Monotone Iterative Techniques for Mildly Nonlinear Differential Equations

YAU SHU WONG AND XINGZHI JI
Department of Mathematics, University of Alberta
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Abstract. This paper discusses from the computational point of view the convergence and error bounds of monotone iterative methods for mildly nonlinear differential equations.

1. INTRODUCTION
The method of upper and lower solutions is an important and fruitful technique for proving existence results for nonlinear differential equations. It also leads to a construction of monotone iterative procedures for numerical solutions of nonlinear problems, for instance [1-4].

In this paper, we examine monotone schemes for a second order elliptic boundary value problem (BVP)

\[-Lu = f(x,u), \text{ in } \Omega \subset \mathbb{R}^m \]
\[Bu = \phi, \quad \text{on } \partial \Omega \]  

where \(L = \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i} + c(x)\) is strictly uniformly elliptic in \(\Omega\), and \(Bu = p(x)u + q(x) \frac{\partial u}{\partial n}\) (\(\partial u/\partial n\) denotes the normal derivative of \(u\)). Our results, especially Theorem 2.1, however, can be easily extended to the corresponding mildly nonlinear problems of ordinary differential equations, or parabolic and hyperbolic partial differential equations.

2. CONVERGENT PROPERTY AND ERROR BOUND
Consider the BVP problem (1), we assume that the boundary \(\partial \Omega\) and the coefficient functions \(a_{ij}, b_i, c, f, p, q, \phi\) satisfy certain appropriate conditions such that the lower and upper solution sequences \(\{v_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}\) can be constructed by the following iterative scheme [5, Ch. 3]

\[-Lu_{n+1} = f(x,v_n) - M(v_{n+1} - v_n), \text{ in } \Omega \]
\[Bu_{n+1} = \phi, \quad \text{on } \partial \Omega \]  

where \(M\) satisfies the relation

\[f(x,v_1) - f(x,v_2) \geq -M(v_1 - v_2), \]

whenever \(v_0 \leq u_1 \leq u_2 \leq w_0\). \(v_0, w_0 \in C^2(\bar{\Omega})\) are the starting lower and upper solutions in which \(v_0 \leq w_0\) and

\[0 \leq Lv_0 + f(x,v_0), \quad Bv_0 \leq \phi, \]
\[0 \geq Lw_0 + f(x,w_0), \quad Bw_0 \geq \phi. \]

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Then it can be proven that the sequences \( \{v_n\} \) and \( \{w_n\} \) converge monotonically to \( \rho, r \) respectively. Moreover, \( \rho, r \) are the minimal and maximal solutions of the BVP (1).

Now we are concerned with the numerical solution of the BVP (1) by a monotone technique. For the sake of simplicity, we consider only the lower solution sequence. Assume that \( v_0 \) is a given lower solution and \( \Omega_h \) is a uniform mesh for \( \Omega \). By the application of a discretization technique to the problem (2), we obtain the difference equations (including the boundary conditions) as follows

\[
(-L_h + M)v_{h+1} = f_h(x_h, v_h) + M v_h, \quad n = 0, 1, \ldots.
\]

For the convergence and error bound of the grid function sequence \( \{v_h\}_{h=0}^{\infty} \), we have

**Theorem 2.1.** Suppose that the right-hand side function \( f(x, u) \) of Eq. (1) satisfies the Lipschitz condition about \( u \) with the Lipschitz constant \( L \), and a consistent and stable difference scheme which is accurate of order \( k \) is employed for Eq. (4). Then \( \{v_h\}_{h=0}^{\infty} \) converges monotonically (with oscillation \( O(h^k) \)) to the grid function \( v_h(x_h) \), where \( v_h(x_h) \) is the restriction of the minimal solution \( \rho(x) \) of Eq. (1) on the mesh \( \Omega_h \). Moreover, when \( n \) is sufficiently large, we have the error estimate

\[
\|v_h - \rho_h(x_h)\| \leq c h^k,
\]

where \( c \) is a constant, and \( \| \cdot \| \) denotes a vector norm.

**Proof:** Let \( \bar{v}_{h+1} \) be the solution of

\[
(-L_h + M)v_{h+1} = f_h(x_h, v_h) + M v_h, \quad n = 0, 1, \ldots,
\]

where \( v_n \) is the restriction of \( v_n \) on \( \Omega_h \).

Without loss of generality, we assume that \( \|v_0 - v_0\| \leq c_1 h^k \). Then

\[
\|v_{h+1} - \rho_h(x_h)\| \leq \|v_{h+1} - v_{1,h}\| + \|v_{1,h} - \rho_h\| \\
\leq \|v_{h+1} - \bar{v}_{h+1}\| + \|\bar{v}_{h+1} - v_{1,h}\| + \|v_{1,h} - \rho_h\|.
\]

By the stability of the difference scheme

\[
\|v_{h+1} - \bar{v}_{h+1}\| \leq c_2(\|f_h(x_h, v_0) + M v_0 - f_h(x_h, v_0) + M v_0\|) \\
\leq c_2(L + M)\|v_0 - v_0\| \\
\leq c_3 h^k.
\]

Since the difference equation (6) is accurate of order \( k \)

\[
\|\bar{v}_{h+1} - v_{1,h}\| \leq c_4 h^k.
\]

Hence

\[
\|v_{h+1} - \rho_h(x_h)\| \leq \|v_{1,h} - \rho_h(x_h)\| + c h^k.
\]

It is easy to verify by induction that

\[
\|v_n - \rho_h(x_h)\| \leq \|v_n, - \rho_h(x_h)\| + c h^k.
\]

Since \( \{v_n,h\} \) is pointwise increasingly convergent to \( \rho_h(x_h) \) on the \( \Omega_h \), \( \{v_n,h\} \) will monotonically converge to \( \rho_h(x_h) \) (with oscillation size \( O(h^k) \)). When \( n \) is sufficiently large, we obtain the error estimation Eq. (5). This completes the proof.

Even though the rate of convergence of a monotone iteration is linear. In practice, there exists some techniques to accelerate the convergence [1,2,4]. The key of accelerating the convergent rate depends on the determination of the operator \( M \) in Eq. (2), which in turns strongly depends upon the problem itself. In Theorem 2.2, we discuss one special case, in which the coefficient functions and the boundary \( \partial \Omega \) of the problem (1) are sufficiently smooth, \( c(x) \leq 0 \), \( p \) and \( q \) are nonnegative and do not vanish simultaneously. Under these conditions, the BVP (1) has a unique classical solution \( u* \) and we can apply the global a prior Schauder estimate for the monotone solutions of Eq. (1).
THEOREM 2.2. Let \( \{v_n\} \) be a lower solution sequence defined as in Eq. (2). Suppose that \( f(x, u) \) is a nonincreasing function with respect to \( u \). At the nth iterative step, let 
\[ M = M_n(x) = -\min_{v_n \leq u \leq u^*} f_u(x, u) = -f_u(x, u_n), \]
where \( v_n \leq u_n \leq u^* \). Then the rate of convergence of \( \{v_n\} \) is quadratic.

PROOF: The choice of \( M \) satisfies the condition stated in Eq. (3). Now, introduce the error functions \( e_n = v_n - u^* \). It is evident that \( Be_n = 0, \quad n = 1, 2, \ldots \). From Eqs. (1) and (2), we have
\[
(-\mathcal{L} + M_n(x))e_{n+1} = f(x, v_n) - f(x, u^*) + M_n(x)e_n
= (f_u(x, v_n + \theta_1 e_n) - f_u(x, u_n))e_n = \theta_3 f_{uu}(x, v_n + \theta_2 e_n)e_{2n},
\]
where \(-1 < \theta_1, \theta_2 < 0, -1 < \theta_3 < 1\).

Therefore, we conclude by Schauder's estimate [6] that there exists a constant \( c \) such that
\[
||e_{n+1}||_2 \leq c||f_{uu}(x, v_n + \theta_2 e_n)e_{2n}||_2 \leq c||e_n||_2^2,
\]
where \( ||e_n(x)|| = \sum_{0 \leq |i| \leq 2} \sup_{x \in \Omega} ||Ju e_n(x)||. \)

REFERENCES


Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1