Touching Points of a Star-Shaped Set With an Affine Subspace

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ABSTRACT

The paper studies the following problem. Given a linear subspace $L \subset R^n$ of dimension k, $1 \leqslant k \leqslant n-1$, the point $x \in R^n$, $x \notin L$, and a star-shaped set $A \subset R^n$, characterize those $\tau > 0$ for which $L + x$ touches τA, and, if this is the case, describe the set $\tau\,\mathrm{cl}(A) \cap (L+x)$. Here A is *star-shaped* if $\lambda A \subseteq A$ for all $0 \leqslant \lambda \leqslant 1$, $L + x$ *touches* τA if $L + x$ meets τA only on the boundary, and $\mathrm{cl}(A)$ means the closure of A. The problem is solved for two kinds of sets: convex A such that the origin θ is contained in the relative interior of A, and A equals to the closed unit l_p-ball G_p for some $0 < p < 1$. For convex A the set $L^{\perp} \cap A^*$ plays a crucial role, where $A^* := \{z \in R^n : \langle z, y\rangle \leqslant 1$ for all $y \in A\}$ is the polar of A, and $L^{\perp}$ is the orthogonal complement subspace to L. For G_p the problem is solved by a special geometrical construction based on "coordinate" subspaces R_T such that $L \cap R_T = \{\theta\}$.

1. INTRODUCTION

For any set $A \subset R^n$, denote by $L(A)$ the linear hull of A (the linear subspace of smallest dimension containing A). Let $\mathrm{ri}(A)$ denote the relative interior of A [the set of interior points of A in $L(A)$], and let $\mathrm{cl}(A)$ be the closure of A. Let $L \subset R^n$ be a linear subspace of dimension less than n, and let $x \in R^n$, $x \notin L$. By tA we mean the dilation of A by $t > 0$, i.e. $tA := \{ta : a \in A\}$.

A general problem we are dealing with is: characterize those $\tau > 0$ for which $L + x$ touches τA, and if this is the case, describe the set $\tau\,\mathrm{cl}(A) \cap (L + x)$. Here, by definition, $L + x$ touches τA if $\tau\,\mathrm{cl}(A) \cap (L + x) \neq \varnothing$ but $\tau\,\mathrm{ri}(A) \cap (L + x) = \varnothing$. Of course, the whole problem is meaningful only if

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there is $t > 0$ such that $t\,\mathrm{cl}(A) \cap (L + x) \neq \emptyset$. In what follows we always assume that this is true.

In the above general setting the problem is almost meaningless. We feel the minimal assumption such that the problem starts to be meaningful is that A is star-shaped, i.e. such that $\lambda A \subseteq A$ for all $0 \leqslant \lambda \leqslant 1$.

We restrict ourselves to two kinds of star-shaped sets: either A is convex with $\theta \in \mathrm{ri}(A)$, or A is a closed unit l_p-ball for $0 < p < 1$. While the results for convex sets are sufficiently general to include many practically interesting cases, those for l_p-balls, $0 < p < 1$, seem to have only a "demonstrative" power in the sense that they are examples of star-shaped nonconvex sets for which the above general problem can be solved.

The paper is divided into three more sections. Section 2 studies the cases of convex sets. Section 3 deals with the l_p-balls, $0 < p < 1$. Finally, Section 4 contains some concluding remarks.

There is a class of full-dimensional convex sets that is of basic importance in the theory of discrete approximation: the l_p-balls, $1 \leqslant p \leqslant \infty$. For this class the general results of Section 2 can be made more exact and explicit. Moreover, this can be done somewhat "independently" of general convexity theory, using only elementary methods; see [1]. In [1] also the connections of l_p-ball cases, $0 < p < 1$, with the l_1-ball case are discussed in detail.

2. CONVEX SETS

Let $L^{\perp}$ mean the orthogonal complement subspace to L, and for any set $A \subset R^n$, let $A^* := \{z \in R^n : \langle z, y \rangle \leqslant 1 \text{ for all } y \in A\}$ be the polar of A, where $\langle z, y \rangle$ is the scalar product in R^n.

LEMMA 2.1. *Let $K \subset R^n$ be a convex set so that $\theta \in \mathrm{ri}(K)$. Then, $\mathrm{ri}(K) \cap (L + x) \neq \emptyset$ if and only if $\langle z, x \rangle < 1$ for all $z \in L^{\perp} \cap K^*$.*

Proof. The condition $\mathrm{ri}(K) \cap (L + x) \neq \emptyset$ is clearly equivalent to the condition

$$x \in \mathrm{ri}(K) + L, \tag{2.1}$$

where $\mathrm{ri}(K) + L$ is the algebraic sum of the sets. Assume first that $L(K) = R^n$ (i.e., K is full-dimensional). If (2.1) is true, then clearly

$$x + G \subset \mathrm{ri}(K) + L, \tag{2.2}$$

where $G \subset R^n$ is a sufficiently small Euclidean ball centered at the origin θ. Let $\theta \neq y \in K^* \cap L^\perp$. Then $\varepsilon y \in G$ for some $\varepsilon > 0$; hence $x + \varepsilon y \in \mathrm{ri}(K) + L$, which implies $\langle x, y \rangle + \varepsilon \langle y, y \rangle = \langle a, y \rangle + \langle b, y \rangle$, where $x + \varepsilon y = a + b$, $a \in \mathrm{ri}(K)$, $b \in L$. But $\langle b, y \rangle = 0$, and $\langle a, y \rangle \leqslant 1$; hence $\langle x, y \rangle = \langle a, y \rangle - \varepsilon \langle y, y \rangle < 1$.

Conversely, let $x \notin \mathrm{ri}(K) + L$. Then the point x may be separated from the convex set $K + L$, i.e., there are $\theta \neq y \in R^n$ and $\gamma \in R^1$ such that

$$\begin{aligned} &\langle y, x \rangle \geqslant \gamma, \\ &\langle y, z \rangle \leqslant \gamma \qquad \forall z \in K + L. \end{aligned} \tag{2.3}$$

The condition $\theta \in \mathrm{ri}(K)$ implies $G \subset \mathrm{ri}(K) + L$ for a sufficiently small Euclidean ball centered at θ; hence $\varepsilon y \in \mathrm{ri}(K) + L$ for some $\varepsilon > 0$. This implies $0 < \varepsilon \langle y, y \rangle \leqslant \gamma$. Let $\bar{y} := y/\gamma$. Then

$$\begin{aligned} &\langle \bar{y}, x \rangle \geqslant 1 \\ &\langle \bar{y}, z \rangle \leqslant 1 \qquad \forall z \in K + L. \end{aligned} \tag{2.4}$$

The second condition in (2.4) is possible only if $\bar{y} \in K^* \cap L^\perp$. But $\langle \lambda \bar{y}, x \rangle = 1$ for some $0 < \lambda \leqslant 1$, and clearly $\lambda \bar{y} \in K^* \cap L^\perp$ (both K^* and $L^\perp$ are convex and contain the origin θ). This proves the lemma in the case $L(K) = R^n$.

The correctness of the lemma in the case $L(K) \subset R^n$ can be derived from the full-dimensional case in the following way. It is clear that we can assume w.l.o.g. that $x \in L^\perp$. Further, one can find $\tilde{x} \in L(K) \cap (L + x) \cap [L^\perp + L(K)^\perp]$ such that

$$\mathrm{ri}(K) \cap (L + x) = \mathrm{ri}(K) \cap (\tilde{L} + \tilde{x}), \tag{2.5}$$

where $\tilde{L} = L(K) \cap L$.

Denote $\tilde{K}^* := \{z \in L(K) : \langle z, y \rangle \leqslant 1 \ \forall y \in K\}$ and $\tilde{L}^\perp := \{z \in L(K) : \langle z, y \rangle = 0 \ \forall y \in \tilde{L}\}$. By the proved full-dimensional case we get

$$\mathrm{ri}(K) \cap (\tilde{L} + \tilde{x}) \neq \varnothing$$

if and only if

$$\langle \tilde{x}, \tilde{\eta} \rangle < 1 \qquad \text{for all} \quad \tilde{\eta} \in \tilde{K}^* \cap \tilde{L}^\perp. \tag{2.6}$$

The equality (2.5) implies that the lemma will be proved after proving that the condition (2.6) is equivalent to the condition

$$\langle x, y\rangle < 1 \qquad \text{for all} \quad y \in K^* \cap L^\perp. \tag{2.7}$$

It is clear that $K^* = \tilde{K}^* + L(K)^\perp$ and $\tilde{L}^\perp = [L^\perp + L(K)^\perp] \cap L(K)$. These show that

$$\begin{aligned} \tilde{K}^* \cap \tilde{L}^\perp &= \tilde{K}^* \cap \left[L^\perp + L(K)^\perp\right] \\ &\subseteq \left[\tilde{K}^* + L(K)^\perp\right] \cap L^\perp + L(K)^\perp = K^* \cap L^\perp + L(K)^\perp \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} K^* \cap L^\perp &= \left[\tilde{K}^* + L(K)^\perp\right] \cap L^\perp \\ &\subseteq \left[L^\perp + L(K)^\perp\right] \cap \tilde{K}^* + L(K)^\perp = \tilde{K}^* \cap \tilde{L}^\perp + L(K)^\perp. \end{aligned} \tag{2.9}$$

We know that

$$\tilde{x} = x + a = b + c, \qquad \text{where} \quad a \in L, \quad b \in L^\perp, \quad c \in L(K)^\perp. \tag{2.10}$$

The conditions (2.8), (2.9), and (2.10) imply the equivalence of (2.6) and (2.7). ■

Lemma 2.2 (The characterization lemma). *Let $\tau > 0$ be such that*

$$\tau\, \mathrm{cl}(K) \cap (L + x) \neq \varnothing. \tag{2.11}$$

Then the space $L + x$ is a touching space to τK if and only if

$$\langle \eta, x\rangle = \tau \qquad \textit{for some} \quad \eta \in K^* \cap L^\perp. \tag{2.12}$$

Proof. It is clear that the condition

$$\tau\, \mathrm{ri}(K) \cap (L + x) = \varnothing \tag{2.13}$$

is equivalent to the condition

$$\mathrm{ri}(K)\cap\left(L+\frac{x}{\tau}\right)=\varnothing. \tag{2.14}$$

The sets K^* and $L^\perp$ are convex and contain θ. Hence the condition

$$\langle\eta,z\rangle\geqslant 1 \qquad \text{for some} \quad \eta\in K^*\cap L^\perp \tag{2.15}$$

is equivalent to

$$\langle\eta,z\rangle=1 \qquad \text{for some} \quad \eta\in K^*\cap L^\perp. \tag{2.16}$$

Using these observations, Lemma 2.1 yields the result. ∎

Denote

$$\mathrm{Ext}(K,L,x,\tau):=\left\{(z,\eta):z\in\mathrm{cl}(K)\cap\left(L+\frac{x}{\tau}\right),\ \eta\in K^*\cap L^\perp,\ \langle z,\eta\rangle=1\right\}. \tag{2.17}$$

[This is a set of ordered pairs (z,η).]

THEOREM 2.3 (The description theorem). *$L+x$ is a touching space to τK if and only if* $\mathrm{Ext}(K,L,x,\tau)$ *is nonempty. If this is the case, then*

$$\tau\,\mathrm{cl}(K)\cap(L+x)=\{\tau z:(z,\eta)\in\mathrm{Ext}(K,L,x,\tau)\}. \tag{2.18}$$

Proof. The condition (2.11) is equivalent to

$$\mathrm{cl}(K)\cap\left(L+\frac{x}{\tau}\right)\neq\varnothing. \tag{2.19}$$

Let $L+x$ be a touching space to τK. Then $L+x/\tau$ is a touching space to K; hence by Lemma 2.2 there is $\eta\in K^*\cap L^\perp$ such that $\langle\eta,x/\tau\rangle=1$. Let $z\in\mathrm{cl}(K)\cap(L+x/\tau)$. Then $z=a+x/\tau$, where $a\in L$; hence $\langle z,\eta\rangle=\langle a,\eta\rangle+\langle x/\tau,\eta\rangle=\langle x/\tau,\eta\rangle=1$. We see that $(z,\eta)\in\mathrm{Ext}$. Conversely, if $(z,\eta)\in\mathrm{Ext}$, then $z=a+x/\tau$, $a\in L$, $\langle z,\eta\rangle=\langle x/\tau,\eta\rangle=1$, and again by Lemma 2.2 $L+x/\tau$ is a touching space to K; hence $L+x$ is a touching space to τK.

Let $(z,\eta)\in$ Ext. Then clearly $\tau z\in\tau\,\mathrm{cl}(K)\cap(L+x)$. Conversely, if $w\in\tau\,\mathrm{cl}(K)\cap(L+x)$ and we know that $L+x$ touches τK, then by the Lemma 2.2 there is $\eta\in K^*\cap L^\perp$ such that $\langle\eta,x\rangle=\tau$. But $w=\tau z=a+x$, $a\in L$; hence $z=\alpha/\tau+x/\tau$ and $\langle z,\eta\rangle=\langle x/\tau,\eta\rangle=1$, showing that $(z,\eta)\in$ Ext. ∎

For fixed x, denote

$$t(K):=\inf\{t>0:t\,\mathrm{cl}(K)\cap(L+x)\neq\varnothing\} \tag{2.20}$$

and

$$s(K):=\sup\{\langle\eta,x\rangle:\eta\in K^*\cap L^\perp\}. \tag{2.21}$$

The definitions of $t(K)$, $s(K)$, K^*, and $L^\perp$ show that

$$s(K)\leqslant t(K). \tag{2.22}$$

In general, neither $t(K)$ nor $s(K)$ is attained, but we have

Corollary 2.4. *The infimum in* (2.20) *is attained if and only if* $\mathrm{Ext}(K,L,x,t(K))$ *is nonempty. If this is the case, then also the supremum in* (2.21) *is attained and*

$$s(K)=t(K). \tag{2.23}$$

Proof. The first part of the corollary is true because if $t(K)\mathrm{cl}(K)\cap(L+x)\neq\varnothing$, then clearly $L+x$ touches $t(K)K$. As to the second part, if Ext is nonempty, then $\langle z,\bar\eta\rangle=1$ for some $\bar\eta\in K^*\cap L^\perp$. But $z=a+x/t(K)$, $a\in L$; hence $\langle z,\bar\eta\rangle=\langle x,\bar\eta\rangle/t(K)=1$; consequently $\langle\bar\eta,x\rangle=t(K)$, implying $t(K)\geqslant s(K)$, so (2.23) holds by (2.22). ∎

The condition (2.23) is true without any additional assumptions.

Theorem 2.5 (The duality theorem).

$$s(K)=t(K). \tag{2.24}$$

Proof. Clearly, for any sufficiently small $\varepsilon > 0$ we have

$$[t(K)-\varepsilon]\operatorname{cl}(K)\cap(L+x)=\varnothing; \tag{2.25}$$

hence

$$[t(K)-\varepsilon]\operatorname{ri}(K)\cap(L+x)=\varnothing, \tag{2.26}$$

i.e.,

$$\operatorname{ri}(K)\cap\left(L+\frac{x}{t(K)-\varepsilon}\right)=\varnothing. \tag{2.27}$$

By Lemma 2.1 there is $\eta(\varepsilon)\in K^*\cap L^{\perp}$ such that

$$\left\langle\eta(\varepsilon),\frac{x}{t(K)-\varepsilon}\right\rangle=1;$$

hence $\langle\eta(\varepsilon),x\rangle=t(K)-\varepsilon$. This implies that for any $\varepsilon>0$, $s(K)\geqslant t(K)-\varepsilon$, yielding (2.24) by (2.22). ■

There are easy examples to show that the infimum in (2.20) is not attained but the supremum in (2.21) is. It is clear that this can happen only if K is not bounded. Theorem 2.3 gives a "parametric" description of touching points, where the Ext plays the role of the parameter set. There are two "levels" of investigation of this set. The first is to find those $z\in\operatorname{cl}(K)$, $\eta\in K^*$ such that $\langle z,\eta\rangle=1$ [it is clear that $\langle z,\eta\rangle\leqslant 1$ for any $z\in\operatorname{cl}(K)$, $\eta\in K^*$]. This may give useful criteria for the touching points. The second step is to find those pairs determined in the first step such that $z\in L+x/\tau$, $\eta\in L^{\perp}$. The more "concrete" is the K, the more "concrete" description of Ext can be given. This approach leads to surprisingly good results in the case when K is the unit closed l_p-ball, $l\leqslant p\leqslant\infty$ (hence K^* equals the unit closed l_q-ball, $1/p+1/q=1$) [1].

3. THE l_p-BALL, $0<p<1$

Let $K:=K_p:=\{z\in R^n:\sum_{i=1}^n|z_i|^p\leqslant 1\}$, $0<p<1$. This set is closed, bounded, and star-shaped.

Given $L \subset R^n$, $x \in R^n$, $x \notin L$, for any index set $T \subseteq \{1,2,\dots,n\}$ denote

$$R_T := \{x \in R^n : x_i = 0 \text{ for } i \notin T\}, \tag{3.1}$$

$$\mathscr{T}(L) := \{T \subseteq \{1,2,\dots,n\} : L \cap R_T = \{\theta\}\}, \tag{3.2}$$

$$\mathscr{T}_{\min}(L) := \{T \in \mathscr{T}(L) : R_T \cap (L+x) \neq \varnothing \text{ and}$$

$$R_{T'} \cap (L+x) = \varnothing \text{ for all } T' \subset T,\ T' \in \mathscr{T}(L)\}, \tag{3.3}$$

and

$$\mathscr{T}_p(L) := \left\{T \in \mathscr{T}_{\min}(L) : \|R_T \cap (L+x)\|_p = \min_{T' \in \mathscr{T}_{\min}(L)} \|R_{T'} \cap (L+x)\|_p\right\}, \tag{3.4}$$

$0 < p < 1$, where

$$\|z\|_p := \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/p}, \qquad z \in R^n. \tag{3.5}$$

One can see easily that if $R_T \cap (L+x)$ is nonempty, then it consists of a unique point. Denote this point by $y(T)$.

THEOREM 3.1. *Let* $0 < p < 1$. *Let* $L + x$ *be a touching space to* τK_p, $\tau > 0$. *Then*

$$\tau K_p \cap (L+x) = \{y(T) : T \in \mathscr{T}_p(L)\}. \tag{3.6}$$

Proof. Denote $H_p := \tau K_p \cap (L+x)$. First we prove that

$$H_p \subseteq \bigcup_{T \in \mathscr{T}(L)} [R_T \cap (L+x)]. \tag{3.7}$$

Let $z \in L + x$ be such that

$$z \notin \bigcup_{T \in \mathscr{T}(L)} [R_T \cap (L+x)]. \tag{3.8}$$

We claim that there is $w \in L + x$ such that

$$\|w\|_p < \|z\|_p. \tag{3.9}$$

The (3.9) implies (3.7). Indeed, that $L + x$ touches τK_p means that

$$\tau' K_p \cap (L + x) = \varnothing \qquad \text{for all} \quad \tau' < \tau. \tag{3.10}$$

Hence, for any point $y \in H_p$, necessarily $\|y\|_p = \tau$. So, if z belonged to H_p, then $\|z\|_p = \tau$ and by (3.9) we would have $\|w\|_p = \tau' < \tau$, which contradicts (3.10).

To prove (3.9) denote $I_0(z) := \{i : z_i = 0\}$, $I(z) := \{i : z_i \neq 0\}$, $I_+(z) := \{i : z_i > 0\}$, $I_-(z) := \{i : z_i < 0\}$. The condition (3.8) implies $I(z) \notin \mathscr{T}(L)$; consequently

$$L \cap R_{I(z)} \neq \{\theta\}. \tag{3.11}$$

One can see easily that there is $\theta \neq u \in L \cap R_{I(z)}$ satisfying the following conditions:

$$I(u) \subseteq I(z), \tag{3.12}$$

$$z + \beta u \in L + x, \qquad -1 \leqslant \beta \leqslant 1, \tag{3.13}$$

and

$$z_i + \beta u_i > 0 \quad \forall i \in I_+(z), \qquad z_i + \beta u_i < 0 \quad \forall i \in I_-(z) \tag{3.14}$$

for all $-1 \leqslant \beta \leqslant 1$.

Denote

$$\varphi(\beta) := \sum_{i=1}^{n} |z_i + \beta u_i|^p, \qquad -1 \leqslant \beta \leqslant 1. \tag{3.15}$$

For this function and its first two derivatives we have

$$\varphi(\beta) = \sum{}'(z_i + \beta u_i)^p + \sum{}''(-z_i - \beta u_i)^p, \tag{3.16}$$

$$\varphi'(\beta) = p\Big(\sum{}'(z_i + \beta u_i)^{p-1} u_i + \sum{}''(-z_i - \beta u_i)^{p-1}(-u_i)\Big), \tag{3.17}$$

$$\varphi''(\beta) = p(p-1)\Big(\sum{}'(z_i + \beta u_i)^{p-2} u_i^2 + \sum{}''(-z_i - \beta u_i)^{p-2} u_i^2\Big), \tag{3.18}$$

where Σ' and Σ'' mean summation over index sets $I_+(z), I_-(z)$ respectively.

If $\varphi'(0) > 0$, then $\varphi(\beta) < \varphi(0)$ for $\beta < 0$ sufficiently near to zero. If $\varphi'(0) < 0$, then $\varphi(\beta) < \varphi(0)$ for sufficiently small $\beta > 0$. If $\varphi'(0) = 0$, then $\varphi''(0) < 0$ implies that $\varphi(\beta)$ has a local maximum at 0; consequently $\varphi(\beta) < \varphi(0)$ for small β. We see that there is β such that $w := z + \beta u$ fulfills (3.9). This proves (3.7).

We can easily see that

$$\bigcup_{T \in \mathscr{T}(L)} [R_T \cap (L + x)] = \{y(T) : T \in \mathscr{T}_{\min}(L)\}. \tag{3.19}$$

Indeed, if $T \in \mathscr{T}(L)$, then there is $T' \subset T$ such that $T' \in \mathscr{T}_{\min}(L)$ and $R_{T'} \cap (L + x) \neq \varnothing$. But $R_{T'} \subset R_T$; hence $y(T) = y(T')$.

It is clear that $\|y(T)\|_p$ is constant for all $T \in \mathscr{T}_p(L)$. Denote this constant by $\bar{\tau}$. The (3.10) implies that $\bar{\tau} \geqslant \tau$. But $\|y(T)\|_p > \bar{\tau}$ for all $T \in \mathscr{T}_{\min}(L) \setminus \mathscr{T}_p(L)$; consequently, by (3.7) and (3.19) we see that $y(T) \notin H_p$ for such T. This implies $H_p \subseteq \{y(T) : T \in \mathscr{T}_p(L)\}$; hence $\bar{\tau} = \tau$, and from this $\{y(T) : T \in \mathscr{T}_p(L)\} \subseteq H_p$. ■

Let K_1 be the K_p for $p = 1$ (the closed unit l_1-ball), and let $\mathscr{T}_1(L)$ be the set (3.4) for $p = 1$. Let extr(C) denote the set of extreme points of a compact convex set C. A result of [1] implies that if $L + x$ is a touching space to τK_1, then

$$\operatorname{extr}(\tau K_1 \cap (L + x)) = \{y(T) : T \in \mathscr{T}_1(L)\}; \tag{3.20}$$

hence

$$\tau K_1 \cap (L + x) = \operatorname{conv}(\{y(T) : T \in \mathscr{T}_1(L)\}). \tag{3.21}$$

This description can be compared to that deriving from the general "theory" in the previous section.

4. REMARKS

Let $\|\cdot\|$ be a norm in R^n, and $\|\cdot\|_*$ be its dual norm. Let K be the closed unit ball of $\|\cdot\|$. Then K^* is the closed unit ball of $\|\cdot\|_*$. Let L, x be as in the previous sections. For $\eta \in K^*$, denote

$$\bar{E}(\eta) := \{z \in R^n : z \in K, \langle z, \eta \rangle = 1\}. \tag{4.1}$$

The main result of [2] can be transformed into the following form: if $L + x$ touches τK, then

$$\tau K \cap (L+x) = (L+x) \cap \left(\bigcup_{\eta \in L^{\perp}} \tau \tilde{E}(\eta) \right). \tag{4.2}$$

For the proof of (4.2) the following classical duality theorem of the $\|\cdot\|$-norm approximation (see, e.g., [3]) has been used:

$$\min_{y \in L+x} \|y\| = \max_{\eta \in L^{\perp}} \frac{\langle \eta, x \rangle}{\|\eta\|_*}. \tag{4.3}$$

The first part of the paper [1] deals with a deeper study of the whole problem for l_p-norms, $1 \leqslant p \leqslant \infty$. In these cases the general result (4.2) can be made more exact and explicit via the precise description of the $\tilde{E}(\eta)$. An interesting feature of the results in [1] is that their proofs do not use the duality theorem (4.3) but only elementary considerations.

The second part of [1] deals with the l_p-norms for $0 < p \leqslant 1$. For $0 < p < 1$ the methods and results of [1] are analogous to those of Section 3. In [1] a deeper study of the case $p = 1$ and its relation to the cases $0 < p < 1$ can also be found.

The results of [1] are especially sharp, "complete," and convincing (for all $0 < p \leqslant \infty$) when the dimension of L is $n - 1$. For lower-dimensional L the "formulas" of [1] are not so complete. This phenomenon can be observed also in the more general result (2.18). Namely, if L is $(n-1)$-dimensional, then $L^{\perp}$ is one-dimensional; hence one can easily prove, changing the τ, the nonemptiness of Ext. We can say that the "complexity" of proving the nonemptiness of Ext and finding all elements of it decreases with increasing dimension of L.

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