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The application of bifurcation method to a higher-order KdV equation ☆

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Abstract

Bifurcation method of dynamical systems is employed to investigate bifurcation of solitary waves in the higher-order KdV equation

$$u_t + au^n u_x + u_{xxx} = 0,$$

where $n \geqslant 1$ and $a \in R$. Numbers of solitary waves are given for each parameter condition. Under some parameter conditions, explicit solitary wave solutions are obtained. Specially, some new solitary wave solutions are found for KdV or MKdV equation.

Keywords: Solitary waves; Explicit solutions; Bifurcation of phase portraits

1. Introduction

In [1,2] higher-order KdV equation

$$u_t + au^n u_x + u_{xxx} = 0 \tag{1.1}$$

is introduced. It is well known that (1.1) incorporates the KdV ($n = 1$) and MKdV ($n = 2$) equation and the two equations are studied by a lot of mathematicians and

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physicists. In [3] a class of fully discrete schemes for numerical simulation of solution of the periodic initial-value problem for (1.1) is analysed, implemented and tested by Bona et al. When $a = n + 1$ and integral constant is zero, the solitary wave solution is given by Fornberg and Whitham [4].

In this paper, we employ bifurcation method of dynamical systems to investigate bifurcation of solitary waves of Eq. (1.1). Numbers of solitary waves are given for each parameter condition. Under some parameter conditions, explicit solitary wave solutions are obtained. Specially, some new solitary wave solutions are found for KdV or MKdV equation (see (3.12), (3.20), (3.21) and (3.29)).

To find solitary wave of Eq. (1.1), substituting the traveling wave solution $u(x,t) = \phi(x - ct) = \phi(\xi)$ into (1.1) for constant wave speed c, we have the following ordinary differential equation:

$$-c\phi'(\xi) + a\phi^n(\xi)\phi'(\xi) + \phi'''(\xi) = 0. \tag{1.2}$$

Integrating (1.2) once, it leads to

$$-c\phi(\xi) + \frac{a}{n+1}\phi^{n+1}(\xi) + \phi''(\xi) = g, \tag{1.3}$$

where $g \in R$ is integral constant. Let $\phi'(\xi) = y$; then we have the planar autonomous system

$$\phi'(\xi) = y, \qquad y'(\xi) = g + c\phi - \frac{a}{n+1}\phi^{n+1}. \tag{1.4}$$

Obviously, (1.4) is Hamiltonian system of Hamiltonian

$$H(\phi, y) = \frac{1}{2}y^2 - \phi\left(g + \frac{c}{2}\phi - \frac{a}{(n+1)(n+2)}\phi^{n+1}\right) = h. \tag{1.5}$$

All level sets $H(\phi, y) = h$, $h \in R$, give the invariant curves of system (1.4). In other words, the phase orbits of the vector fields defined by system (1.4) determine all traveling wave solutions of Eq. (1.1). Thus, to investigate the bifurcation of solitary waves of Eq. (1.1), we have to know the dynamical behaviour of system (1.4). The bifurcation theory of dynamical systems [5,6] play on important role to our study. From the bifurcation theory we know, for a singular point of Hamiltonian system, there are three possibilities, that is, it is center point or saddle point or degenerate saddle point.

This paper is organized as follows: In Section 2 we study the bifurcation of solitary waves. In Section 3 we give explicit solitary wave solutions under some parameter conditions.

2. Bifurcation of solitary waves

In this section, we study the bifurcation of solitary waves. To clear, we give a definition and a lemma for Eq. (1.1).

Definition 2.1. A traveling wave solution $u(x,t) = \phi(x - cx)$ of Eq. (1.1) is called a solitary wave solution if $\phi(\xi)$ has a well-defined limit $\lim_{|\xi|\to\infty} \phi(\xi)$, which is the same at both $\pm\infty$. In three-dimensional space (x,t,u) the graph of solitary wave solution is called solitary wave.

According to above definition, we have the following lemma:

Lemma 2.1. (i) *If Eq.* (1.1) *has one solitary wave, then system* (1.4) *has at least one homoclinic orbit.* (ii) *If system* (1.4) *has one homoclinic orbit, then Eq.* (1.1) *has one solitary wave when n is odd, two solitary waves when n is even.*

Proof. Firstly, we prove (i). Let $u(x,t) = \phi(x - ct)$ be a solitary wave solution of Eq. (1.1). It leads to $\phi(\xi)$, $y = \phi'(\xi)$ satisfy system (1.4). Because all singular points of system (1.4) are in ϕ-axis and $\phi(\xi)$ has a well-defined limit as $|\xi|$ approaches to infinity and the limit is the same at both $\pm\infty$, the orbit of $(\phi(\xi), y(\xi))$ is a homoclinic orbit of system (1.4). The (i) is proved. Secondly, we prove (ii). Suppose $\phi = \phi(\xi)$, $y = y(\xi)$ is the parameter expression of homoclinic orbit of system (1.4). Then $\phi(\xi)$ has a well-defined limit as $|\xi|$ tend infinity and the limit is the same at both $\pm\infty$. On the other hand, $\phi(\xi)$ and $y(\xi)$ satisfy system (1.4), that is, $\phi(\xi)$ satisfies (1.3). Further $\phi(\xi)$ is solution of (1.2). Thus $u(x,t) = \phi(x - ct)$ is solitary wave solution of Eq. (1.1). When n is even, if $u(x,t)$ is solution of Eq. (1.1), so dose $-u(x,t)$. This completes the proof. □

From the above definition and lemma we can prove the following bifurcation theorem about solitary wave of Eq. (1.1).

Theorem 2.1. *In (c,a)-parameter plane, let the coordinate axes c^-, a^-, c^+ and a^+ divide the plane into four regions* (I), (II), (III) *and* (IV), *that is, four quadrants* (*see Fig.* 1). *Let g_0 be as follows*:

$$g_0 = \frac{n|c|}{n+1}\left|\frac{c}{a}\right|^{1/n}, \tag{2.1}$$

and let g be the integral constant in system (1.4). *For given g, from* (1.4) *we have the following results*:

(1) *When n is even and $(c,a) \in$* (I) *in Fig.* 1, (i) *if g equals one of $-g_0$,* 0 *and g_0, then Eq.* (1.1) *has a valley form and a peak form solitary waves*; (ii) *if $|g| < g_0$ and $g \neq 0$, then Eq.* (1.1) *has two valley form and two peak form solitary waves.*
(2) *When n is even and (c,a) is in one of c^+, c^-, a^+, a^-,* (II) *and* (IV), *Eq.* (1.1) *has no solitary wave.*

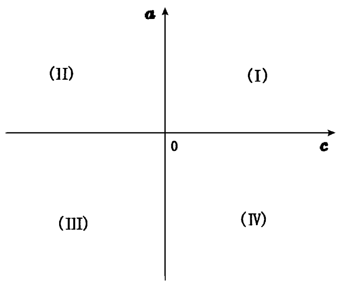


Fig. 1. Parameter plane.

(3) *When* n *is even and* $(c,a)\in$ (III), (i) *if* $|g|<g_0$ *and* $g\neq 0$, *then Eq.* (1.1) *has one valley form and one peak form solitary waves*; (ii) *if* $g=0$, *then Eq.* (1.1) *has two kink waves* [7,8].
(4) *When* n *is odd and* $(c,a)\in$ (I) *or* (II), (i) *if* $g>-g_0$, *then Eq.* (1.1) *has one peak form solitary wave*; (ii) *if* $g\leqslant -g_0$, *then Eq.* (1.1) *has no solitary wave.*
(5) *When* n *is odd and* $(c,a)\in a^+$, *if* $g>0$, *then Eq.* (1.1) *has one peak form solitary wave.*
(6) *When* n *is odd and* $(c,a)\in$ (III) *or* (IV), *if* $g<g_0$, *then Eq.* (1.1) *has one valley form solitary wave.*
(7) *When* n *is odd and* $(c,a)\in a^-$, *if* $g<0$, *then Eq.* (1.1) *has one valley form solitary wave.*
(8) *When* n *is odd and* $(c,a)\in c^+$ *or* c_-, *then Eq.* (1.1) *has no solitary wave.*

Proof. Let function $f(\phi)$ be as follows:

$$f(\phi)=g+c\phi-\frac{a}{n+1}\phi^{n+1}. \tag{2.2}$$

It follows

$$f'(\phi)=c-a\phi^n. \tag{2.3}$$

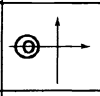
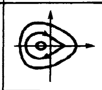
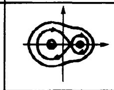
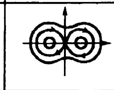
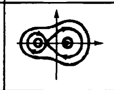
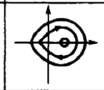
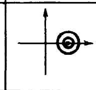
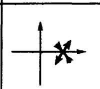
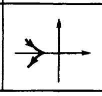
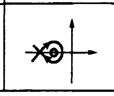
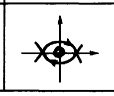
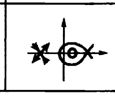
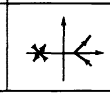
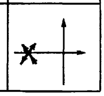
Let $(\phi_0,0)$ be one of singular points of system (1.4). Then at the singular point $(\phi_0,0)$ the characteristic values of linearized system of system (1.4) are

$$\lambda_{1,2}=\pm\sqrt{f'(\phi_0)}. \tag{2.4}$$

From (2.4) we know that if $f'(\phi_0)>0$, then $(\phi_0,0)$ is saddle point; if $f'(\phi_0)<0$, then $(\phi_0,0)$ is center point; if $f'(\phi_0)=0$, then $(\phi_0,0)$ is degenerate saddle point. From $H(\phi,y)$ in (1.5) and above analysis, we obtain the bifurcation of phase portraits of system (1.4), shown in Table 1, when n is even and (c,a) is in (I) or (III).

When n is even and $(c,a)\in$ (I), from Table 1 we see that if $|g|=g_0$, then system (1.4) has one homoclinic orbit; if $g=0$, then system (1.4) has two symmetric homoclinic orbits; if $|g|<g_0$ and $g\neq 0$, then system (1.4) has two nonsymmetric

Table 1
Bifurcation of phase portraits of system (1.4) when n is even and (c, a) is in (I) or (III)

(c,a) \ g	$g<-g_0$	$g=-g_0$	$-g_0<g<0$	$g=0$	$-g_0<g<0$	$g=g_0$	$g>g_0$
$(c,a)\in$(I)							
$(c,a)\in$(III)							

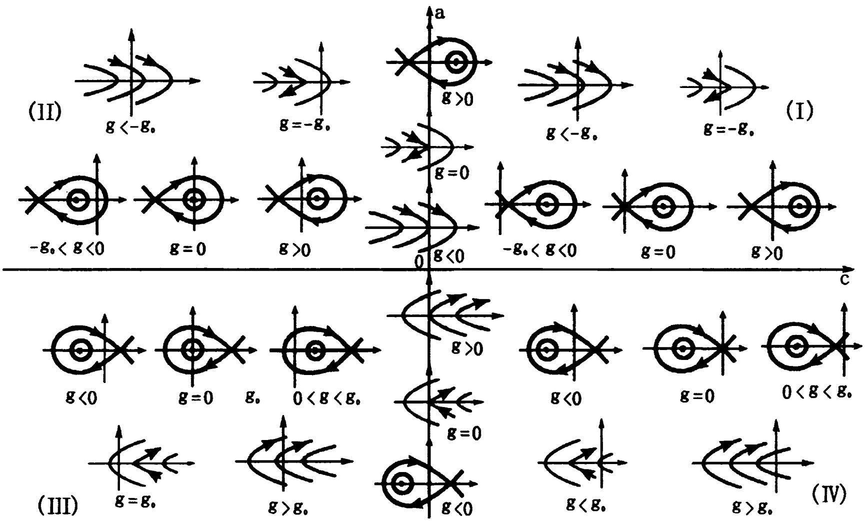


Fig. 2. Bifurcation of phase portraits of system (1.4) when n is odd.

homoclinic orbits; if $|g| > g_0$, then system (1.4) has no homoclinic orbit. When n is even and $(c, a) \in$ (III), from Table 1 we see that if $|g| < g_0$ and $g \neq 0$, then system (1.4) has one homoclinic orbit; (ii) if $g = 0$, then system (1.4) has two heteroclinic orbits; (iii) if $|g| \geqslant g_0$, then system (1.4) has no homoclinic orbit. When n is even and (c, a) is in one of c^+, c^-, a^+, a^-, (II) and (IV), system (1.4) has only a singular point and it is center point or saddle point, thus system (1.4) has no homoclinic orbit. From Lemma 2.1 the (1), (2) and (3) are proved. When n is odd, we have bifurcation of phase portraits of system (1.4) shown in Fig. 2. To discuss Fig. 2 similarly to above, the proof of (4)–(8) can be completed. □

3. Solitary wave solutions of Eq. (1.1) under some parameter conditions

In this section we give solitary wave solutions of Eq. (1.1) for case $n = 1$ or $n = 2$ or general n and $g = 0$. For system (1.4), let $(\phi_s, 0)$ be a saddle point connected homoclinic orbit $\Gamma(h_s)$. Thus from (1.5) we can write the expression of homoclinic orbit $\Gamma(h_s)$ in (ϕ, y) plane as follows:

$$y^2 = 2\left(h_s + g\phi + \frac{c}{2}\phi^2 - \frac{a}{(n+1)(n+2)}\phi^{n+2}\right), \tag{3.1}$$

where

$$h_s = \phi_s\left(\frac{a}{(n+1)(n+2)}\phi_s^{n+1} - \frac{c}{2}\phi_s - g\right). \tag{3.2}$$

Substituting (3.1) into $d\phi/d\xi = y$, it follows

$$\frac{d\phi}{\left(h_s + g\phi + \frac{c}{2}\phi^2 - \frac{a}{(n+1)(n+2)}\phi^{n+2}\right)^{1/2}} = \pm\sqrt{2}\,d\xi. \tag{3.3}$$

Under general conditions it is difficult to integrate the left of (3.3). But under some conditions the integral can be finished. Thus the solitary wave solutions of Eq. (1.1), can be obtained.

3.1. Solitary wave solutions of Eq. (1.1) *when $n = 1$*

When $n = 1$, suppose $(\phi_0, 0)$ is saddle point connected homoclinic orbit of system (1.4). Then ϕ_0 is as follows:

$$\phi_0 = \begin{cases} \frac{c}{a} - \left(\frac{c^2}{a^2} + \frac{2g}{a}\right)^{1/2}, & \text{for } a > 0, \\ \frac{c}{a} + \left(\frac{c^2}{a^2} + \frac{2g}{a}\right)^{1/2}, & \text{for } a < 0. \end{cases} \tag{3.4}$$

Let h_0 be as follows:

$$h_0 = \phi_0\left(\frac{a}{6}\phi_0^2 - \frac{c}{2}\phi_0 - g\right). \tag{3.5}$$

Then the expression of homoclinic orbit is

$$y^2 = 2\left(h_0 + g\phi + \frac{c}{2}\phi^2 - \frac{a}{6}\phi^3\right) = \frac{a}{3}(\phi - \phi_0)^2\left(\frac{3c}{a} - 2\phi_0 - \phi\right). \tag{3.6}$$

Substituting (3.6) into $d\phi/d\xi = y$ and integrating along homoclinic orbit, we obtain the parameter expression of homoclinic orbit of system (1.4):

$$\begin{cases} \phi(\xi) = \frac{1}{a}\left[c + 2(c^2 + 2ag)^{1/2} \right. \\ \qquad\qquad \left. - 3(c^2 + 2ag)^{1/2}\tanh^2\frac{(c^2+2ag)^{1/4}\xi}{2}\right], \\ y(\xi) = \phi'(\xi). \end{cases} \tag{3.7}$$

From (3.7) and Lemma 2.1 we know that equation

$$u_t + auu_x + u_{xxx} = 0 \tag{3.8}$$

has solitary wave solution

$$u(x,t) = \frac{1}{a}\bigg[c + 2(c^2 + 2ag)^{1/2} - 3(c^2 + 2ag)^{1/2}\tanh^2\frac{(c^2 + 2ag)^{1/4}(x - ct)}{2}\bigg], \tag{3.9}$$

where $a \neq 0$ and $c^2 + 2ag > 0$.

Example 1. Let $a = 6$ and $g = 0$. (3.9) is reduced to

$$u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right). \tag{3.10}$$

(3.10) is well known as a solitary wave solution [9] of classical KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{3.11}$$

Obviously, the function

$$u_0(x,t) = \frac{1}{6}\bigg[c + 2(c^2 + 12g)^{1/2} - 3(c^2 + 12g)^{1/2}\tanh^2\frac{(c^2 + 12g)^{1/4}(x - ct)}{2}\bigg] \tag{3.12}$$

is solitary wave solution of KdV equation (3.11) and more general than (3.10), where $g > -c^2/12$.

We have not seen $u_0(x,t)$ in any others papers. So we think that it is a new solitary wave solution of KdV equation.

3.2. The solitary wave solutions of Eq. (1.1) *when* $n = 2$

(i) Given $a > 0$, $c > 0$, $|g| < g_0$ and $g \neq 0$, then $f(\phi) = g + c\phi - (a/3)\phi^3$ has three real zero points, and they can be found out by the finding root formula of cubic equation, say ϕ_0, ϕ_1, ϕ_2 and $\phi_1 < \phi_0 < \phi_2$. Because $g \neq 0$, we know that $\phi_0 \neq 0$ and $(\phi_0, 0)$ is saddle point of system (1.4) (see Fig. 3).

At saddle point $(\phi_0, 0)$ the invariant h_1 is

$$h_1 = \frac{a}{12}\phi_0^4 - \frac{c}{2}\phi_0^2 - g\phi_0. \tag{3.13}$$

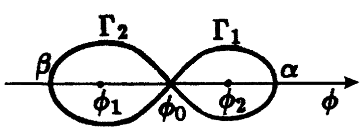


Fig. 3. Homoclinic orbits of system (1.4) when $n = 2$, $a > 0$, $c > 0$, $|g| < g_0$ and $g \neq 0$.

Thus we have the expression of two homoclinic orbits connected at $(\phi_0, 0)$:

$$y^2 = 2h_1 + 2\left(g + \frac{c}{2}\phi - \frac{a}{12}\phi^4\right) = \frac{a}{6}(\phi - \phi_0)^2(\alpha - \phi)(\phi - \beta), \tag{3.14}$$

where

$$\alpha = \sqrt{\left|\frac{6g}{a\phi_0}\right|} - \phi_0, \qquad \beta = -\phi_0 - \sqrt{\left|\frac{6g}{a\phi_0}\right|}. \tag{3.15}$$

Substituting (3.14) into $d\phi/d\xi = y$, then integrating along homoclinic orbit Γ_1 and letting

$$a_0 = \left|\frac{6g}{a\phi_0}\right| - 4\phi_0^2, \qquad \alpha_0 = \sqrt{\left|\frac{aa_0}{6}\right|}, \qquad \beta_0 = 2\sqrt{\left|\frac{6g}{a\phi_0}\right|}, \tag{3.16}$$

we obtain the parameter expression of Γ_1 as follows:

$$\Gamma_1\colon\quad \phi(\xi) = \frac{2a_0}{\beta_0 \cosh \alpha_0\xi + 4\phi_0} + \phi_0, \quad y(\xi) = \phi'(\xi). \tag{3.17}$$

By the same method we get the parameter expression of Γ_2:

$$\Gamma_2\colon\quad \phi(\xi) = \phi_0 - \frac{2a_0}{\beta_0 \cosh \alpha_0\xi - 4\phi_0}, \quad y(\xi) = \phi'(\xi). \tag{3.18}$$

From (3.17), (3.18) and Lemma 2.1 we know that when $a > 0$, $c > 0$, $|g| < g_0$ and $g \neq 0$, equation

$$u_t + au^2u_x + u_{xxx} = 0 \tag{3.19}$$

has four solitary wave solutions:

$$u(x,t) = \pm\left(\phi_0 - \frac{2a_0}{\beta_0 \cosh \alpha_0(x - ct) - 4\phi_0}\right) \tag{3.20}$$

and

$$u(x,t) = \pm\left(\phi_0 + \frac{2a_0}{\beta_0 \cosh \alpha_0(x - ct) + 4\phi_0}\right). \tag{3.21}$$

Example 2. Taking $(c, a) = (7, 3)$ and $|g| = 6$, it follows $\phi_0 = -1$ or $\phi_0 = 1$. Further we have $a_0 = 8$, $\alpha_0 = 2$, $\beta_0 = 4\sqrt{3}$. Thus we know that equation

$$u_t + 3u^2u_x + u_{xxx} = 0 \tag{3.22}$$

has four solitary wave solutions:

$$u(x,t) = \pm\left(1 - \frac{4}{\sqrt{3}\cosh(2x - 14t) - 1}\right) \tag{3.23}$$

and

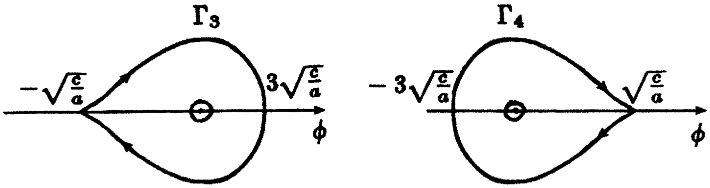


Fig. 4. Homoclinic orbits when $n = 2$, $a > 0$, $c > 0$ and $|g| = g_0$.

$$u(x,t) = \pm\left(1 + \frac{4}{\sqrt{3}\cosh(2x - 14t) + 1}\right). \tag{3.24}$$

(ii) Given $c > 0$, $a > 0$ and $|g| = g_0$, then the singular point connected homoclinic orbit is a degenerate saddle point (see Fig. 4).

In Fig. 4 the expression of homoclinic orbits is

$$y^2 = \frac{a}{6}\left(3\sqrt{\frac{c}{a}} - \phi\right)\left(\phi + \sqrt{\frac{c}{a}}\right)^3, \tag{3.25}$$

or

$$y^2 = \frac{a}{6}\left(\phi + 3\sqrt{\frac{c}{a}}\right)\left(\sqrt{\frac{c}{a}} - \phi\right)^3. \tag{3.26}$$

Substituting y into $d\phi/d\xi = y$ and integrating it along homoclinic orbits, we have the parameter expressions of homoclinic orbits as follows:

$$\Gamma_3\colon\quad \phi(\xi) = \sqrt{\frac{c}{a}}\frac{9 - 2c\xi^2}{3 + 2c\xi^2}, \quad y(\xi) = \phi'(\xi), \tag{3.27}$$

$$\Gamma_4\colon\quad \phi(\xi) = \sqrt{\frac{c}{a}}\frac{2c\xi^2 - 9}{2c\xi^2 + 3}, \quad y(\xi) = \phi'(\xi). \tag{3.28}$$

From (3.27) and (3.28) we obtain others two wave solutions of Eq. (3.19):

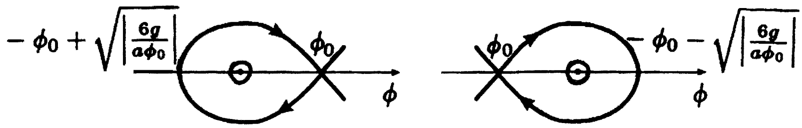
$$u(x,t) = \pm\sqrt{\frac{c}{a}}\frac{2c(x - ct)^2 - 9}{2c(x - ct)^2 + 3}, \tag{3.29}$$

where $a > 0$, $c > 0$.

(iii) Given $c > 0$, $a > 0$ and $g = 0$, then the singular point connected two symmetric homoclinic orbits is (0, 0). The expression of two homoclinic orbits is

$$y^2 = \phi^2\left(c - \frac{a}{6}\phi^2\right). \tag{3.30}$$

From (3.30) and $d\phi/d\xi = y$ we get others two solitary wave solutions of Eq. (3.19):



(i) $0 < g < g_0$ (ii) $-g_0 < g < 0$

Fig. 5. Homoclinic orbits when $n = 2$, $a < 0$, $c < 0$, $|g| < g_0$ and $|g| \neq 0$.

$$u(x,t) = \pm\sqrt{\frac{6c}{a}}\,\operatorname{sech}\sqrt{c}(x-ct), \tag{3.31}$$

where $a > 0$, $c > 0$.

Example 3. Taking $a = 6$ and $n = 2$, Eq. (1.1) reduces to classical MKdV equation

$$u_t + 6u^2u_x + u_{xxx} = 0. \tag{3.32}$$

It is well known that (3.32) has solitary wave solutions [9]

$$u = \pm\sqrt{c}\,\operatorname{sech}\big(\sqrt{c}x - c\sqrt{c}t\big), \quad c > 0. \tag{3.33}$$

On the other hand, if $0 < g < \sqrt{6}c^{3/2}/9$, the MKdV equation (3.32) has four solitary wave solutions as (3.20) and (3.21), where $a = 6$ and ϕ_0 is the biggest between two negative roots of $g + c\phi - 2\phi^3 = 0$. When $g = \sqrt{6}c^{3/2}/9$, MKdV equation (3.32) has two solitary wave solutions

$$u(x,t) = \pm\sqrt{\frac{c}{6}}\frac{9-2c(x-ct)^2}{3+2c(x-ct)^2}, \tag{3.34}$$

where $c > 0$.

Clearly, the solitary wave solutions in (3.20), (3.21) and (3.29) are different from (3.31) and we have not found them in any other paper. So we think that they are new solitary wave solutions of MKdV equation.

(iv) Given $c < 0$, $a < 0$, $|g| < g_0$ and $|g| \neq 0$, then the saddle point and homoclinic orbits are given in Fig. 5.

In Fig. 5 the ϕ_0 is the biggest among three real roots of $y + c\phi - (a/3)\phi^3 = 0$ when $0 < g < g_0$, and the ϕ_0 is the smallest among three real roots of $y + c\phi - (a/3)\phi^3 = 0$ when $-g_0 < g < 0$. Thus ϕ_0 is not zero and expression of homoclinic orbits is

$$y^2 = \frac{|a|}{6}(\phi-\phi_0)^2\left(\phi+\phi_0-\sqrt{\left|\frac{6g}{a\phi_0}\right|}\right)\left(\phi+\phi_0+\sqrt{\left|\frac{6g}{a\phi_0}\right|}\right). \tag{3.35}$$

Substituting (3.35) into $d\phi/d\xi = y$ and integrating it along homoclinic orbit, according to Lemma 2.1, we get two solitary wave solutions of Eq. (3.19):

$$u(x,t) = \pm\left(|\phi_0| - \frac{2a_1}{\beta_1 \cosh \alpha_1(x-ct) + 4\phi_0}\right), \tag{3.36}$$

where

$$a_1 = 4\phi_0^2 - \left|\frac{6g}{a\phi_0}\right|, \qquad \alpha_1 = \sqrt{\left|\frac{aa_1}{6}\right|}, \qquad \beta_1 = 2\sqrt{\left|\frac{6g}{a\phi_0}\right|}. \tag{3.37}$$

Example 4. Taking $a = -3$, $c = -7$ and $|g| = 6$, it follows $\phi_0 = 2$ or $\phi_0 = -2$, $a_1 = 10$, $\alpha_1 = \sqrt{5}$, $\beta_1 = 2\sqrt{6}$. Thus equation

$$u_t - 3u^2 u_x + u_{xxx} = 0 \tag{3.38}$$

has two solitary wave solutions:

$$u = \pm\left(2 - \frac{10}{\sqrt{6}\cosh\sqrt{5}(x+7t) + 4}\right). \tag{3.39}$$

3.3. The solitary wave solutions of Eq. (1.1) *for* $n \geqslant 1$ *and* $g = 0$.

(i) When n is even number, given $a > 0$, $c > 0$ and $g = 0$, then $(0,0)$ is saddle point connected two symmetric homoclinic orbits of system (1.4) (see Table 1). At $(0,0)$ the invariant h at $(0,0)$ is zero. Thus the two homoclinic orbits have expression

$$y^2 = \phi^2\left(c - \frac{2a}{(n+1)(n+2)}\phi^n\right). \tag{3.40}$$

Substituting (3.40) into $d\phi/d\xi = y$ and integrating it along the homoclinic orbit, we obtain two solitary wave solution of Eq. (1.1):

$$u(x,t) = \pm\left(\frac{(n+1)(n+2)c}{2a}\operatorname{sech}^2\frac{n\sqrt{c}(x-ct)}{2}\right)^{1/n}, \tag{3.41}$$

where $a > 0$, $c > 0$.

(ii) When n is add number, given $g = 0$ and $a > 0$, $c > 0$ or $a < 0$, $c > 0$, the solitary wave solution of Eq. (1.1) is as (3.41) when taking "+."

Example 5. Taking $n = p$, $a = p + 1$ and $c = \alpha^2$, (3.41) transforms into the solitary wave solution in [4]:

$$u^p(x,t) = \frac{(p+2)\alpha^2}{2}\operatorname{sech}^2\frac{p}{2}(\alpha x - \alpha^3 t). \tag{3.42}$$

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