BULLETIN
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Matrix valued Jacobi polynomials

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Received 15 January 2003; accepted 15 January 2003

Abstract

For every value of the parameters $\alpha, \beta > -1$ we find a matrix valued weight whose orthogonal polynomials satisfy an explicit differential equation of Jacobi type.

Résumé

Pour chaque choix des parameters $\alpha, \beta > -1$ nous trouvons un poids matriciel pour lequel les polynômes orthogonaux satisfont une équation différentielle de Jacobi.

MSC: 33C45; 22E45

Keywords: Matrix valued orthogonal polynomials; Bochner problem; Jacobi polynomials

1. Introduction

It would be very hard to overstate the role that the Jacobi polynomials

$$P_j^{(\alpha,\beta)}(t)$$

have played in mathematics over the last few centuries.

Their origin is firmly rooted in classical problems in mathematical physics ranging from potential theory to electromagnetism, etc. where most of the time the parameters α, β vanish and give rise to the celebrated special case of the Legendre polynomials already studied by Laplace. The study of spherical harmonics requires the consideration of other values of the parameters. Laplace needed to establish their "addition formula" and did it in 1782. In the case of the Gegenbauer polynomials ($\alpha = \beta$) this was established in 1875.

[1] The author was supported in part by NSF Grant # FD99711151.

0007-4497/03/$ – see front matter
doi:10.1016/S0007-4497(03)00009-5

The extension to an arbitrary choice of these parameters was only accomplished in 1972 by T. Koornwinder. By integrating the addition formula with respect to one of the variables one obtains the so-called product formulas. We return to them later. A large number of other applications of Jacobi polynomials in areas like approximation theory, interpolation, etc. can be considered as the place where lots of functional analysis was born.

For some of these results as well as for a wealth of related facts see [1].

We will not be interested here in special results like those just mentioned, but the dates given above illustrate well an important phenomenon: while Jacobi polynomials may arise first for special values of the parameters in a concrete problem and certain special results are obtained in such a case, it usually takes a long time to extend these results to general values of the parameters.

Another important development, this one from the 20th century, allows one to put under one roof a number of isolated results for several families of special functions, including Jacobi polynomials. This is the theory of spherical functions developed initially by E. Cartan and H. Weyl.

The product formula satisfied by the Jacobi polynomials was taken by E. Cartan as the inspiration for his definition of spherical functions related to a symmetric space G/K where G is Lie group and K a compact subgroup of it. The definition adopted by Cartan will be given below in a way that gives the background for this paper.

There are *two* directions in which Jacobi polynomials have been extended in recent decades.

On the one hand there is the work of Askey and Wilson, see [2], where the hypergeometric character of the Jacobi polynomials is taken over by the so called basic-hypergeometric functions and one obtains the so-called Askey–Wilson polynomials. They have been seen to play many of the roles formerly played by the Jacobi polynomials: for instance they give matrix entries for representations of quantum groups. On a different vein, one may wonder for the case where instead of one variable t one allows for several variables. This would be needed for instance if one were considering symmetric spaces of rank higher than one. Here there is extremely important work by E. Opdam and G. Heckman. These polynomials are very likely to play the role of higher-dimensional analogs of the Jacobi ones. It may be relevant to notice that although a fully developed theory exist in this case, with the analogs of a recursion relation as well as a differential equation, one still *does not* have some of the explicit formulas that are so useful in the case of one variable. For more details of this work see [19,23]. The work of Opdam and Heckman is part of a larger program known under the general name of Macdonald polynomials. For a view of these topics, see [8].

After one definition we will be in a position to explain the results of this paper. They give rise to a *third* extension of Jacobi polynomials, namely the case when they take values in a set of matrices of a given size. Our polynomials will depends on only one variable t but it is clear that all the extensions alluded to above and the one considered here could be combined in several ways. In conjunction with a viewpoint that regards Jacobi polynomials (and similar special functions) as part of a fascinating game with many possible applications, as given for instance in [27], this could turn out to be an interesting way to combine the old and the new.

2. Matrix valued spherical functions

Consider functions defined on G with values in $\mathrm{End}(V)$, where V is any finite-dimensional complex vector space. Let $\widehat{K}$ denote the set of all equivalence classes of complex finite-dimensional irreducible representations of K; for each $\delta \in \widehat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, and $\chi_\delta = d(\delta)\xi_\delta$. A spherical function Φ on G (see [26]) of type $\delta \in \widehat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that $\Phi(e) = 1$ and

$$\Phi(x)\Phi(y) = \int_K \chi_\delta\big(k^{-1}\big)\Phi(xky)\,\mathrm{d}k, \quad x, y \in G.$$

In the case when the representation of K is the trivial one, this gives the original definition of Cartan and as long as one is dealing with the rank one case the Jacobi polynomials (in the compact case) and more generally the hypergeometric function (for certain values of the parameters, the so-called group values) account for the spherical functions. This is the content of some fundamental papers of Harish-Chandra.

In [12] one finds a detailed elaboration of this theory for any K-type when the symmetric space G/K is the complex projective plane.

The results in [12–14] yield, apparently for the first time, examples of matrix valued Jacobi polynomials where the parameters α, β take the values $\alpha \in Z_{\geqslant 0}$, $\beta = 1$. This comes about, as stated above, from the study of the matrix valued spherical functions of a specific symmetric space, namely the complex projective space $P_2(\mathbb{C})$. The choice of this example allows for fairly explicit computations carried out in [12]. While one could in principle attempt the same computations for higher-dimensional projective spaces, where higher integer values of β could play a role, this has not been done yet.

The point of the present paper is to give a completely explicit description of an extension of the theory in the papers mentioned above to arbitrary values of the parameters α, β where there is no symmetric space around. For simplicity we give all results quite explicitly in the case of two-by-two matrices. The case of matrices of arbitrary size will be dealt with in [15] where a number of difficulties that are absent in the two-by-two case require special attention. For emphasis we stress that in the case of $\beta = 1$, corresponding to the case of the complex projective plane we obtain examples of matrices of arbitrary size, with the classical case corresponding to size one and the further specialization $\alpha = 0$.

3. The Jacobi polynomials: matrix valued classical polynomials

Consider the matrix valued polynomials given by the 3-term recursion relation

$$A_n\Phi_{n-1}(t) + B_n\Phi_n(t) + C_n\Phi_{n+1}(t) = t\Phi_n(t), \quad n \geqslant 0,$$
$$\Phi_{-1}(t) = 0, \quad \Phi_0(t) = I,$$

where the entries in A_n, B_n, C_n are given in a section at the end of the paper. This is not the symmetric form of the recursion relation, but it has the advantage of avoiding square roots. The reader will notice that, as in [12] and [13], the expressions for the entries of A_n

and C_n are simpler than those corresponding to the (diagonal of) the matrix B_n. There are simple expressions for the diagonal matrices that conjugate the recursion relation above into a symmetric form.

If the matrix $\Psi_0(t)$ is given by

$$\Psi_0(t) = \begin{pmatrix} 1 & 1 \\ 1 & \frac{(\beta+2\alpha+3)t}{\beta+1} - \frac{2(\alpha+1)}{\beta+1} \end{pmatrix}$$

one can see that the polynomials $\Phi_n(t)$ satisfy the orthogonality relation

$$\int_0^1 \Phi_i(t) M(t) \Phi_j^{\mathrm{T}}(t)\,\mathrm{d}t = 0 \quad \text{if } i \neq j,$$

where

$$M(t) = \Psi_0(t) \begin{pmatrix} (1-t)^\beta t^{\alpha+1} & 0 \\ 0 & (1-t)^\beta t^\alpha \end{pmatrix} \Psi_0^{\mathrm{T}}(t).$$

The polynomials $\Phi_n(t)$ are "classical" in the sense that they are eigenfunctions of a fixed second order differential operator. More precisely, we have

$$\mathcal{F}\Phi_n^{\mathrm{T}} = \Phi_n^{\mathrm{T}} \Lambda_n$$

where

$$\Lambda_n = \begin{pmatrix} -n^2 - (\alpha+\beta+2)n + \alpha + 1 + \frac{\beta+1}{2} & 0 \\ 0 & -n^2 - (\alpha+\beta+3)n \end{pmatrix}$$

and $\mathcal{F}$ is given by

$$\begin{aligned} \mathcal{F} \;=\; & t(1-t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \\ & + \begin{pmatrix} \frac{(\alpha+1)(\beta+2\alpha+5)}{\beta+2\alpha+3} - (\alpha+\beta+3)t & \frac{2\alpha+2}{2\alpha+\beta+3} + t \\ \frac{\beta+1}{\beta+2\alpha+3} & \frac{(\alpha+2)\beta+2\alpha^2+5\alpha+4}{\beta+2\alpha+3} - (\alpha+\beta+4)t \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \\ & + \begin{pmatrix} \alpha + 1 + \frac{\beta+1}{2} & 0 \\ 0 & 0 \end{pmatrix} I. \end{aligned}$$

We have only given the 3-term recursion relation as well as the second order differential equation satisfied by these polynomials. For certain uses this is the only thing one needs. For instance the beautiful electrostatic interpretation of the zeros of the scalar Jacobi polynomials due to Stieltjes, [25], depends only on the differential equation. We give below, as an illustration, the values of our polynomials at $t = 0$ and at $t = 1$. These values play an important role in the general expression for $\Phi_i(t)$ which is given in [15]. For the two-by-two case manuscript submitted.

For the value of $(-1)^i\ \Phi_i(t)$ at $t = 0$ we have

$$\begin{pmatrix} \frac{(\alpha+1)_i(\beta+2i+1)(\beta+2\alpha+2i+3)}{(\beta+1)(\beta+2)_i(\beta+2\alpha+3)} & \frac{2i(\alpha+2)_i(\alpha+\beta+i+2)}{(\beta+2)_i(\alpha+i+1)(\beta+2\alpha+3)} \\ 0 & \frac{(\alpha+2)_i}{(\beta+2)_i} \end{pmatrix}$$

and for the value of $\Phi_i(t)$ at $t = 1$ we have

$$\begin{pmatrix} \frac{(\beta+i+2)(\beta+2\alpha+i+3)+i(i-1)}{(\beta+2)(\beta+2\alpha+3)} & \frac{-2i(\alpha+\beta+i+2)}{(\beta+2)(\beta+2\alpha+3)} \\ \frac{-2i(\alpha+\beta+i+3)}{(\beta+2)(\beta+2\alpha+3)} & \frac{(\beta+i+2)(\beta+2\alpha+i+3)+(i+1)i}{(\beta+2)(\beta+2\alpha+3)} \end{pmatrix}.$$

Here the symbol $(a)_i$ denotes the usual ascending factorial.

4. The problem of Bochner and the Darboux process

In the scalar valued case, Bochner [3] posed and solved the problem of finding all families of orthogonal polynomials that are eigenfunctions of some fixed second order differential operator. Among the possible families one finds the Jacobi polynomials as the most interesting ones since they depend on two arbitrary parameters. In [10] one sees how the so-called Darboux process consisting in the factorization of a tridiagonal matrix into the poduct of an upper times a lower factor followed by the multiplication of the factors in the reverse order can be used to obtain solutions of the Krall problem, namely a matrix version of the *bispectral problem* posed in the purely continuous case in [5]. This is an extension of the Bochner problem in that the differential operator sought for is allowed to be of arbitrary finite order.

While this classification should be attempted in the matrix-valued case, see [6] for pioneering work in this direction, it is clear that this is a very hard problem.

One possible way of looking at the results in [12] is to say that by going into a situation with a high degree of symmetry, namely an underlying symmetric space, we were able to uncover a bispectral situation that had not been found before. In that vein, the results here show that the situation is fairly robust and one can do without so much symmetry, getting the same results even away from *group values* for the parameters.

It should be pointed out that the polynomials constructed here, for arbitrary values of the parameters α, β may not be the only ones naturally associated with a Jacobi weight. The construction given here is just an extension of the one arrived at in [12–14]. The full story will be contained in an extension to the non-commutative case of the Bochner's classification. For previous work in this direction see [6,20,21] and [4]. For a general reference to matrix valued orthogonal polynomials, see [7,22,24].

If one considers the case of doubly infinite matrices, and thus abandons the insistence on polynomials $p_n(t)$, one finds that Gauss' hypergeometric equation plays a crucial role, see [9,17]. For some first results and examples in the doubly infinite matrix valued case involving differential operators of first order, see [11]. This and similar versions of the "bispectral problem", see [5], could play a role in some nonabelian versions of Toda and KdV systems. One can also speculate that the analysis of this nonabelian version of the bispectral problem could lead to some matrix valued form of Gauss' equation as indicated above. If this were the case then one could try to isolate a regular solution of it and use this matrix valued function in the fashion that one used the scalar hypergeometric function. This could give a nice expression for the matrices $\Phi_n(t)$ given above. Note that even in the special case discussed in [12] the entries of the spherical functions given there required the use of *generalized* hypergeometric functions. One could hope that a good matrix analog of

Gauss' hypergeometric function could do a better job. For an important recent development in this area see [27].

Jacobi polynomials serve as good starting places when one wants to apply the Darboux process to obtain solutions of the problem of Bochner involving higher order differential operators, see [16]. Attempting this in the matrix valued case remains an interesting challenge. As one sees in [16] it is very important to be able to deal with more or less general values of the parameters α, β.

In conclusion it is important to point out that extensions of Jacobi polynomials either to the case of q-difference equations, see the pioneering paper [2], or to the case of several variables, see [8,19,23] are very important topics of current research. For a very nice survey of many developments in the q-world starting with [9] see the recent work [18]. One can only hope that the extensions discussed here, maybe in different combinations with the ones mentioned above, would continue to enrich a set of tools that has proved useful in the past.

5. The coefficients in the recursion relation

Here we give the explicit expressions of the (matrix) coefficients that appear in the three term recursion relation that defines the Jacobi polynomials in the two-by-two case. Notice that the matrices A_n and C_n are upper and lower triangular respectively.

$$
\begin{aligned}
A_n^{11} &:= \frac{n(\alpha+n)(\beta+2\alpha+2n+3)}{(\beta+\alpha+2n+1)(\beta+\alpha+2n+2)(\beta+2\alpha+2n+1)},\\
A_n^{12} &:= \frac{2n(\beta+1)}{(\beta+2n+1)(\beta+\alpha+2n+2)(\beta+2\alpha+2n+1)},\\
A_n^{21} &:= 0,\\
A_n^{22} &:= \frac{n(\alpha+n+1)(\beta+2n+3)}{(\beta+2n+1)(\beta+\alpha+2n+2)(\beta+\alpha+2n+3)},\\
B_n^{11} &:= 1+\frac{n(\beta+n+1)(\beta+2n-1)}{(\beta+2n+1)(\beta+\alpha+2n+1)}-\frac{(n+1)(\beta+n+2)(\beta+2n+1)}{(\beta+2n+3)(\beta+\alpha+2n+3)}\\
&\quad -\frac{2(\beta+1)^2}{(\beta+2n+1)(\beta+2n+3)(\beta+2\alpha+2n+3)},\\
B_n^{12} &:= \frac{2(\beta+1)(\alpha+\beta+n+2)}{(\beta+2n+3)(\beta+\alpha+2n+2)(\beta+2\alpha+2n+3)},\\
B_n^{21} &:= \frac{2(\alpha+n+1)(\beta+1)}{(\beta+2n+1)(\beta+\alpha+2n+3)(\beta+2\alpha+2n+3)},\\
B_n^{22} &:= 1+\frac{n(\beta+n+1)(\beta+2n+3)}{(\beta+2n+1)(\beta+\alpha+2n+2)}-\frac{(n+1)(\beta+n+2)(\beta+2n+5)}{(\beta+2n+3)(\beta+\alpha+2n+4)}\\
&\quad +\frac{2(\beta+1)^2}{(\beta+2n+1)(\beta+2n+3)(\beta+2\alpha+2n+3)},
\end{aligned}
$$

and finally for the entries of C_n we have

$$
\begin{aligned}
C_n^{11} &:= \frac{(\beta+n+2)(\beta+2n+1)(\beta+\alpha+n+2)}{(\beta+2n+3)(\beta+\alpha+2n+2)(\beta+\alpha+2n+3)},\\
C_n^{12} &:= 0,\\
C_n^{21} &:= \frac{2(\beta+1)(\beta+n+2)}{(\beta+2n+3)(\beta+\alpha+2n+3)(\beta+2\alpha+2n+5)},\\
C_n^{22} &:= \frac{(\beta+n+2)(\beta+\alpha+n+3)(\beta+2\alpha+2n+3)}{(\beta+\alpha+2n+3)(\beta+\alpha+2n+4)(\beta+2\alpha+2n+5)}.
\end{aligned}
$$

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