Strong and weak convergence theorems for asymptotically nonexpansive mappings

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Abstract

Suppose $K$ is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space $E$ with $P$ as a nonexpansive retraction. Let $T : K \to E$ be an asymptotically nonexpansive nonself-map with sequence $\{k_n\}_{n \geq 1} \subset [1, \infty)$, $\lim k_n = 1$, $F(T) := \{x \in K : Tx = x \} \neq \emptyset$. Suppose $\{x_n\}_{n \geq 1}$ is generated iteratively by

$$x_1 \in K, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(P(x)_{n-1})x_n), \quad n \geq 1,$$

where $\{\alpha_n\}_{n \geq 1} \subset (0, 1)$ is such that $\epsilon < 1 - \alpha_n < 1 - \epsilon$ for some $\epsilon > 0$. It is proved that $(I - T)$ is demiclosed at 0. Moreover, if $\sum_{n \geq 1}(k_n^2 - 1) < \infty$ and $T$ is completely continuous, strong convergence of $\{x_n\}$ to some $x^* \in F(T)$ is proved. If $T$ is not assumed to be completely continuous but $E$ also has a Fréchet differentiable norm, then weak convergence of $\{x_n\}$ to some $x^* \in F(T)$ is obtained.

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1. Introduction

Let \( K \) be a nonempty subset of a real normed linear space \( E \). A self-mapping \( T : K \to K \) is called asymptotically nonexpansive if there exists a sequence \( \{k_n\} \subset [1, \infty) \), \( k_n \to 1 \) as \( n \to \infty \) such that \( \forall x, y \in K \), the following inequality holds:

\[
\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall n \geq 1.
\] (1.1)

\( T \) is called uniformly \( L \)-Lipschitzian if there exists a constant \( L > 0 \) such that \( \forall x, y \in K \),

\[
\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall n \geq 1.
\] (1.2)

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [9] as an important generalization of the class of nonexpansive maps (i.e., mappings \( T : K \to K \) such that \( \|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K \)) who proved that if \( K \) is a nonempty closed convex subset of a real uniformly convex Banach space and \( T \) is an asymptotically nonexpansive self-mapping of \( K \), then \( T \) has a fixed point.

Iterative methods for approximating fixed points of nonexpansive mappings have been studied by various authors (see, e.g., [3–7,11,12,15–25]), using the Mann iteration process (see, e.g., [14]) or the Ishikawa iteration process (see, e.g., [12]).

In 1978, Bose [2] proved that if \( K \) is a bounded closed convex nonempty subset of a uniformly convex Banach space \( E \) satisfying Opial’s condition and \( T : K \to K \) is an asymptotically nonexpansive mapping, then the sequence \( \{T^n x\} \) converges weakly to a fixed point of \( T \) provided \( T \) is asymptotically regular at \( x \in K \), i.e., \( \lim_{n \to \infty} \|T^n x - T^{n+1} x\| = 0 \). Passy [17] and also Xu [28] proved that the requirement that \( E \) satisfies Opial’s condition can be replaced by the condition that \( E \) has a Fréchet differentiable norm. Furthermore, Tan and Xu [22,23] later proved that the asymptotic regularity of \( T \) at \( x \) can be weakened to the weakly asymptotic regularity of \( T \) at \( x \), i.e., \( \omega - \lim_{n \to \infty} (T^n x - T^{n+1} x) = 0 \).

In [20,21], Schu introduced a modified Mann process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space \( H \). More precisely, he proved the following theorems:

**Theorem JS1** [20, Theorem 1.5, p. 409]. Let \( H \) be a Hilbert space, \( K \) a nonempty closed convex and bounded subset of \( H \). Let \( T : K \to K \) be completely continuous asymptotically nonexpansive mapping with sequence \( \{k_n\} \subset [1, \infty) \) for all \( n \geq 1 \), \( \lim k_n = 1 \), and \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \). Let \( \{\alpha_n\}_{n=1}^{\infty} \) be a real sequence in \( [0, 1] \) satisfying the condition \( \epsilon \leq \alpha_n \leq 1 - \epsilon \) for all \( n \geq 1 \) and for some \( \epsilon > 0 \). Then the sequence \( \{x_n\} \) generated from arbitrary \( x_1 \in K \) by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1,
\] (1.3)

converges strongly to some fixed point of \( T \).

**Theorem JS2** [21, Theorem 2.1, p. 156]. Let \( E \) be a uniformly convex Banach space satisfying Opial’s condition and let \( K \) be a nonempty closed convex and bounded subset of \( E \). Let \( T : K \to K \) be an asymptotically nonexpansive mapping with sequence \( \{k_n\} \subset
for all \( n \geq 1 \), \( \lim_{n \to \infty} k_n = 1 \), and \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \). Let \( \{\alpha_n\}_{n=1}^{\infty} \) be a real sequence in \( [0, 1] \) satisfying the condition \( 0 < a \leq \alpha_n \leq b < 1 \), \( n \geq 1 \), for some constants \( a, b \). Then the sequence \( \{x_n\} \) generated from arbitrary \( x_1 \in K \) by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1,
\]

converges weakly to some fixed point of \( T \).

In [19], Rhoades extended Theorem JS1 to uniformly convex Banach space using a modified Ishikawa iteration method. In [15], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on \( K \), provided that \( F(T) = \{ x \in K : Tx = x \} \neq \emptyset \). In [25], Tan and Xu extended Theorem JS2 to the so-called Ishikawa iteration scheme. Recently, Chang et al. [4] proved convergence theorems for asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces without assuming any of the following conditions: (a) \( E \) satisfies Opial’s condition; (b) \( T \) is asymptotically regular or weakly asymptotically regular; (c) \( K \) is bounded. Their results improve and generalize the corresponding results of Bose [2], Gornicki [10], Passty [17], Reich [18], Schu [21], Tan and Xu [22,23,25], and Xu [28].

In all the above results, the operator \( T \) remains a self-mapping of a nonempty closed convex subset \( K \) of a uniformly convex Banach space. If, however, the domain of \( T \), \( D(T) \), is a proper subset of \( E \) (and this is the case in several applications), and \( T \) maps \( D(T) \) into \( E \), then the iteration processes of Mann and Ishikawa studied by these authors; and their modifications introduced by Schu may fail to be well defined.

It is our purpose in this paper first to introduce the class of asymptotically nonexpansive nonself-maps and prove demiclosed principle for such maps. Then, an iteration scheme for approximating a fixed point of any map belonging to this class (when such a fixed point exists) is constructed; and strong and weak convergence theorems are proved. Our theorems improve and generalize important related results of Chang et al. and other authors.

2. Preliminaries

Let \( E \) be a real Banach space and \( J \) denote the normalized duality mapping from \( E \) to \( 2^{E^*} \) given by

\[
J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\| \},
\]

where \( E^* \) denotes the dual space of \( E \) and \( \langle \cdot, \cdot \rangle \) denotes the generalized duality pairing between elements of \( E \) and \( E^* \).

Let \( E \) be a real normed linear space. The modulus of convexity of \( E \) is the function \( \delta_E : (0, 2] \to [0, 1] \) defined by

\[
\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \epsilon = \|x - y\| \right\}.
\]

\( E \) is uniformly convex if and only if \( \delta_E(\epsilon) > 0 \), \( \forall \epsilon \in (0, 2] \).

It is well known (see, e.g., [8,13,27]) that in a uniformly convex space, \( \delta_E \) is continuous, strictly increasing, and \( \delta_E(0) = 0 \).
A subset $K$ of $E$ is said to be a retract of $E$ if there exists a continuous map $P : E \to K$ such that $Px = x$, $\forall x \in K$. Every closed convex set of a uniformly convex Banach space is a retract. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map $P$ is a retraction, then $Py = y$ for all $y$ in the range of $P$.

A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be demiclosed at $p$ if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to $p$, then $Tx^* = p$.

In what follows, we shall make use of the following lemmas.

**Lemma 2.1** (see, e.g., [1]). Let $E$ be a real uniformly convex Banach space, $\lambda \in [0, 1]$, $x, y \in E$. Then

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - 2\lambda(1 - \lambda)C^2\delta_E\left(\frac{\|x - y\|}{2C}\right),$$

where $C = \sqrt{(\|x\|^2 + \|y\|^2)/2}$.

**Remark 2.2.** If $\|x\| \leq R$ and $\|y\| \leq R$, where $R$ is some positive number, then

$$C \leq R \quad \text{and} \quad 2C^2\delta_E\left(\frac{\|x - y\|}{2C}\right) \geq \frac{R^2\delta_E(\|x - y\|/2R)}{2L^*},$$

where $L^*$ is a constant (the Figiel constant) such that $1 < L^* < 1.7$ (see, e.g., [1,8]).

**Lemma 2.3** (see, e.g., [24]). Let $\{\lambda_n\}_{n \geq 1}$ and $\{\sigma_n\}_{n \geq 1}$ be sequences of nonnegative real numbers such that $\lambda_{n+1} \leq \lambda_n + \sigma_n$, $\forall n \geq 1$, and $\sum_{n \geq 1} \sigma_n < \infty$. Then $\lim_{n \to \infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$, then $\lambda_n \to 0$ as $n \to \infty$.

**Lemma 2.4** (see, e.g., [4, p. 1251]). Let $E$ be a uniformly convex Banach space, $K$ a nonempty bounded closed convex subset of $E$. Then there exists a strictly increasing continuous convex function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for any Lipschitzian mapping $T : K \to E$ with Lipschitz constant $L \geq 1$, and elements $\{x_j\}_{j=1}^n$ in $K$ and any nonnegative numbers $\{t_j\}_{j=1}^n$ with $\sum_{j=1}^n t_j = 1$, the following inequality holds:

$$\left\|T\left(\sum_{j=1}^n t_jx_j\right) - \sum_{j=1}^n t_jTx_j\right\| \leq L\phi^{-1}\left\{\max_{1 \leq j, k \leq n} (\|x_j - x_k\| - L^{-1}\|Tx_j - Tx_k\|)\right\}.$$

### 3. Main results

In this section, we give new definitions and prove our main theorems.

**Definition 3.1.** Let $E$ be a real normed linear space, $K$ a nonempty subset of $E$. Let $P : E \to K$ be the nonexpansive retraction of $E$ onto $K$. A map $T : K \to E$ is said to be
asymptotically nonexpansive if there exists a sequence \( \{k_n\} \subset [1, \infty) \), \( k_n \to 1 \) as \( n \to \infty \) such that the following inequality holds:

\[
\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n \|x - y\|, \quad \forall x, y \in K, \ n \geq 1. \tag{3.1}
\]

\( T \) is called uniformly \( L \)-Lipschitzian if there exists \( L > 0 \) such that

\[
\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L \|x - y\|, \quad \forall x, y \in K, \ n \geq 1. \tag{3.2}
\]

Let \( K \) be a nonempty closed convex subset of a real uniformly convex Banach space \( E \). The following iteration scheme is studied:

\[
x_1 \in K, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \tag{3.3}
\]

where \( \{\alpha_n\}_{n \geq 1} \) is a sequence in \((0, 1)\), and \( P \) is as in Definition 3.1.

**Remark 3.2.** If \( T \) is a self-map, then \( P \) becomes the identity map so that (3.1) and (3.2) coincide with (1.1) and (1.2), respectively. Moreover, (3.3) reduces to a Mann-type iteration scheme (see [14]).

In the sequel, we shall need the following lemma.

**Lemma 3.3.** Let \( E \) be a normed linear space, \( K \) nonempty closed convex subset of \( E \), \( T : K \to E \) uniformly \( L \)-Lipschitzian, \( \{\alpha_n\}_{n \geq 1} \subset (0, 1) \), \( x_1 \in K \). Define \( x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n) \) and set \( c_n = \|x_n - T(PT)^{n-1}x_n\|, \forall n \geq 1 \). Then \( \|x_n - Tx_n\| \leq c_n + c_{n-1}L(1 + 2L) \).

The proof follows basically as in the proof of Lemma 1.2 of [20] and is, therefore, omitted.

We now state and prove the following theorems.

**Theorem 3.4** (Demiclosed principle for nonself-map). Let \( E \) be a uniformly convex Banach space, \( K \) a nonempty closed convex subset of \( E \), and let \( T : K \to E \) be an asymptotically nonexpansive mapping with a sequence \( \{k_n\} \subset [1, \infty) \) and \( k_n \to 1 \) as \( n \to \infty \). Then \( I - T \) is demiclosed at zero.

**Proof.** This is basically the proof of Theorem 1 of Chang et al. [4]. For completeness and because of more general nature of our map, we sketch the details. Let \( \{x_n\} \) converge weakly to \( x^* \in K \), and \( \{(I - T)x_n\} \) converge strongly to 0. We prove \( (I - T)x^* = 0 \). Clearly, \( \{x_n\} \) is bounded. So, there exists \( r > 0 \) such that \( \{x_n\} \subset C := K \cap \overline{B_r}(0) \), where \( \overline{B_r}(0) \) is the closed ball in \( E \) with center 0 and radius \( r \). Thus, \( C \) is a nonempty, closed, bounded and convex subset in \( K \).

**Claim.**

\[
T(PT)^{n-1}x^* \to x^* \quad \text{as} \quad n \to \infty. \tag{3.4}
\]
In fact, since \( \{x_n\} \) converges weakly to \( x^* \), by Mazur’s theorem (see, e.g., [26]), we have that for all \( n > 1 \), there exists a convex combination

\[
y_n = \sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n}, \quad t_i^{(n)} \geq 0 \quad \text{and} \quad \sum_{i=1}^{m(n)} t_i^{(n)} = 1,
\]

such that

\[
\|y_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}
\]

Moreover, since \( \{(I - T)x_n\} \) converges strongly to 0, then for \( \epsilon > 0 \) and positive integer \( j \geq 1 \), there exists \( N = N(\epsilon, j) > 0 \) such that

\[
\|(I - T)x_n\| \leq \frac{1}{(1+k_1+k_2+\cdots+k_{j-1})} < \epsilon, \quad \forall n \geq N.
\]

Hence, \( \forall n \geq N \), using Definition 3.1 and the fact that \( P \) is nonexpansive, we have the following estimates:

\[
\|(I - T(PT)^j)x_n\| \leq \|(I - T)x_n\| + \|(T - T(PT))x_n\| + \cdots + \|(T(PT)^{j-2} - T(PT)^{j-1})x_n\|
\]

\[
\leq (1 + k_1 + k_2 + \cdots + k_{j-1})\|x_n - Tx_n\| < \epsilon. \tag{3.6}
\]

Now, since \( T : K \to E \) is asymptotically nonexpansive, so is \( T : C \to E \). Therefore, \( \forall j \geq 1, T(PT)^{j-1} : C \to E \) is a Lipschitzian mapping with the Lipschitz constant \( k_j \geq 1 \). Furthermore,

\[
\|(PT)^{j-1}y_n - y_n\| \leq \|(PT)^{j-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T(PT)^{j-1}x_{i+n}\|
\]

\[
+ \sum_{i=1}^{m(n)} t_i^{(n)} \|T(PT)^{j-1}x_{i+n} - x_{i+n}\|. \tag{3.7}
\]

Using inequality (3.6), we obtain that

\[
\sum_{i=1}^{m(n)} t_i^{(n)} \|T(PT)^{j-1}x_{i+n} - x_{i+n}\| < \epsilon, \quad n \geq N. \tag{3.8}
\]

Furthermore, by Lemma 2.4 and inequality (3.6) there exists an increasing function \( \phi : [0, \infty) \to [0, \infty) \) with \( \phi(0) = 0 \) such that for \( n \geq N \) we have that

\[
\|(PT)^{j-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T(PT)^{j-1}x_{i+n}\| \leq k_j \phi^{-1}(2\epsilon + 2r(1 - k_j^{-1})k_j), \tag{3.9}
\]

since \( x_{i+n} \) and \( x_{k+n} \) are both in \( C \). Substituting (3.8) and (3.9) into (3.7), we have that

\[
\|(PT)^{j-1}y_n - y_n\| \leq k_j \phi^{-1}(2\epsilon + 2r(1 - k_j^{-1})k_j) + \epsilon.
\]

Taking \( \limsup_{n \to \infty} \) of both sides and noting that \( \epsilon > 0 \) is arbitrary, we have that
Theorem 3.5. Let E be a real uniformly convex Banach space, K a closed convex nonempty subset of E. Let \( T : K \rightarrow E \) be asymptotically nonexpansive with sequence \( \{k_n\} \subset [1, \infty) \) such that \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \) and \( x^* \in F(T) := \{x \in K : Tx = x\} \). Let \( \{\alpha_n\} \subset (0, 1) \) be such that \( \epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1 \) and some \( \epsilon > 0 \). Starting from arbitrary \( x_1 \in K \), define \( \{x_n\} \) by (3.3). Then \( \lim_{n \rightarrow \infty} \|x_n - x^*\| \) exists.

Proof. Set \( k_n = 1 + \mu_n \). From (3.3) we have that
\[
\|x_{n+1} - x^*\| = \|P((1 - \alpha_n)x_n + \alpha_n T(P)^{n-1}x_n) - Px^*\|
\leq \|(1 - \alpha_n)x_n + \alpha_n T(P)^{n-1}x_n - x^*\|
\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|T(P)^{n-1}x_n - T(P)^{n-1}x^*\|
\leq (1 + \mu_n)\|x_n - x^*\| \leq \epsilon \sum_{i=1}^{n} \mu_i \|x_1 - x^*\|
\leq \epsilon \sum_{i=1}^{\infty} \mu_i \|x_1 - x^*\|.
\]
(3.11)
So from (3.11), we get that \( \{x_n\} \) is bounded and by Lemma 2.3 we conclude that \( \lim_{n \rightarrow \infty} \|x_n - x^*\| \) exists. This completes the proof. \( \square \)

Theorem 3.6. Let E be a real uniformly convex Banach space, K closed convex nonempty subset of E. Let \( T : K \rightarrow E \) be asymptotically nonexpansive with sequence \( \{k_n\} \subset [1, \infty) \) such that \( \sum_{n=1}^{\infty} (k_n^2 - 1) < \infty \) and \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \subset (0, 1) \) be such that \( \epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1 \) and some \( \epsilon > 0 \). From arbitrary \( x_1 \in K \), define the sequence \( \{x_n\} \) by (3.3). Then \( \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \).
Lemma 3.8. Let \( n>M \) for some \( M > 0 \).

**Proof.** Let \( x^* \in F(T) \). Then from (3.3), Lemma 2.1 and Remark 2.2, we get that

\[
\|x_{n+1} - x^*\|^2 = \|P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n) - x^*\|^2 \\
\leq \| (1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n - x^*\|^2 \\
\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n \|T(PT)^{n-1}x_n - x^*\|^2 \\
- 2\alpha_n(1 - \alpha_n)C^2\delta_E\left(\|x_n - T(PT)^{n-1}x_n\|\right) \\
\leq (1 - \alpha_n)\|x_n - x^*\|^2 + \alpha_n k_n^2\|x_n - x^*\|^2 \\
- \alpha_n(1 - \alpha_n) M^2\delta_E(\|x_n - T(PT)^{n-1}x_n\|/2M) \\
\leq \|x_n - x^*\|^2 + (k_n^2 - 1)M - \epsilon^2 M^2\delta_E(\|x_n - T(PT)^{n-1}x_n\|/2M)
\]

for some \( M > 0 \). This implies that

\[
e^2 M^2\delta_E(\|x_n - T(PT)^{n-1}x_n\|/2M) \\
\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (k_n^2 - 1)M
\]

so that

\[
e^2 M^2 \sum_{n \geq 1} \delta_E\left(\|x_n - T(PT)^{n-1}x_n\|/2M\right) \leq \|x_1 - x^*\|^2 + M \sum_{n \geq 1} (k_n^2 - 1) < \infty.
\]

Thus, since \( \delta_E \) is strictly increasing and continuous, and \( \delta_E(t) \to 0 \) as \( t \to 0 \),

\[
\lim_{n \to \infty} \|x_n - T(PT)^{n-1}x_n\| = 0.
\]

Since \( T \) is uniformly \( L \)-Lipschitzian for some \( L > 0 \), it follows from Lemma 3.3 that \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \). This completes the proof. \( \square \)

**Theorem 3.7.** Let \( E \) be a real uniformly convex Banach space, \( K \) closed convex nonempty subset of \( E \). Let \( T : K \to E \) be completely continuous and asymptotically nonexpansive map with sequence \( \{k_n\} \subset [1, \infty) \) such that \( \sum_{n \geq 1} (k_n^2 - 1) < \infty \) and \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \subset (0, 1) \) be such that \( \epsilon \leq 1 - \alpha_n \leq 1 - \epsilon \), \( \forall n \geq 1 \) and some \( \epsilon > 0 \). From arbitrary \( x_1 \in K \), define the sequence \( \{x_n\} \) by (3.3). Then \( \{x_n\} \) converges strongly to some fixed point of \( T \).

**Proof.** Since by Theorem 3.5 \( \{x_n\} \) is bounded and by hypothesis \( T \) is completely continuous, there exists a subsequence \( \{Tx_{n_j}\} \) of \( \{Tx_n\} \) such that \( Tx_{n_j} \to y^* \) as \( j \to \infty \). Moreover, by Theorem 3.6 we have that \( \|T^jx_{n_j} - x_{n_j}\| \to 0 \) which implies that \( x_{n_j} \to y^* \) as \( j \to \infty \). Thus by continuity of \( T \) we get that \( Ty^* = y^* \). Furthermore, since \( \lim_{n \to \infty} \|x_n - y^*\| \) exists by Theorem 3.5, the conclusion holds. This completes the proof. \( \square \)

**Lemma 3.8.** Let \( E \) be a real uniformly convex Banach space, \( K \) a closed convex nonempty subset of \( E \). Let \( T : K \to E \) be asymptotically nonexpansive with sequence \( \{k_n\}_{n \geq 1} \subset [1, \infty) \) such that \( \sum_{n \geq 1} (k_n - 1) < \infty \) and \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \subset (0, 1) \) be such that
\[ \epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1 \] and some \( \epsilon > 0 \). From arbitrary \( x_1 \in K \), define \( \{x_n\} \) by (3.3). Then for all \( u, v \in F(T) \), the limit
\[ \lim_{n \to \infty} \|tx_n + (1 - t)u - v\| \]
e 0 exists for all \( t \in [0, 1] \).

**Proof.** Let \( \alpha_n(t) = \|tx_n + (1 - t)u - v\| \). Then \( \lim_{n \to \infty} \alpha_n(0) = \|u - v\| \), and from Theorem 3.5, \( \lim_{n \to \infty} \alpha_n(1) = \lim_{n \to \infty} \|x_n - v\| \) exists. It, therefore, remains to prove the lemma for \( t \in (0, 1) \). Define \( T_n : K \to E \) by
\[ T_n x = P((1 - \alpha_n)x + T(P)\alpha_n^{-1} x), \quad x \in K. \]
Then the rest of the proof follows as in the proof of Lemma 3 of [16]. \( \square \)

**Lemma 3.9.** Let \( E \) be a real uniformly convex Banach space which has a Fréchet differentiable norm, \( K \) a closed convex nonempty subset of \( E \). Let \( T : K \to E \) be asymptotically nonexpansive map with sequence \( \{\alpha_n\}_{n \geq 1} \subset [1, \infty) \) such that \( \sum_{n \geq 1} (k_n - 1) < \infty \) and \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \subset (0, 1) \) be such that \( \epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1 \) and some \( \epsilon > 0 \). From arbitrary \( x_1 \in K \), define \( \{x_n\} \) by (3.3). Then for all \( u, v \in F(T) \), the limit
\[ \lim_{n \to \infty} \langle x_n, j(u - v) \rangle \]
e 0 exists. Furthermore, if \( \omega_u(x_n) \) denotes the set of weak subsequential limits of \( \{x_n\} \), then \( \langle x^* - y^*, j(u - v) \rangle = 0, \forall u, v \in F(T) \) and \( \forall x^*, y^* \in \omega_u(x_n) \).

**Proof.** This follows basically as in the proof of Lemma 4 of [16] using Lemma 3.8 instead of Lemma 3 of [16]. \( \square \)

**Theorem 3.10.** Let \( E \) be a real uniformly convex Banach space which has a Fréchet differentiable norm, \( K \) closed convex nonempty subset of \( E \). Let \( T : K \to E \) be asymptotically nonexpansive map with sequence \( \{\alpha_n\} \subset [1, \infty) \), \( \sum_{n \geq 1} (k_n^2 - 1) < \infty \), \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \subset (0, 1) \) be such that \( \epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1 \) and some \( \epsilon > 0 \). From arbitrary \( x_1 \in K \), define the sequence \( \{x_n\} \) by (3.3). Then \( \{x_n\} \) converges weakly to some fixed point of \( T \).

**Proof.** By Theorem 3.5, \( \{x_n\} \) is bounded. Since \( E \) is reflexive, there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) which converges weakly to some \( x^* \in K \). Furthermore, \( \|x_{n_j} - T x_{n_j}\| \to 0 \) as \( j \to \infty \) by Theorem 3.6. Therefore, Theorem 3.4 implies that \( (I - T)x^* = 0 \), i.e., \( T x^* = x^* \). Now, we show that \( \{x_n\} \) converges weakly to \( x^* \). Suppose \( \{x_{m_j}\} \) is another subsequence of \( \{x_n\} \) which converges weakly to some \( y^* \). Then as for \( x^* \), \( y^* \) must be in \( K \) and \( y^* \in F(T) \). Thus, \( x^* = y^* \) by Lemma 3.9. Hence, \( \omega_u(x_n) \) is a singleton, so that \( \{x_n\} \) converges weakly to a fixed point of \( T \). \( \square \)

The following corollary follows from Theorem 3.10.

**Corollary 3.11.** Let \( E \) be a real uniformly convex Banach space which has a Fréchet differentiable norm, \( K \) a closed convex nonempty subset of \( E \). Let \( T : K \to E \) be a
nonexpansive map. Let \( \{ \alpha_n \} \subset (0, 1) \) be such that \( \epsilon \leq 1 - \alpha_n \leq 1 - \epsilon, \forall n \geq 1 \) and some \( \epsilon > 0 \). For arbitrary \( x_1 \in K \), define the sequence \( \{ x_n \} \) by (3.3). Then \( \{ x_n \} \) converges weakly to some \( x^* \in F(T) \).

**Remark 3.12.** Theorems 3.5–3.7, 3.10, and Corollary 3.11 are also valid for the so-called Ishikawa type iteration method with parameters \( \{ \alpha_n \}, \{ \beta_n \} \). Under the conditions on \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) in which generally they are independent, there is no further generality obtained in using the Cumberson–Ishikawa scheme, rather than the scheme considered in this paper with recurrence relation (3.3). In fact, in this case, setting \( \beta_n = 0 \) for all integers \( n \geq 0 \), the so-called Ishikawa-type method reduces to the Mann scheme.

**Remark 3.13.** Theorem 3.7 extends Theorem JS1 and the corresponding results of Rhoades [19] and Osilike and Aniagbosor [15] to the more general class of nonself-maps. Furthermore, no boundedness condition is imposed on \( K \) as in [14]. Theorem 3.4 extends Theorem 1 of Chang et al. [4] to the more general case of nonself-maps. Under the additional hypothesis that \( E \) has a Fréchet differentiable norm, Theorem 3.10 generalizes Theorem JS2 to the case of nonself-maps with Opial’s condition and boundedness of \( K \) dispensed and also improves the results of [15] to the more general case of nonself-maps, and without the assumption that \( (I - T) \) is demiclosed and that \( E \) satisfies Opial condition. We remark that Theorem 3.10 is applicable in \( L_p \) spaces, \( 1 < p < \infty \), while Theorem JS2 and the results of [15] are not.

**References**