ in l_1, we define a functional $L_\alpha\colon l_1 \to \mathbf{C}$ by $L_\alpha(\{w_k\}) = \sum_{k=1}^\infty \hat{p}_\alpha^{(j)}(\lambda_k) w_k$. Since $\sum_{k=1}^\infty \hat{p}_\alpha^{(j)}(\lambda_k) w_k$ converges for all $\{w_k\}$ in l_1, it follows from the Banach–Steinhaus theorem that $\{\hat{p}_\alpha^{(j)}(\lambda_k)\}_{k=1}^\infty$ is in l_∞. Hence L_α is a bounded linear functional on l_1, and, moreover, $\|L_\alpha\|_{l_1^*} = \|\{\hat{p}_\alpha^{(j)}(\lambda_k)\}\|_{l_\infty}$. By Eq. (7), $L_\alpha(\{w_k\}) = \sum_{k=1}^\infty \hat{p}_\alpha^{(j)}(\lambda_k) w_k$ is bounded in α for all $\{w_k\}$ in l_1. Hence the sequence $\{L_\alpha\}$ of functionals on l_1 is pointwise bounded and so uniformly bounded by the principle of uniform boundedness. That is, $\sup_\alpha \|L_\alpha\|_{l_1^*} = \sup_\alpha \|\{\hat{p}_\alpha^{(j)}(\lambda_k)\}\|_{l_\infty}$ is finite for all $j \ge 0$. The result follows.

In general, the weak closure and the weak sequential closure of the set $\{p(J) : p \text{ is a polynomial}\}$ of polynomials in J do not coincide, even if J is a diagonalizable operator $J = \oplus J(\lambda_n, 1)$ (see Wermer [14], the Corollary to Theorem 1 of Sarason [11, p. 511], and the remarks following Lemma 7). In the following theorem, we give sufficient conditions for J to be the weak sequential limit of polynomials in J by applying Mergelyan's theorem on polynomial approximation.

THEOREM 5. *Let $\{\lambda_n\}$ be a bounded sequence of distinct complex numbers, let $\{m_n\}$ be a bounded sequence of positive integers, and let $J = \oplus J(\lambda_n, m_n)$ be a Jordan operator acting on a Hilbert space $\mathscr{H} = \oplus_{n=1}^\infty \mathscr{H}_n$. Suppose that for each positive integer n, λ_n is in the unbounded component of $(\overline{\{\lambda_m : m \ne n\}})^c$. Then there exists a sequence $\{p_{n,i}\}$ of polynomials such that $\{p_{n,i}(J)\}$ converges in the weak operator topology to the projection operator $P_{\mathscr{H}_n}$ for each positive integer n. In particular, $J = \oplus J(\lambda_n, m_n)$ admits spectral synthesis.*

Proof. Let n be a fixed positive integer. By Theorem 4, it suffices to show that there exist polynomials $\{p_{n,i}\}$ such that

(i) $\lim_{i\to\infty} p_{n,i}(\lambda_k) = \begin{cases} 0 & \text{if } k \ne n, \\ 1 & \text{if } k = n, \end{cases}$

(ii) $\lim_{i\to\infty} \hat{p}_{n,i}^{(j)}(\lambda_k) = 0$ for all $j, k \ge 1$, and

(iii) $\sup_{i,k} |\hat{p}_{n,i}^{(j)}(\lambda_k)| < \infty$ for all $j \ge 0$

where

$$\hat{p}_{n,i}^{(j)}(\lambda_k) \equiv \begin{cases} 0 & \text{if } j \geq m_k, \\ p_{n,i}^{(j)}(\lambda_k) & \text{if } j < m_k. \end{cases}$$

We apply Mergelyan's theorem (see Rudin [9, p. 390]). Define $K_n = \overline{\{\lambda_m : m \neq n\}}$. By hypothesis λ_n is not in K_n and so there exists $\epsilon > 0$ such that $\overline{B(\lambda_n, \epsilon)} \cap K_n = \varnothing$ where here $B(\lambda_n, \epsilon)$ denotes the open ball of radius ϵ with center λ_n. Let J_n be any Jordan curve whose interior $\operatorname{int} J_n$ contains K_n and whose exterior $\operatorname{ext} J_n$ contains $\overline{B(\lambda_n, \epsilon)}$. Define $h(z) \equiv 0$ on $(J_n \cup \operatorname{int} J_n)$ and $h(z) \equiv 1$ on $\overline{B(\lambda_n, \epsilon)}$. Since h is continuous on the compact set $C_n = \overline{B(\lambda_n, \epsilon)} \cup \{J_n \cup \operatorname{int} J_n\}$ and analytic on the interior $C_n^0 = B(\lambda_n, \epsilon) \cup \operatorname{int} J_n$ of C_n, by Mergelyan's theorem there exist polynomials $\{\tilde{p}_{n,i}\}$ converging uniformly to h on C_n.

We show that the polynomials $p_{n,i} \equiv \tilde{p}_{n,i}^N$ satisfy properties (i), (ii), and (iii) where here $N \equiv \sup m_n$. It suffices to show that for each $j \geq 0$, $\{p_{n,i}^{(j)}\}$ converges uniformly to $h^{(j)}$ on an open set containing $B(\lambda_n, \epsilon/2) \cup K_n$. We map to the open unit disc $\mathbf{D}$ and apply Cauchy's integral formula. Since $K_n \subseteq \operatorname{int} J_n$ and K_n and J_n are compact, there exists an open set θ_n for which $K_n \subseteq \theta_n \subseteq \overline{\theta_n} \subseteq \operatorname{int} J_n$. So $\overline{\theta}_n \cap J_n = \varnothing$. By the Riemann mapping theorem (see Burckel [1, Theorem 9.7, pp. 299 and 303]), there exists a conformal map ϕ from $\operatorname{int} J_n$ onto $\mathbf{D}$. Moreover, ϕ extends to a continuous map $\phi\colon \overline{(\operatorname{int} J_n)} \to \overline{\mathbf{D}}$ which is one-to-one on $\overline{(\operatorname{int} J_n)}$ (see Burckel [1, Lemma 9.13, p.307]). Since ϕ is one-to-one, it follows that $\overline{\phi(\theta_n)} \subseteq \mathbf{D}$. Hence there exists r in $(0,1)$ for which $\overline{\phi(\theta_n)} \subseteq B(0, r)$. For any $r' \in (r, 1)$, we have that $K_n \subseteq \theta_n \subseteq \phi^{-1}(B(0, r'))$. Since $\{\tilde{p}_{n,i}\}$ converges uniformly to h on $\overline{(\operatorname{int} J_n)}$, $\{p_{n,i}\}$ converges uniformly to $h^N = h$ on $\overline{(\operatorname{int} J_n)}$, and so $\{p_{n,i} \circ \phi^{-1}\}$ converges uniformly to $h \circ \phi^{-1}$ on $B(0, r')$. Hence, by Cauchy's integral formula, $\frac{d}{dz}(p_{n,i} \circ \phi^{-1}) = (p'_{n,i} \circ \phi^{-1}) \cdot (\phi^{-1})'$ converges uniformly to $\frac{d}{dz}(h \circ \phi^{-1}) = (h' \circ \phi^{-1}) \cdot (\phi^{-1})'$ on $B(0, r')$ for all $r' \in (r, 1)$.

Since ϕ^{-1} is a conformal map from $\mathbf{D}$ to $\overline{(\operatorname{int} J_n)}$ and continuous on $B(0, r')$ for all $r' \in (r, 1)$, we have that $\inf\{|(\phi^{-1})'| : z \in \overline{B(0, r')}\} > 0$. Hence $\{p'_{n,i} \circ \phi^{-1}\}$ converges uniformly to $h' \circ \phi^{-1}$ on $B(0, r')$ for all $r' \in (r, 1)$. So by Cauchy's integral formula, $\frac{d}{dz}(p'_{n,i} \circ \phi^{-1}) = (p''_{n,i} \circ \phi^{-1}) \cdot (\phi^{-1})'$ converges uniformly to $h' \circ \phi^{-1} = (h'') \circ (\phi^{-1})'$ on $\phi^{-1}(\overline{B(0, r')})$ for all $r' \in (r, 1)$. Since $\inf\{|(\phi^{-1})'| : z \in \overline{B(0, r')}\} > 0$ for all $r' \in (r, 1)$, we have that $\{p''_{n,i}\}$ converges uniformly to h'' on $\phi^{-1}(\overline{B(0, r')})$ for all $r' \in (r, 1)$. Induction yields that $\{p_{n,i}^{(j)}\}$ converges uniformly to $h^{(j)}$ on $\phi^{-1}(\overline{B(0, r')})$ for all $r' \in (r, 1)$ and hence on $\phi^{-1}(\overline{B(0, r')}) \supseteq \overline{\theta}_n$.

Similarly, $\{p_{n,i}^{(j)}\}$ converges uniformly to $h^{(j)}$ on $B(\lambda_n, 3\epsilon/4)$. Hence $\{p_{n,i}^{(j)}\}$ converges uniformly to $h^{(j)}$ on an open set containing $B(\lambda_n, \epsilon/2) \cup K_n$ for each $j \geq 0$. The result follows.

4. A NECESSARY CONDITION

In this section we give a necessary condition for a Jordan operator to admit spectral synthesis. We begin with the following notation.

Let $\{\lambda_n\}$ be a bounded sequence of distinct complex numbers and let $\{m_n\}$ be a bounded sequence of positive integers. Define $R \equiv \sup m_n - 1$. We denote by $\mathscr{A}$ the subalgebra

$$\mathscr{A} \equiv \left\{ f \in H^\infty : \left\{ \hat{f}^{(i)}(\lambda_n) \right\} \in l_\infty \text{ for all } i = 0, 1, 2, \dots, R \right\}$$

of H^∞ where for each positive integer n,

$$\hat{f}^{(i)}(\lambda_n) \equiv \begin{cases} 0 & \text{if } i \geq m_n \\ f^{(i)}(\lambda_n) & \text{if } i < m_n \end{cases}$$

and norm $\mathscr{A}$ by defining $\|f\|_{\mathscr{A}} \equiv \|f\|_{H^\infty} + \sum_{i=1}^{R} \|\{\hat{f}^{(i)}(\lambda_n)\}\|_{l_\infty}$. We denote by $\mathscr{B}$ the subalgebra

$$\mathscr{B} \equiv \big\{ (\{a_{n,0}\}, \{a_{n,1}\}, \dots, \{a_{n,R}\} : \{a_{n,i}\} \in l_\infty \text{ for all } i = 0, 1, 2, \dots, R \text{ and }$$
$$a_{n,i} = 0 \text{ for all } i > m_n \big\}$$

of $\oplus_{i=0}^{R} l_\infty$ and norm $\mathscr{B}$ by defining $\|(\{a_{n,0}\}, \dots, \{a_{n,R}\})\|_{\mathscr{B}} \equiv \sum_{i=0}^{R} \|\{a_{n,i}\}\|_{l_\infty}$.

LEMMA 1. *If the bounded linear operator $T: \mathscr{A} \to \mathscr{B}$ defined by*

$$T : f \to \left(\{f(\lambda_n)\}, \{\hat{f}^{(1)}(\lambda_n)\}, \dots, \{\hat{f}^{(R)}(\lambda_n)\} \right)$$

is an isometry, then the Jordan operator $J = \oplus(\lambda_n, m_n)$ fails to admit spectral synthesis.

Proof. We first show that $T: \mathscr{A} \to \mathscr{B}$ is not onto. By means of contradiction, suppose that T maps $\mathscr{A}$ onto $\mathscr{B}$. So for each sequence $\{a_n\}$ in l_∞ there exists a function f in $\mathscr{A} \subseteq H^\infty$ for which $T(f) = (\{a_n\}, \{a_{n,1}\}, \{a_{n,2}\}, \dots, \{a_{n,R}\})$ where $a_{n,i} \equiv 0$ for all $n \geq 1$ and all $i = 1, 2, \dots, R$. Hence $f(\lambda_n) = a_n$ for all positive integers n, and so $\{\lambda_n\}$ is an interpolating sequence for H^∞. Let B denote any interpolating Blaschke product having simple zeros $\{\lambda_n\}$. Then $B^R(z)$ is in $\mathscr{A}$ and since T is an isometry, we have that $1 = \|B^R\|_{\mathscr{A}} = \|T(B^R)\|_{\mathscr{B}} = \|(\{0\}, \dots, \{0\})\|_{\mathscr{B}} = 0$, a contradiction. Hence $T: \mathscr{A} \to \mathscr{B}$ is not onto.

Let $\mathscr{C}$ denote the subalgebra

$$\mathscr{C} \equiv \big\{ (\{a_{n,0}\}, \{a_{n,1}\}, \dots, \{a_{n,R}\}) : \{a_{n,i}\} \in l_1 \text{ for all } i = 0, 1, 2, \dots, R \text{ and }$$
$$a_{n,i} = 0 \text{ for all } i > m_n \big\}$$

of $\oplus_{i=0}^{R} l_\infty$ and norm $\mathscr{C}$ by defining $\|(\{a_{n,0}\},\dots,\{a_{n,R}\})\|_{\mathscr{C}} \equiv \sum_{i=0}^{R}\|\{a_{n,i}\}\|_{l_1}$. Then $\mathscr{B} = \mathscr{C}^*$. We show that the range of T is weak-star closed in $\mathscr{B}$. By the Krein–Smulian theorem (see Conway [3, Corollary 12.7, p. 165]), it suffices to show that the range of T is weak-star sequentially closed. To this end, let $\{f_k\}$ be any sequence of functions in $\mathscr{A}$ for which $\{T(f_k)\}$ converges weak-star to some vector $(\{a_{n,0}\},\{a_{n,1}\},\dots,\{a_{n,R}\})$ in $\mathscr{B}$. Since $\{T(f_k)\}$ converges weak-star, $\{\|T(f_k)\|_{\mathscr{B}}\}$ is bounded. But T is an isometry, and so $\{\|f_k\|_{\mathscr{A}}\}$ is bounded. Since $\|f\|_{H^\infty} \le \|f\|_{\mathscr{A}}$, $\{\|f_k\|_{H^\infty}\}$ is bounded and so by Montel's theorem, there exists a subsequence $\{f_{k_r}\}$ of $\{f_k\}$ which converges uniformly to some function g on every compact subset of the unit disc. Since $\{T(f_k)\}$ converges weak-star to $(\{a_{n,0}\},\{a_{n,1}\},\dots,\{a_{n,R}\})$ in $\mathscr{B}$, it follows that $a_{n,i} = \lim_{r\to\infty} \hat{f}_{k_r}^{(i)}(\lambda_n) = \hat{g}^{(i)}(\lambda_n)$ for all positive integers n and i. Hence $(\{a_{n,0}\},\{a_{n,1}\},\dots,\{a_{n,R}\}) = Tg$ is in the range of T and so the range of T is weak-star closed.

Since $T\colon \mathscr{A} \to \mathscr{B}$ is not onto and the range of T is weak-star closed in $\mathscr{B}$, by the Hahn–Banach theorem there exists a nonzero weak-star continuous functional on $\mathscr{B}$ annihilating the range of T. That is, there exists a nonzero vector $(\{u_{n,0}\},\{u_{n,1}\},\dots,\{u_{n,R}\})$ in $\mathscr{C}$ such that

$$0 = \sum_{n=1}^{\infty}\sum_{k=0}^{R} u_{n,k}\hat{f}^{(k)}(\lambda_n) \tag{8}$$

for all f in $\mathscr{A}$. Define $w_{n,j} \equiv u_{n,j-1}(j-1)!$ for all positive integers n and all $j = 1,2,\dots,R$. Hence

$$\sum_{n=1}^{\infty}\sum_{j=1}^{m_n}|w_{n,j}| = \sum_{n=1}^{\infty}\sum_{k=0}^{R}|u_{n,k}|k! \le R!\|(\{u_{n,0}\},\dots,\{u_{n,R}\})\|_{\mathscr{C}} < \infty.$$

For each nonnegative integer i, the function $f_i(z) \equiv z^i$ is in $\mathscr{A}$. Moreover,

$$\hat{f}_i^{(k)}(\lambda_n) = \begin{cases} \dfrac{i!}{(i-k)!}\lambda_n^{i-k} & \text{if } 0 \le i-k \text{ and } k < m_n \\ f^{(i)}(\lambda_n) & \text{otherwise.} \end{cases}$$

Hence for each nonnegative integer i, we have by Eq. (8) that

$$0 = \sum_{n=1}^{\infty}\sum_{k=0}^{R} u_{n,k}\hat{f}_i^{(k)}(\lambda_n) = \sum_{n=1}^{\infty}\sum_{j=1}^{\min(i+1,m_n)} w_{n,j}\binom{i}{j-1}\lambda_n^{i-j+1},$$

and so J fails to admit spectral synthesis by Theorem 2.

It is not known if the converse of Lemma 1 is true.

5. SUNDBERG'S EXAMPLE

In this section, we outline an example due to Sundberg [13] of a bounded sequence $\{\lambda_n\}$ of distinct complex numbers for which the diagonalizable operator $D = \oplus J(\lambda_n, 1)$ admits spectral synthesis but the corresponding Jordan operator $J = \oplus J(\lambda_n, 2)$ consisting of two-by-two Jordan cells does not.

LEMMA 2. *Let $n \geq 2$ and d in $[1/2, 1]$ be given, and let k and δ be positive numbers for which $\ln \delta \leq \ln(2n) - \ln 2 \cdot \ln d\{\ln n + \ln(2n)\}$ and $k \geq 16\sqrt{2}\,n/\delta$. Define*

$$S_k = \{(a + ib)/k : a, b \in \mathbf{Z};\ |(a + ib)/k| \leq 1/2\}.$$

If f is any function in H^∞ for which $\|f\|_\infty \leq n$ and $|f'(z)| \leq \delta$ for all $z \in S_k$, then $|f(z) - f(0)| \leq 1/n$ for $|z| \leq d$.

Proof. By Cauchy's integral formula, we have that $|f''(z)| \leq 16n$ for $|z| \leq 1/2$. If $|z| \leq 1/2$, then there exists a point a in S_k such that $|z - a| \leq 1/(\sqrt{2}\,k)$. It follows from Cauchy's integral formula that $|f'(z)| < 2\delta$ for $|z| \leq 1/2$, and so $|f(z) - f(0)| \leq \delta$ for $|z| \leq 1/2$.

The function $g(z) \equiv \ln(2n) + \ln(2n/\delta)\ln|z|/\ln 2$ is harmonic on $\{z : 1/2 < |z| < 1\}$. Since g is increasing in $|z|$ for $1/2 \leq |z| \leq d$, we have that $g(z) \leq -\ln n$ for $1/2 \leq |z| \leq d$. Since $\ln|f(z) - f(0)|$ is subharmonic on $\{z : 1/2 < |z| < 1\}$, $\ln|f(z) - f(0)| \leq \ln \delta = g(z)$ for $|z| = 1/2$, and $\ln|f(z) - f(0)| \leq \ln(2n) = g(z)$ for $|z| = 1$, we have that $\ln|f(z) - f(0)| \leq g(z) \leq -\ln n$ for $1/2 \leq |z| \leq d$. Hence $|f(z) - f(0)| \leq 1/n$ for $1/2 \leq |z| \leq d$.

Construction of the Points $\{\lambda_n\}$

For each $n = 2, 3, 4, \ldots,$ define $d_n = 2^{n+1}\pi/\sqrt{1 + 4^{n+1}\pi^2}$ and choose any pair of positive numbers δ_n and k_n satisfying the hypotheses of Lemma 2. Since $\gamma_n \equiv \{(1 + r^4)(1 - d_n^2) - (1 - r^2)^2\}/\{2r^2(1 - d_n^2)\}$ tends to one as r tends to one, there exist numbers r_n in $(0, 1)$ for which $8n(1 - r_n) < \delta_n$ and

$$m_n \equiv 2\pi/\cos^{-1}\Big\{\big((1 + r_n^4)(1 - d_n^2) - (1 - r_n^2)^2\big)/\big(2r_n^2(1 - d_n^2)\big)\Big\} \tag{9}$$

is an integer. Define $z_{j,k} = r_j e^{2\pi i k/m_j}$ for $k = 0, 1, \ldots, m_j - 1$ and define S_{k_n} as in Lemma 2. Let $\{\lambda_n\}$ be an enumeration of $E \equiv \bigcup_{n=2}^{\infty} \bigcup_{j=0}^{m_j - 1} E_{n,j}$ where $E_{n,j} \equiv \{(z + z_{n,j})/(1 + \bar{z}_{n,j} z) : z \in S_{k_n}\}$.

LEMMA 3. *If f is any function in H^∞ for which $\|f\|_\infty \leq n$, and $|f(z)| \leq 1$ and $|f'(z)| \leq n$ for all z in $\bigcup_{j=0}^{m_n - 1} E_{n,j}$, then $|f(z)| \leq 1 + 1/n$ for $|z| = r_n$.*

Proof. We first show that for all $n \geq 2$ and all $j = 0, 1, \ldots, m_n - 1$, the hyperbolic distance $\rho_n \equiv |(z_{n,j} - z_{n,j+1})/(1 - \bar{z}_{n,j} z_{n,j+1})|$ between $z_{n,j}$ and $z_{n,j+1}$ is d_n. One readily checks that $\cos(2\pi/m_n) = \{(1 + r_n^4)(1 - \rho_n^2) - (1 - r_n^2)^2\}/\{2r_n^2(1 - \rho_n^2)\}$. Moreover, by Eq. (9), $\cos(2\pi/m_n) = \{(1 + r_n^4)(1 - d_n^2) - (1 - r_n^2)^2\}/\{2r_n^2(1 - d_n^2)\}$. Hence $\{(1 + r_n^4)(1 - \rho_n^2) - (1 - r_n^2)^2\}(1 - d_n^2) = \{(1 + r_n^4)(1 - d_n^2) - (1 - r_n^2)^2\}(1 - \rho_n^2)$ from which it follows that $d_n = \rho_n$ as asserted.

Let f be any function in H^∞ for which $\|f\|_\infty \leq n$, and $|f(z)| \leq 1$ and $|f'(z)| \leq n$ for all z in $\bigcup_{j=0}^{m_n - 1} E_{n,j}$. Define $g(z) \equiv f((z + z_{n,j})/(1 + \bar{z}_{n,j} z))$ so that $f(z) = g((z - z_{n,j})/(1 - \bar{z}_{n,j} z))$. If z is in S_{k_n}, then $(z + z_{n,j})/(1 + \bar{z}_{n,j} z)$ is in $E_{n,j}$ and so

$$|g'(z)| = \left| f'\left(\frac{z + z_{n,j}}{1 + \bar{z}_{n,j} z} \right) \right| \cdot \frac{1 - |z_{n,j}|^2}{|1 + \bar{z}_{n,j} z|^2} \leq n \frac{1 - r_n^2}{(1 - |z|)^2} \leq \delta_n$$

by choice of r_n. So by Lemma 2, $|g(z) - g(0)| < 1/n$ for $|z| \leq d_n$. Hence for all z between $z_{n,j}$ and $z_{n,j+1}$ with $|z| = d_n$, we have that

$$\begin{aligned} |f(z)| &\leq |f(z_{n,j})| + |f(z) - f(z_{n,j})| \\ &\leq 1 + \left| g\left(\frac{z - z_{n,j}}{1 - \bar{z}_{n,j} z} \right) - g(0) \right| \leq 1 + 1/n \end{aligned}$$

since $|(z - z_{n,j})/(1 - \bar{z}_{n,j} z)| \leq \rho_n = d_n$. Hence $|f(z)| \leq 1 + 1/n$ for $|z| = r_n$.

LEMMA 4. *If f is any function in H^∞ and $|f'(z)|$ is bounded on $E = \cup \cup E_{n,j}$, then $\|f\|_\infty = \sup_E |f(z)|$. In particular, the Jordan operator $J = \oplus J(\lambda_n, 2)$ having eigenvalues $\{\lambda_n\} = E$ fails to admit spectral synthesis.*

Proof. Let f be any function in H^∞ for which $\sup_E |f'| < \infty$. The function $g(z) \equiv f(z)/\sup_E |f|$ is in H^∞ with $\sup_E |g| = 1$. For any $n > \max(\|f\|_\infty / \sup_E |f|; \sup_E |f'| / \sup_E |f|)$, we have that $\|g\|_\infty < n$ and $\sup_E |g'| < n$. Hence by the preceding lemma, $|g(z)| \leq 1 + 1/n$ for $|z| = r_n$. By the maximum modulus principle, $|g(z)| \leq 1 + 1/n$ for $|z| \leq r_n$. Letting n tend to ∞ yields $|g| \leq 1$ on the unit disc. That is, $\|f\|_\infty \leq \sup_E |f|$ and so J fails to admit spectral synthesis by Lemma 1.

LEMMA 5. *The diagonalizable Jordan operator $D = \oplus J(\lambda_n, 1)$ having eigenvalues $\{\lambda_n\} = E$ admits spectral synthesis.*

Proof. One readily checks that $\Sigma\Sigma(1 - |z_{n,j}|^2) = \Sigma m_n(1 - r_n^2) < \infty$. That is, the points $\{z_{n,j}\}$ form a Blaschke sequence and so the nontangential cluster set of $\{z_{n,j}\}$ has measure zero on the unit circle (see Brown et al. [2, Remark 2, p. 170]). Since the points comprising $E_{n,j}$ are hyperbolically within $1/2$ of $z_{n,j}$, it follows that the nontangential cluster set of E has measure zero on the unit circle. Hence the diagonalizable operator $D = \oplus J(\lambda_n, 1)$ admits spectral synthesis by [2, Theorem 3, p. 167].

6. ALGEBRAS ASSOCIATED WITH JORDAN OPERATORS

In this section, we study some algebras of operators associated with a Jordan operator J. In particular, we identify the commutant and double commutant of J and the weakly closed C^*-algebra $\mathscr{W}^*(J) \equiv \overline{\{p(J, J^*) : p \text{ is a polynomial}\}}$ generated by J and the identity. We also give sufficient conditions for the weakly closed algebra $\mathscr{W}(J) \equiv \overline{\{p(J) : p \text{ is a polynomial}\}}$ generated by J and the identity to coincide with the commutant of J.

The proof of the following result identifying the commutant of a Jordan operator, being routine, is omitted.

LEMMA 6. *Let $\{\lambda_n\}$ be a bounded sequence of distinct complex numbers, let $\{m_n\}$ be a bounded sequence of positive integers, and let $J = \oplus J(\lambda_n, m_n)$ be a Jordan operator acting on a Hilbert space $\mathscr{H} = \oplus_{n=1}^{\infty} \mathscr{H}_n = \oplus_{n=1}^{\infty} \operatorname{span}\{e_{n,i} : 1 \le i \le m_n\}$. A bounded linear operator $T\colon \mathscr{H} \to \mathscr{H}$ on $\mathscr{H}$ commutes with J if and only if $T = \oplus T_n$ where for each positive integer n, the matrix representation for $T_n \equiv T\,|\mathscr{H}_n$ with respect to $\{e_{n,i} : 1 \le i \le m_n\}$ is upper triangular and constant on diagonals.*

As a consequence of Lemma 6, we have that the double commutant $\{J\}''$ of J consists of those operators $S \equiv \oplus S_n$ on $\mathscr{H}$ for which the matrix representation for each $S_n = S|_{\mathscr{H}_n}$ with respect to $\{e_{n,i} : 1 \le i \le m_n\}$ is upper triangular and constant on diagonals. Also as a consequence of Lemma 6, we have that the commutant of the adjoint $J^* = \oplus J^*(\lambda_n, m_n)$ of the Jordan operator J consists of those operators $S \equiv \oplus S_n$ on $\mathscr{H}$ for which the matrix representation for each $S_n \equiv S|_{\mathscr{H}_n}$ with respect to $\{e_{n,i} : 1 \le i \le m_n\}$ is lower triangular and constant on diagonals. Hence a bounded linear operator T_n on $\mathscr{H}_n$ commutes with both $J(\lambda_n, m_n)$ and $J^*(\lambda_n, m_n)$ if and only if T_n is a multiple $\alpha_n I$ of the identity I on $\mathscr{H}_n$. By Lemma 6, the double commutant $\{J, J^*\}''$ of $\{J, J^*\}$ is $\{J, J^*\}'' = \{\oplus \alpha_n I : \alpha_n \in \mathbf{C}\}' = \{\oplus T_n : T_n$ is a bounded linear operator on $\mathscr{H}_n\}$. Hence by the double commutant theorem (see Radjavi and Rosenthal [8, Theorem 7.5, p. 119]), the weakly closed C^*-algebra $\mathscr{W}^*(J)$ generated by J and the identity is

$$\mathscr{W}^*(J) = \{J, J^*\}'' = \{\oplus T_n : T_n \text{ is a bounded linear operator on } \mathscr{H}_n\}.$$

A related problem is to identify the weakly closed algebra $\mathscr{W}(J)$ generated by the Jordan operator J and the identity. Certainly, $\mathscr{W}(J) \subseteq \{J\}'$.

Since each subspace $\mathscr{H}_i$ reduces J, each orthogonal projection $P_{\mathscr{H}_i}: \mathscr{H} \to \mathscr{H}_i$ commutes with J. The converse is also true.

LEMMA 7. *Let $\{\lambda_n\}$ be a bounded sequence of distinct complex numbers, let $\{m_n\}$ be a bounded sequence of positive integers, and let $J = \oplus J(\lambda_n, m_n)$ be a Jordan operator acting on a Hilbert space $\mathscr{H} = \oplus_{n=1}^{\infty} \mathscr{H}_n$. Then $\{J\}' = \mathscr{W}\{J\}$ if and only if for each positive integer i, the orthogonal projection $P_{\mathscr{H}_i}: \mathscr{H} \to \mathscr{H}_i$ is in $\mathscr{W}(J)$.*

Proof. Suppose that $\{J\}' = \mathscr{W}(J)$. For each positive integer i, the subspace $\mathscr{H}_i$ reduces J and so $P_{\mathscr{H}_i}$ is in $\{J\}' = \mathscr{W}(J)$.

Conversely, if $P_{\mathscr{H}_i}$ is in $\{J\}' = \mathscr{W}(J)$ for each positive integer i, then $J(\lambda_i, m_i) = JP_{\mathscr{H}_i}$ is in $\mathscr{W}(J)$ for each positive integer i. Hence $\{J(\lambda_i, m_i)\}' = \{p(J(\lambda_i, m_i)) : p \text{ is a polynomial}\} \subseteq \mathscr{W}(J)$ for each positive integer i and so $\{J\}' = \{\oplus T_i : T_i \in \{J(\lambda_i, m_i)\}'\} \subseteq \mathscr{W}(J)$.

By Theorem 3 and Lemma 7, a necessary condition for $\{J\}' = \mathscr{W}(J)$ is that J admit spectral synthesis. The converse is true for any diagonalizable Jordan operator $D \equiv \oplus J(\lambda_n, 1)$. (By the remarks following Lemma 6, $\{D\}' = \mathscr{W}^*(D)$, and by Sarason [11, corollary to Theorem 1, p. 511], the weakly closed algebra generated by a normal operator is a star-algebra if and only if every invariant subspace for the normal operator is reducing. Hence $\mathscr{W}(D) = \{D\}'$ if and only if D admits spectral synthesis.) So, in general, $\mathscr{W}(J) \neq \{J\}'$. Sufficient conditions for $\mathscr{W}(J) = \{J\}'$ are given in Theorem 5. An open question is under what additional conditions on $\{\lambda_n\}$ and $\{m_n\}$, if any, does the spectral synthesis of J imply $\{J\}' = \mathscr{W}(J)$?

REFERENCES

1. R. B. Burckel, "An Introduction to Classical Complex Analysis: Volume 1," Academic Press, New York, 1979.
2. L. Brown, A. Shields, and K. Zeller, On absolutely convergent exponential sums, *Trans. Amer. Math. Soc.* **96** (1960), 162–183.
3. J. B. Conway, "A Course in Functional Analysis," Graduate Texts in Mathematics, No. 96, Springer-Verlag, New York, 1985.
4. J. P. Lesko and S. M. Seubert, Cyclicity results for Jordan and compressed Toeplitz operators, *Integral Equations Operator Theory* **31** (1998), 338–352.
5. A. S. Markus, The problem of spectral synthesis for operators with point spectrum, *Math. USSR Izv.* **4** (1970), 670–696.
6. N. K. Nikolskii, The present state of the analysis-synthesis problem. I, *in* "Fifteen Papers on Functional Analysis," Amer. Math. Soc. Trans., Ser. 2, Vol. 124, pp. 97–130, Am. Math. Soc., Providence, 1984.
7. N. K. Nikolskii, "Treatise on the Shift Operator," A Series of Comprehensive Studies in Mathematics, No. 273, Springer-Verlag, Berlin/Heidelberg, 1986.
8. H. Radjavi and P. Rosenthal, "Invariant Subspaces," Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77, Springer-Verlag, Berlin/Heidelberg, 1973.

9. W. Rudin, "Real and Complex Analysis," 3rd ed., McGraw-Hill, New York, 1987.
10. D. Sarason, Weak-star density of polynomials, *J. Reine Angew. Math.* **252** (1972), 1–15.
11. D. Sarason, Invariant subspaces and unstarred operators algebras, *Pacific J. Math.* **17** (1966), 511–517.
12. J. E. Scroggs, Invariant subspaces of normal operators, *Duke Math. J.* **26** (1959), 95–112.
13. C. Sundberg, private communication, University of Tennessee, 1995.
14. J. Wermer, On invariant subspaces of normal operators, *Proc. Amer. Math. Soc.* **3** (1952), 270–277.