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The group of isometries of a Banach space and duality

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Dedicated to my wife Mariola

Abstract

We construct an example of a real Banach space whose group of surjective isometries has no uniformly continuous one-parameter semigroups, but the group of surjective isometries of its dual contains infinitely many of them. Other examples concerning numerical index, hermitian operators and dissipative operators are also shown.

Keywords: Isometries; Duality; Numerical range; Daugavet equation; Numerical index; Hermitian operators; Dissipative operators

1. Introduction

Given a real or complex Banach space X, its dual space is denoted by X^* and the Banach algebra of all bounded linear operators on X by $L(X)$. If $T \in L(X)$, $T^* \in L(X^*)$ denotes the adjoint operator of T.

To understand the geometry of a Banach space, it is very useful to know the structure of its *surjective isometries*, i.e. surjective linear applications which preserve the norm. We refer the reader to the recent books by R. Fleming and J. Jamison [13,14] and references therein for background.

In this paper, we would like to investigate the relationship between the group $\operatorname{Iso}(X)$ of all surjective isometries on a Banach space X and the one of its dual. It is well known that the map $T \mapsto (T^*)^{-1}$ is a group monomorphism from $\operatorname{Iso}(X)$ into $\operatorname{Iso}(X^*)$. There are situations in which

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this application is an isomorphism. Namely, when X is reflexive (obvious) and, more generally, when X is M-embedded [16, Proposition III.2.2] and, even more generally, when there exists a unique norm-one projection $\pi : X^{***} \to X^*$ with w^*-closed kernel [15, Proposition VII.1]. Other condition to get that all the surjective isometries of X^* are w^*-continuous is to assure that X has a shrinking 1-unconditional basis [26]. On the other hand, the above map is not always surjective, i.e. it is possible to find surjective isometries on the dual of a Banach space which are not w^*-continuous (see [26, p. 184] for an easy example on $C[0,1]$).

Therefore, in some sense, the group $\mathrm{Iso}(X^*)$ is bigger than $\mathrm{Iso}(X)$. Our aim in this paper is to show that this phenomenon could be even stronger. We construct a Banach space X such that the geometry of $\mathrm{Iso}(X)$ around the identity is trivial (the tangent space to $\mathrm{Iso}(X)$ at Id is zero), while the geometry of $\mathrm{Iso}(X^*)$ around the identity is as rich as the one of a Hilbert space (the tangent space to $\mathrm{Iso}(X^*)$ at Id is infinite-dimensional). By the *tangent space* of the group of surjective isometries on a Banach space Z at Id we mean the set

$$\mathcal{T}\big(\mathrm{Iso}(Z),\mathrm{Id}\big) = \big\{f'(0)\colon\ f:[-1,1]\longrightarrow \mathrm{Iso}(Z),\ f(0)=\mathrm{Id},\ f \text{ differentiable at } 0\big\}.$$

Equivalently (see Proposition 2.1), $\mathcal{T}(\mathrm{Iso}(Z),\mathrm{Id})$ is the set of the generators of uniformly continuous one-parameter semigroup of isometries. By a *uniformly continuous one-parameter semigroup of surjective isometries* on a Banach space Z we mean a continuous function $\Phi : \mathbb{R}_0^+ \to L(Z)$ valued in $\mathrm{Iso}(Z)$ such that $\Phi(t+s) = \Phi(t)\Phi(s)$ for every $s,t \in \mathbb{R}$; equivalently (see [17, Chapter IX] for instance), a group of the form

$$\big\{\exp(\rho T)\colon\ \rho \in \mathbb{R}\big\}$$

for some $T \in L(Z)$ which is contained in $\mathrm{Iso}(Z)$, where $\exp(\cdot)$ denotes the exponential function, i.e. $\exp(T) = \sum_{n=0}^{\infty} \frac{T^n}{n!} \in L(Z)$.

The main aim of this paper is to make a construction which allows us, among others, to present the following result (see Example 4.1).

> *There exists a real Banach space X such that* $\mathrm{Iso}(X)$ *does not contain any non-trivial uniformly continuous one-parameter subgroup, while* $\mathrm{Iso}(X^*)$ *contains infinitely many different uniformly continuous one-parameter subgroups. Equivalently,* $\mathcal{T}(\mathrm{Iso}(X),\mathrm{Id}) = \{0\}$ *but* $\mathcal{T}(\mathrm{Iso}(X^*),\mathrm{Id})$ *is infinite-dimensional.*

The paper is divided into four sections, including this introduction. The second one is devoted to explain the main tool we are using, the numerical range of operators on Banach spaces and its relationship with isometries. In the third section, for every separable Banach space E, we construct a C-rich subspace $X(E)$ of $C[0,1]$ (see Definition 3.1) such that E^* is (isometrically isomorphic to) an L-summand of $X(E)^*$. These spaces inherit some properties from $C[0,1]$ while their dual spaces share properties with the particular spaces E. This allows us to present, in Section 4, the aforementioned example and other ones concerning the numerical index, hermitian operators, and dissipative operators.

We finish the introduction with some notation. Given a real or complex Banach space X, we write B_X for the closed unit ball and S_X for the unit sphere of X. We write $\mathbb{T}$ to denote the set of all modulus-one scalars, i.e. $\mathbb{T} = \{-1,1\}$ in the real case and $\mathbb{T} = \{e^{i\theta}\colon\ \theta \in \mathbb{R}\}$ in the complex case. A closed subspace Z of X is an *L-summand* if $X = Z \oplus_1 W$ for some closed subspace W of X, where $\oplus_1$ denotes the ℓ_1-sum. A closed subspace Y of a Banach space X is said to be an *M-ideal* of X if the annihilator $Y^{\perp}$ of Y is an L-summand of X^*. We refer the reader to the monograph by P. Harmand, D. Werner and W. Werner [16] for background on L-summands and M-ideals.

2. The tool: numerical range and isometries

The main tool we are using in the paper is the relationship between isometries and the numerical range of operators on Banach spaces, a concept independently introduced by F. Bauer [5] and G. Lumer [27] in the sixties to extend the classical field of values of matrices (O. Toeplitz, 1918 [34]). We refer the reader to the monographs by F. Bonsall and J. Duncan [7,8] from the seventies for background and more information. Let X be a real or complex Banach space. The *numerical range* of an operator $T \in L(X)$ is the subset $V(T)$ of the scalar field defined by

$$V(T) = \big\{x^*(Tx):\ x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\big\}.$$

This is Bauer's definition, while Lumer's one depends upon the election of a compatible semi-inner product on the space (a notion also introduced by Lumer [27]). But concerning our applications, both of them are equivalent in the sense that they have the same closed convex hull. The *numerical radius* is the seminorm defined by

$$v(T) = \sup\big\{|\lambda|:\ \lambda \in V(T)\big\}$$

for every $T \in L(X)$. It is clear that v is continuous; actually, $v(T) \leqslant \|T\|$ for every $T \in L(X)$. We say that $T \in L(X)$ is *skew-hermitian* if $\operatorname{Re} V(T) = \{0\}$; we write $\mathcal{A}(X)$ for the closed subspace of $L(X)$ consisting of all skew-hermitian operators on X, which is called the *Lie algebra* of X by H. Rosenthal [31]. In the real case, $T \in \mathcal{A}(X)$ if and only if $v(T) = 0$.

Let us give a clarifying example. If H is a n-dimensional Hilbert space, it is easy to check that $\mathcal{A}(H)$ is the space of skew-symmetric operators on H (i.e. $T^* = -T$ in the Hilbert space sense), so it identifies with the space of skew-symmetric matrices. It is a classical result from the theory of linear algebra that a $n \times n$ matrix A is skew-symmetric if and only if $\exp(\rho A)$ is an orthogonal matrix for every $\rho \in \mathbb{R}$ (see [4, Corollary 8.5.10] for instance). The same is true for an infinite-dimensional Hilbert space by just replacing orthogonal matrices by unitary operators (i.e. surjective isometries).

The above fact extends to general Banach spaces. Indeed, for an arbitrary Banach space X and an operator $T \in L(X)$, by making use of the *exponential formula* [7, Theorem 3.4]

$$\sup \operatorname{Re} V(T) = \lim_{\beta \downarrow 0} \frac{\|\mathrm{Id} + \beta T\| - 1}{\beta} = \sup_{\alpha > 0} \frac{\log\|\exp(\alpha T)\|}{\alpha},$$

the following known result is easy to prove.

Proposition 2.1. *(See [31, Theorem 1.4].) Let X be a real or complex Banach space and $T \in L(X)$. Then, the following are equivalent.*

(i) *T is skew-hermitian.*
(ii) *$\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.*

(iii) $\{\exp(\rho T)\colon \rho \in \mathbb{R}\} \subset \mathrm{Iso}(X)$, *i.e. T is the generator of a uniformly continuous one-parameter subgroup of isometries.*
(iv) $T \in \mathcal{T}(\mathrm{Iso}(X), \mathrm{Id})$.

Therefore, $\mathcal{A}(X)$ coincides with $\mathcal{T}(\mathrm{Iso}(X), \mathrm{Id})$ and with the set of generators of uniformly continuous one-parameter subgroups of isometries.

In the real case, the above result leads us to consider when the numerical radius is a norm on the space of operators. A related concept to this fact is the *numerical index* of a (real or complex) Banach space X, introduced by G. Lumer in 1968, which is the constant $n(X)$ defined by

$$n(X) = \inf\bigl\{v(T)\colon\ T \in L(X),\ \|T\| = 1\bigr\}$$

or, equivalently, the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in L(X)$. Let us mention here a couple of facts concerning the numerical index which will be relevant to our discussion. For more information, recent results and open problems, we refer the reader to the very recent survey [23] and references therein. First, real or complex $C(K)$ and $L_1(\mu)$ spaces have numerical index 1, and the numerical index of a Hilbert space of dimension greater than one is 0 in the real case and $1/2$ in the complex case. The set of values of the numerical index is given by the following equalities:

$$\bigl\{n(X)\colon\ X \text{ complex Banach space}\bigr\} = \bigl[\mathrm{e}^{-1}, 1\bigr],$$
$$\bigl\{n(X)\colon\ X \text{ real Banach space}\bigr\} = [0, 1].$$

Finally, since $v(T) = v(T^*)$ for every $T \in L(X)$ (this can be proved by making use of the exponential formula, for example), it follows that $n(X^*) \leqslant n(X)$ for every Banach space X. Very recently, K. Boyko, V. Kadets, D. Werner and the author proved that the reversed inequality does not always hold [9, Example 3.1], answering in the negative way a question which had been latent since the beginning of the seventies.

Let us observe that, for a *real* Banach space X, as a consequence of Proposition 2.1, if $\mathcal{T}(\mathrm{Iso}(X), \mathrm{Id})$ is non-trivial, then $n(X) = 0$. Let us state this result for further reference.

Proposition 2.2. *Let X be a real Banach space with $n(X) > 0$. Then, $\mathcal{T}(\mathrm{Iso}(X), \mathrm{Id})$ reduces to zero.*

In the finite-dimensional case, the above implication reverses and, actually, the numerical index zero characterizes those finite-dimensional real Banach spaces with infinitely many isometries [31, Theorem 3.8]. We refer the reader to the just cited [31] and to [29,32] for further results on finite-dimensional spaces with infinitely many isometries. In the infinite-dimensional case, the situation is different and it is possible to find a real Banach space X such that $n(X) = 0$ but $\mathcal{T}(\mathrm{Iso}(X), \mathrm{Id}) = \{0\}$, i.e. the numerical radius it is a (necessarily non-complete) norm on $L(X)$ which is not equivalent to the usual one.

We will also use another concept related to the numerical range: the so-called Daugavet equation. A bounded linear operator T on a Banach space X is said to satisfy the *Daugavet equation* if

$$\|\mathrm{Id} + T\| = 1 + \|T\|. \tag{DE}$$

This norm equality appears for the first time in a 1963 paper by I. Daugavet [11], where it was proved that every compact operator on $C[0,1]$ satisfies it. Following [21,22], we say that a Banach space X has the *Daugavet property* if every rank-one operator $T \in L(X)$ satisfies (DE). In such a case, every operator on X not fixing a copy of ℓ_1 also satisfies (DE) [33]; in particular, this happens to every compact or weakly compact operator on X [22]. Examples of spaces with the Daugavet property are $C(K)$ when the compact space K is perfect, and $L_1(\mu)$ when the positive measure μ is atomless. An introduction to the Daugavet property can be found in the books by Y. Abramovich and C. Aliprantis [1,2] and the state-of-the-art can be found in the survey paper by D. Werner [35]; for more recent results we refer the reader to [6,10,19,24] and references therein.

The relation between the Daugavet equation and the numerical range is given as follows [12]. Given a Banach space X and $T \in L(X)$,

$$T \text{ satisfies (DE)} \quad \Longleftrightarrow \quad \sup \operatorname{Re} V(T) = \|T\|.$$

The following result is an straightforward consequence of this fact.

Proposition 2.3. *Let X be a real or complex Banach space and $T \in L(X)$. If λT satisfies* (DE)*, then $\|T\| \in \lambda \overline{V(T)}$. In particular, if X has the Daugavet property, then every operator $T \in L(X)$ which does not fix a copy of ℓ_1 satisfies*

$$\|T\|\mathbb{T} \subset \overline{V(T)}.$$

We finish this section collecting some easy results concerning L-summands of Banach spaces, numerical ranges, and isometries. We include a proof for the sake of completeness.

Proposition 2.4. *Let X be a real or complex Banach space and suppose that $X = Y \oplus_1 Z$ for closed subspaces Y and Z.*

(a) *Given an operator $S \in L(Y)$, the operator $T \in L(X)$ defined by*

$$T(y,z) = (Sy, 0) \quad (y \in Y,\ z \in Z)$$

satisfies $\|T\| = \|S\|$ and $V(T) \subset [0,1]\,V(S)$.

(b) *For every $S \in \operatorname{Iso}(Y)$, the operator*

$$T(y,z) = (Sy, z) \quad (y \in Y,\ z \in Z)$$

belongs to $\operatorname{Iso}(X)$.

Proof. (a) It is clear that $\|T\| = \|S\|$. On the other hand, given

$$\lambda = (y^*, z^*)\big(T(y,z)\big) = y^*(Sy) \in V(T),$$

where $(y,z) \in S_X$ and $(y^*, z^*) \in S_{X^*}$ with $(y^*, z^*)(y,z) = 1$, we have

$$1 = (y^*, z^*)(y,z) = y^*(y) + z^*(z) \leqslant \|y^*\|\|y\| + \|z^*\|\|z\| \leqslant \|y\| + \|z\| = 1.$$

We deduce that $y^*(y) = \|y^*\|\|y\|$. If $\|y^*\|\|y\| = 0$, then $\lambda = 0 \in [0,1]V(S)$. Otherwise,

$$\lambda = \|y^*\|\|y\| \frac{y^*}{\|y^*\|}\left(S\frac{y}{\|y\|}\right) \in [0,1]\, V(S). \tag{1}$$

(b) For $(y,z) \in X$, we have

$$\|T(y,z)\| = \|(Sy,z)\| = \|Sy\| + \|z\| = \|y\| + \|z\| = \|(y,z)\|. \qquad \square$$

3. The construction

Our aim here is to construct closed subspaces of $C[0,1]$, which share some properties with it, but such that their duals could be extremely different from $C[0,1]^*$. The idea of our construction is to squeeze the one that was given in [9, Examples 3.1 and 3.2] to show that the numerical index of the dual of a Banach space can be different than the numerical index of the space. Let us comment that all the results in this section are valid in the real and in the complex case. We need one definition.

Definition 3.1. (See [9, Definition 2.3].) Let K be a compact Hausdorff space. A closed subspace X of $C(K)$ is said to be *C-rich* if for every nonempty open subset U of K and every $\varepsilon > 0$, there is a positive continuous function h of norm 1 with support inside U such that the distance from h to X is less than ε.

Our interest in C-rich subspaces of $C(K)$ is that they inherit some geometric properties from $C(K)$, as the following result summarizes.

Proposition 3.2. *(See [9, Theorem 2.4] and [20, Theorem 3.2].) Let K be a perfect compact Hausdorff space and let X be a C-rich subspace of $C(K)$. Then $n(X) = 1$ and X has the Daugavet property.*

We are now able to present the main result of the paper.

Theorem 3.3. *Let E be a separable Banach space. Then, there is a C-rich subspace $X(E)$ of $C[0,1]$ such that $X(E)^*$ contains (an isometrically isomorphic copy of) E^* as an L-summand.*

Proof. By the Banach–Mazur theorem, we may consider E as a closed subspace of $C(\Delta)$, where Δ denotes the Cantor middle third set viewed as a subspace of $[0,1]$. We write $P : C[0,1] \to C(\Delta)$ for the restriction operator, i.e.

$$[P(f)](t) = f(t) \quad (t \in \Delta,\ f \in C[0,1]).$$

We define the closed subspaces $X(E)$ and Y of $C[0,1]$ by

$$X(E) = \{f \in C[0,1]:\ P(f) \in E\}, \qquad Y = \ker P.$$

Since $[0,1] \setminus \Delta$ is open and dense in $[0,1]$, it is immediate to show that $X(E)$ is C-rich in $C[0,1]$. Indeed, for every nonempty open subset U of $[0,1]$, we consider the nonempty open subset $V = U \cap ([0,1] \setminus \Delta)$ and we take a norm-one continuous function $h : [0,1] \to [0,1]$ whose support is contained in V. Therefore, h belongs to $Y \subseteq X(E)$, and the support of h is contained in U.

Since Y is an M-ideal in $C[0,1]$ (see [16, Example I.1.4(a)]), it is a fortiori an M-ideal in $X(E)$ by [16, Proposition I.1.17], meaning that $Y^\perp \equiv (X(E)/Y)^*$ is an L-summand of $X(E)^*$.

It only remains to prove that $X(E)/Y$ is isometrically isomorphic to E. To do so, we define the operator $\Phi : X(E) \to E$ given by $\Phi(f) = P(f)$ for every $f \in X(E)$. Then Φ is well defined, $\|\Phi\| \leqslant 1$, and $\ker \Phi = Y$. To see that the canonical quotient operator $\widetilde{\Phi} : X(E)/Y \to E$ is a surjective isometry, it suffices to show that

$$\Phi\big(\big\{f \in X(E)\colon \|f\| < 1\big\}\big) = \{g \in E\colon \|g\| < 1\}.$$

Indeed, the left-hand side is contained in the right-hand side since $\|\Phi\| \leqslant 1$. On the other hand, for every $g \in E \subset C(\Delta)$ with $\|g\| < 1$, we consider any isometric extension $f \in C[0,1]$ (it is easy to construct it by just considering an affine extension, see [3, p. 18] for instance). It is clear that $f \in X(E)$ with $\|f\| = \|g\| < 1$ and that $\Phi(f) = g$. □

Remarks 3.4.

(a) Let us observe that the spaces $X(E)$ has a strong version of C-richness which can be read as the validity of the Urysohn lemma in $X(E)$. Namely, *for every nonempty open subset U of* $[0,1]$, *there is a non-null positive continuous function $h \in X(E)$ whose support is contained in U.*

(b) Also, following the proof of the theorem, it is easy to check what we have actually proved is that $X(E)^* \equiv E^* \oplus_1 L_1(\mu)$ *for a suitable localizable positive measure* μ. Indeed, we have shown that Y is an M-ideal in $X(E)$ and $Y^\perp = (X(E)/Y)^* \equiv E^*$. By [16, Remark I.1.13], one gets $X(E)^* \equiv E^* \oplus_1 Y^*$. On the other hand, Y is an M-ideal in $C[0,1]$ and so, Y^* is (isometrically isomorphic to) an $L_1(\mu)$ space. To this end, one may make use of [16, Example I.1.6(a)] and of the fact that $C[0,1]^*$ is isometric to an $L_1(\nu)$ space for some localizable positive measure ν.

(c) The above observation leads us to give a direct and simple proof of the fact that $n(X(E)) = 1$ for every E, without calling Proposition 3.2. Indeed, in the identification $X(E)^* \equiv E^* \oplus_1 Y^*$, the evaluation functionals

$$A = \big\{\delta_t\colon t \in [0,1] \setminus \Delta\big\}$$

(where, as usual, $\delta_t(f) = f(t)$) belong to $Y^* \equiv L_1(\mu)$ (see [16, Proposition I.1.12]). Being $[0,1] \setminus \Delta$ dense in $[0,1]$, it follows that $B_{X(E)^*}$ is the w^*-closed convex hull of A. On the other hand, every extreme point of the unit ball of $X(E)^{**} \equiv E^{**} \oplus_\infty L_\infty(\mu)$ is of the form (e^{**}, h) where e^{**} is extreme in $B_{E^{**}}$ and h is extreme in $B_{L_\infty(\mu)}$. It implies that

$$\big|x^{**}(a)\big| = 1 \quad \big(a \in A,\ x^{**} \text{ extreme point of } B_{X^{**}}\big).$$

Now, by just using that $v(T) = v(T^*)$ for every $T \in L(X(E))$, it is easy to check that $n(X(E)) = 1$ (see [23, Proposition 6] for example).

The construction in Theorem 3.3 can be easily extended to the general case in which E is not separable by just replacing $[0,1]$ by a convenient perfect compact space K. The main difference is that, obviously, there is no universal such a K.

Proposition 3.5. *Let E be a Banach space. Then there is a perfect Hausdorff compact space K and a C-rich subspace $X(E)$ of $C(K)$ such that E^* is an L-summand of $X(E)^*$.*

Proof. We consider E as a closed subspace of $C((B_{E^*}, w^*))$, we write K for the perfect compact space $(B_{E^*}, w^*) \times [0,1]$, $P : C(K) \to C((B_{E^*}, w^*))$ for the operator

$$\big[P(f)\big](t) = f(t,0) \quad \big(t \in B_{E^*},\ f \in C(K)\big),$$

and we consider the space

$$X(E) = \big\{f \in C(K):\ P(f) \in E\big\}.$$

Now, it is easy to adapt the proof of Theorem 3.3 to this situation. □

4. The examples

Our aim here is to use Theorem 3.3 with some particular spaces E to produce some interesting examples. The first one is the promised space whose group of isometries is much smaller than the one of its dual.

Example 4.1. The real Banach space $X(\ell_2)$ produced in Theorem 3.3 satisfies that $\mathrm{Iso}(X(\ell_2))$ does not contain any non-trivial uniformly continuous one-parameter subgroup, while $\mathrm{Iso}(X(\ell_2)^*)$ contains infinitely many uniformly continuous one-parameter subgroups. Equivalently, $\mathcal{T}(\mathrm{Iso}(X(\ell_2)), \mathrm{Id}) = \{0\}$ but $\mathcal{T}(\mathrm{Iso}(X(\ell_2)^*), \mathrm{Id})$ is infinite-dimensional.

Proof. Being $n(X(\ell_2)) = 1$ by Proposition 3.2, the tangent space at Id of the group $\mathrm{Iso}(X(\ell_2))$ is null by Proposition 2.2. On the other hand, Proposition 2.4(b) gives us that the group of isometries of $X(\ell_2)^*$ contains $\mathrm{Iso}(\ell_2)$ as a subgroup, and so $\mathcal{T}(X(\ell_2)^*, \mathrm{Id})$ is infinite-dimensional. □

The above example gives us, in particular, with a real Banach space with numerical index 1 such that its dual has numerical index 0. An example of this kind was given in [9, Example 3.2.a] as a c_0-sum of spaces whose duals have positive numerical index. Therefore, the dual of that example does not contain any non-null skew-hermitian operator (see [30, Example 3.b] for a proof). Thus, the existence of a space like the one given in Example 4.1 is not contained in [9, Example 3.2.a]. On the other hand, the new construction can be used to give the following improvement of the examples given in [9]: the dual of a Banach space with numerical index 1 may have any possible value of the numerical index.

Proposition 4.2.

$$\big\{n(X^*):\ X \textit{ complex Banach space with } n(X) = 1\big\} = \big[\mathrm{e}^{-1}, 1\big],$$

$$\big\{n(X^*):\ X \textit{ real Banach space with } n(X) = 1\big\} = [0,1].$$

Proof. Indeed, just take two-dimensional spaces E with any possible value of the numerical index (see [12]) and consider $X(E)$. Then, by Theorem 3.3 and Remark 3.4(b), $n(X(E)) = 1$ while $X(E)^* = E^* \oplus_1 L_1(\mu)$ for a suitable measure μ. Now, [30, Proposition 1] gives

$$n\big(X(E)^*\big) = \min\big\{n(E^*), L_1(\mu)\big\} = \min\big\{n(E^*), 1\big\} = n(E^*) = n(E). \qquad \square$$

In a 1977 paper [18], T. Huruya determined the numerical index of a (complex) C^*-algebra. Part of the proof was recently clarified by A. Kaidi, A. Morales, and A. Rodríguez-Palacios in [25], where the result is extended to preduals of von Neumann algebras. Namely, the numerical index of a C^*-algebra is equal to $1/2$ when it is not commutative and 1 when it is commutative, and the numerical index of the predual of a von Neumann algebra coincides with the numerical index of the algebra. Therefore, if X is a C^*-algebra or the predual of a von Neumann algebra, then $n(X) = n(X^*)$, and the same is true for all the successive duals of X. The following example shows that we can not extend this result to successive preduals.

Example 4.3. There exists a Banach space X such that X^{**} is (isometrically isomorphic to) a C^*-algebra, $n(X) = 1$ and $n(X^*) = 1/2$.

Proof. We consider $E = K(\ell_2)$, the space of compact linear operators on ℓ_2. Then, the space $X(E)$ given in Theorem 3.3 has numerical index 1 and Remark 3.4(b) gives us that

$$X(E)^* \equiv K(\ell_2)^* \oplus_1 L_1(\mu)$$

for a suitable positive localizable measure μ. Then, $n(X(E)^*) = n(K(\ell_2)^*) = 1/2$ and $X(E)^{**}$ is isometrically isomorphic to the C^*-algebra $L(\ell_2) \oplus_\infty L_\infty(\mu)$. $\square$

Let us observe that the adjoint of a skew-hermitian operator on a Banach space X is also skew-hermitian, and Example 4.1 shows that there might be skew-hermitian operators on X^* which are not w^*-continuous. The next two examples show that the same is true for hermitian operators and dissipative operators.

Let X be a complex Banach space. An operator $T \in L(X)$ is said to be *hermitian* if $V(T) \subset \mathbb{R}$ (i.e. the operator iT is skew-hermitian), equivalently (see Proposition 2.1), if $\exp(i\rho T) \in \mathrm{Iso}(X)$ for every $\rho \in \mathbb{R}$. Hermitian operators have been deeply studied and many results on Banach algebras depend on them; for instant, the Vidav–Palmer characterization of C^*-algebras. We refer to [7,8,27] for more information.

Example 4.4. Let us consider the complex space $X(\ell_2)$ produced in Theorem 3.3. Then, every non-null hermitian operator T on X fixes a copy of ℓ_1 whereas $X(\ell_2)^*$ has an infinite-dimensional real subspace of finite-rank hermitian operators.

Proof. The space $X(\ell_2)$ has the Daugavet property by Proposition 3.2, and so Proposition 2.3 gives us $\|T\|\mathbb{T} \subseteq \overline{V(T)}$ for every operator T which does not fix a copy of ℓ_1. This implies that T is not hermitian. On the other hand, being ℓ_2 an L-summand of $X(\ell_2)^*$, every finite-dimensional orthogonal projection on ℓ_2 (which is clearly hermitian) canonically extend to a finite-rank hermitian operator on $X(\ell_2)^*$ by Proposition 2.4. $\square$

Our last example deals with dissipative operators, a concept translated from the Hilbert space setting to general Banach spaces by G. Lumer and R. Phillips [28] in 1961. Let X be a real or complex Banach space. An operator $T \in L(X)$ is said to be *dissipative* if $\operatorname{Re} V(T) \subset \mathbb{R}_0^-$ or, equivalently, if $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}^+$ (i.e. T is the generator of a one-parameter semigroup of contractions). We refer to [7,8,28] for more information and background.

Example 4.5. Let us consider the real or complex space $X(\ell_2)$ produced in Theorem 3.3. Then, every non-null dissipative operator T on X fixes a copy of ℓ_1 whereas $X(\ell_2)^*$ has infinitely many linear independent finite-rank dissipative operators.

Proof. The space $X(\ell_2)$ has the Daugavet property by Proposition 3.2, and so Proposition 2.3 gives us $\|T\|\mathbb{T} \subseteq \overline{V(T)}$ for every operator T which does not fix a copy of ℓ_1. This implies that T is not dissipative. On the other hand, being ℓ_2 an L-summand of $X(\ell_2)^*$, the opposite of every finite-dimensional orthogonal projection on ℓ_2 (which is clearly dissipative) canonically extend by zero to a finite-rank dissipative operator on $X(\ell_2)^*$. □

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References

[1] Y. Abramovich, C. Aliprantis, An invitation to Operator Theory, Grad. Stud. Math., vol. 50, Amer. Math. Soc., Providence, RI, 2002.
[2] Y. Abramovich, C. Aliprantis, Problems in Operator Theory, Grad. Stud. Math., vol. 51, Amer. Math. Soc., Providence, RI, 2002.
[3] F. Albiac, N.J. Kalton, Topics in Banach Space Theory, Grad. Texts in Math., vol. 233, Springer-Verlag, New York, 2006.
[4] M. Artin, Algebra, Prentice Hall, Englewood Cliffs, NJ, 1991.
[5] F.L. Bauer, On the field of values subordinate to a norm, Numer. Math. 4 (1962) 103–111.
[6] J. Becerra Guerrero, M. Martín, The Daugavet property of C^*-algebras, JB^*-triples, and of their isometric preduals, J. Funct. Anal. 224 (2005) 316–337.
[7] F.F. Bonsall, J. Duncan, Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras, London Math. Soc. Lecture Note Ser., vol. 2, Cambridge Univ. Press, Cambridge, 1971.
[8] F.F. Bonsall, J. Duncan, Numerical Ranges II, London Math. Soc. Lecture Note Ser., vol. 10, Cambridge Univ. Press, Cambridge, 1973.
[9] K. Boyko, V. Kadets, M. Martín, D. Werner, Numerical index of Banach spaces and duality, Math. Proc. Cambridge Philos. Soc. 142 (2007) 93–102.
[10] K. Boyko, V. Kadets, D. Werner, Narrow Operators on Bochner L_1-spaces, J. Math. Phys. Anal. Geom. 4 (2006) 358–371.
[11] I.K. Daugavet, On a property of completely continuous operators in the space C, Uspekhi Mat. Nauk 18 (1963) 157–158 (in Russian).
[12] J. Duncan, C. McGregor, J. Pryce, A. White, The numerical index of a normed space, J. London Math. Soc. 2 (1970) 481–488.
[13] R. Fleming, J. Jamison, Isometries on Banach Spaces: Function Spaces, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., vol. 129, Chapman & Hall/CRC, Boca Raton, FL, 2003.
[14] R. Fleming, J. Jamison, Isometries on Banach Spaces: Vector-Valued Function Spaces and Operator Spaces, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., vol. 132, Chapman & Hall/CRC, Boca Raton, FL, 2008.
[15] G. Godefroy, Existence and uniqueness of isometric preduals: A survey, in: Banach Space Theory, Iowa City, IA, 1987, in: Contemp. Math., vol. 85, Amer. Math. Soc., Providence, RI, 1989, pp. 131–193.

[16] P. Harmand, D. Werner, W. Werner, M-Ideals in Banach Spaces and Banach Algebras, Lecture Notes in Math., vol. 1547, Springer-Verlag, Berlin, 1993.

[17] E. Hille, R. Phillips, Functional Analysis and Semi-groups, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, RI, 1957.

[18] T. Huruya, The normed space numerical index of C^*-algebras, Proc. Amer. Math. Soc. 63 (1977) 289–290.

[19] Y. Ivaknho, V. Kadets, D. Werner, The Daugavet property for spaces of Lipschitz functions, Math. Scand. 101 (2007) 261–279.

[20] V.M. Kadets, M.M. Popov, The Daugavet property for narrow operators in rich subspaces of $C[0,1]$ and $L_1[0,1]$, St. Petersburg Math. J. 8 (1997) 571–584.

[21] V.M. Kadets, R.V. Shvidkoy, G.G. Sirotkin, D. Werner, Espaces de Banach ayant la propriété de Daugavet, C. R. Acad. Sci. Paris, Sér. I Math. 325 (1997) 1291–1294.

[22] V. Kadets, R. Shvidkoy, G. Sirotkin, D. Werner, Banach spaces with the Daugavet property, Trans. Amer. Math. Soc. 352 (2000) 855–873.

[23] V. Kadets, M. Martín, R. Payá, Recent progress and open questions on the numerical index of Banach spaces, Rev. R. Acad. Cien. Ser. A Mat. 100 (2006) 155–182.

[24] V. Kadets, V. Shepelska, D. Werner, Quotients of Banach spaces with the Daugavet property, preprint.

[25] A. Kaidi, A. Morales, A. Rodríguez-Palacios, Geometrical properties of the product of a C^*-algebra, Rocky Mountain J. Math. 31 (2001) 197–213.

[26] P. Legisa, Banach spaces with a shrinking hyperorthogonal basis, Acta Math. Hungar. 54 (1989) 81–184.

[27] G. Lumer, Semi-inner-product spaces, Trans. Amer. Math. Soc. 100 (1961) 29–43.

[28] G. Lumer, R.S. Phillips, Dissipative operators in a Banach space, Pacific J. Math. 11 (1961) 679–698.

[29] M. Martín, J. Merí, A. Rodríguez-Palacios, Finite-dimensional Banach spaces with numerical index zero, Indiana Univ. Math. J. 53 (2004) 1279–1289.

[30] M. Martín, R. Payá, Numerical index of vector-valued function spaces, Studia Math. 142 (2000) 269–280.

[31] H.P. Rosenthal, The Lie algebra of a Banach space, in: Banach Spaces, Columbia, MO, 1984, in: Lecture Notes in Math., vol. 1166, Springer-Verlag, Berlin, 1985, pp. 129–157.

[32] H.P. Rosenthal, Functional Hilbertian sums, Pacific J. Math. 124 (1986) 417–467.

[33] R.V. Shvidkoy, Geometric aspects of the Daugavet property, J. Funct. Anal. 176 (2000) 198–212.

[34] O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejer, Math. Z. 2 (1918) 187–197.

[35] D. Werner, Recent progress on the Daugavet property, Irish Math. Soc. Bull. 46 (2001) 77–97.