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Travelling wavefronts of Belousov–Zhabotinskii system with diffusion and delay☆

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ABSTRACT

This paper is concerned with the existence, nonexistence and minimal wave speed of the travelling wavefronts of Belousov–Zhabotinskii system with diffusion and delay.

1. Introduction

In 1959, Belousov [2] proposed the so-called Belousov–Zhabotinskii system to model the chemical reaction, and one of its simplified models takes the form as follows

$$\begin{cases} \dfrac{\partial U(x,t)}{\partial t} = \Delta U(x,t) + U(x,t)\left[1 - U(x,t) - rV(x,t)\right], \\ \dfrac{\partial V(x,t)}{\partial t} = \Delta V(x,t) - bU(x,t)V(x,t), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}, t > 0, r \in (0,1)$, b is a positive constant, and $U, V \in \mathbb{R}$ correspond to the concentration of bromic acid and bromide ion, respectively, Δ is the Laplacian operator on $\mathbb{R}$. Model (1.1), in fact, was also derived in biochemical and biological fields, see [10,11,21,22]. Recalling the chemical and biological backgrounds of (1.1), the following asymptotical boundary conditions were proposed [5,6,17,20]

$$\begin{cases} \lim\limits_{x\to-\infty} U(x,t) = 0, & \lim\limits_{x\to-\infty} V(x,t) = 1, \\ \lim\limits_{x\to\infty} U(x,t) = 1, & \lim\limits_{x\to\infty} V(x,t) = 0. \end{cases} \tag{1.2}$$

On the dynamics of (1.1) and (1.2), *travelling wavefront*, which takes the form of $(U(x,t), V(x,t)) = (\rho(x+ct), \varrho(x+ct))$ for some wave speed $c > 0$ and monotone wave profile function (ρ, ϱ), attracted much attention, see Murray [12], Troy [17], Ye and Wang [20] and the references cited therein. Moreover, from the viewpoint of the chemical reaction, the travelling wavefronts of (1.1) and (1.2) have significant sense, namely, the waves move from a region of higher bromic

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acid concentration to one of lower bromic acid concentration as it reduces the level of bromic ion (we can refer to Wu and Zou [19]).

It is well known that time delay seems to be inevitable in many evolutionary processes, e.g., biological science [4], therefore the time delay was incorporated into (1.1) by Wu and Zou [19], which takes the form as follows

$$\begin{cases} \dfrac{\partial U(x,t)}{\partial t} = \Delta U(x,t) + U(x,t)\left[1 - U(x,t) - rV(x,t-\tau)\right], \\ \dfrac{\partial V(x,t)}{\partial t} = \Delta V(x,t) - bU(x,t)V(x,t), \end{cases} \tag{1.3}$$

where $\tau \geq 0$ denotes a time delay. For model (1.3), some results have been established for the existence of travelling wavefronts, see, for example, Ma [8], and Wu and Zou [19]. In particular, Ma [8] proved the existence of travelling wavefronts of (1.3) with (1.2) by the upper and lower solution and Schauder's fixed point theorem if the wave speed $c > 2\sqrt{1-r}$. But for the case of $c \leq 2\sqrt{1-r}$, the existence of travelling wavefronts of (1.3) remains open. This constitutes the purpose of this paper.

We first change the variables such that $u = U$ and $v = 1 - V$, then (1.3) reduces to

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t)\left[1 - r - u(x,t) + rv(x,t-\tau)\right], \\ \dfrac{\partial v(x,t)}{\partial t} = \Delta v(x,t) + bu(x,t)\left[1 - v(x,t)\right], \end{cases} \tag{1.4}$$

and we are interested in the following asymptotic boundary conditions (see (1.2))

$$\lim_{x\to-\infty} u(x,t) = \lim_{x\to-\infty} v(x,t) = 0, \qquad \lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} v(x,t) = 1. \tag{1.5}$$

Let $(u(x,t), v(x,t)) = (\phi(x+ct), \psi(x+ct))$ be the travelling wavefront of (1.4) and denote $x + ct$ by ξ, then $(\phi(\xi), \psi(\xi))$, $\xi \in \mathbb{R}$ must satisfy

$$\begin{cases} c\phi'(\xi) = \phi''(\xi) + \phi(\xi)\left[1 - r - \phi(\xi) + r\psi(\xi - c\tau)\right], \\ c\psi'(\xi) = \psi''(\xi) + b\phi(\xi)\left[1 - \psi(\xi)\right], \end{cases} \tag{1.6}$$

and the corresponding asymptotic boundary conditions as follows (see (1.5))

$$\lim_{\xi\to-\infty} \phi(\xi) = \lim_{\xi\to-\infty} \psi(\xi) = 0, \qquad \lim_{\xi\to\infty} \phi(\xi) = \lim_{\xi\to\infty} \psi(\xi) = 1. \tag{1.7}$$

By the above notations, our main concern in this paper is to investigate the monotone nondecreasing solutions of (1.6) and (1.7). In Section 2, we prove the existence of travelling wavefronts if $c \geq 2\sqrt{1-r}$ by the method of Ma [8] and an approximation argument used in [3,16]. In Section 3, the nonexistence and minimal wave speed of (1.6) and (1.7) will be proved by the theory of asymptotic spreading [7,16] and comparison principle for partial functional differential equations [9]. This is probably the first time that the nonexistence of travelling wavefronts of (1.3) has been reported, even for the case of $\tau = 0$.

2. Existence of travelling wavefronts

In this section, we shall investigate the existence of monotone solution of (1.6) and (1.7). Throughout this paper, X will be defined by

$$X = C(\mathbb{R}, \mathbb{R}^2) = \left\{u(x) | u(x) : \mathbb{R} \to \mathbb{R}^2 \text{ is uniformly continuous and bounded}\right\},$$

which is a Banach space with the super norm. For $(\phi, \psi) \in X$, denote (H_1, H_2) as follows

$$\begin{cases} H_1(\phi,\psi)(\xi) = 2\phi(\xi) + \phi(\xi)\left[1 - r - \phi(\xi) + r\psi(\xi - c\tau)\right], \\ H_2(\phi,\psi)(\xi) = b\psi(\xi) + b\phi(\xi)\left[1 - \psi(\xi)\right]. \end{cases}$$

Then (1.6) can be rewritten as

$$\begin{cases} c\phi'(\xi) = \phi''(\xi) - 2\phi(\xi) + H_1(\phi,\psi)(\xi), \\ c\psi'(\xi) = \psi''(\xi) - b\psi(\xi) + H_2(\phi,\psi)(\xi). \end{cases}$$

For $c > 0$, define constants as follows

$$\gamma_1(c) = \frac{c - \sqrt{c^2+8}}{2}, \qquad \gamma_2(c) = \frac{c + \sqrt{c^2+8}}{2},$$
$$\gamma_3(c) = \frac{c - \sqrt{c^2+4b}}{2}, \qquad \gamma_4(c) = \frac{c + \sqrt{c^2+4b}}{2}.$$

Moreover, for $(\phi, \psi) \in X$, we denote $F = (F_1, F_2)$ by

$$F_1(\phi, \psi)(\xi) = \frac{1}{\gamma_2(c) - \gamma_1(c)} \left[\int_{-\infty}^{\xi} e^{\gamma_1(c)(\xi - s)} + \int_{\xi}^{\infty} e^{\gamma_2(c)(\xi - s)} \right] H_1(\phi, \psi)(s) \mathrm{d}s,$$

$$F_2(\phi, \psi)(\xi) = \frac{1}{\gamma_4(c) - \gamma_3(c)} \left[\int_{-\infty}^{\xi} e^{\gamma_3(c)(\xi - s)} + \int_{\xi}^{\infty} e^{\gamma_4(c)(\xi - s)} \right] H_2(\phi, \psi)(s) \mathrm{d}s,$$

which is similar to that in [8,19]. Then, it is sufficient to consider the fixed point of the operator F in the space X.

Furthermore, define constants as follows

$$c^* = 2\sqrt{1-r}, \qquad \lambda^*(c) = \frac{c - \sqrt{c^2 - 4(1-r)}}{2}.$$

The following existence result is established in Ma [8].

Theorem 2.1. *Let $c > c^*$ be true such that $be^{-\lambda^*(c)c\tau} \le 1 - r$. Then (1.6) and (1.7) have a monotone solution $(\phi(\xi), \psi(\xi))$, which is a travelling wavefront of (1.4) and (1.5).*

In addition, from the definition of F, the following result is clear.

Lemma 2.2. *For any $c \in (c^*, c^* + 1)$, (ϕ, ψ) formulated by Theorem 2.1 are equicontinuous.*

Theorem 2.3. *Assume that $c = c^*$ and $be^{-\lambda^*(c)c\tau} < 1 - r$. Then (1.6) has a monotone solution $(\phi(\xi), \psi(\xi))$ such that*

$$\lim_{\xi \to \infty} (\phi(\xi), \psi(\xi)) = (1, 1), \qquad \lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) = (0, \alpha) \tag{2.1}$$

for some constant $\alpha \in [0, 1]$.

Proof. We prove this by an approximation method used in [3,16]. Since $be^{-\lambda^*(c^*)c^*\tau} < 1 - r$, there exists a $\delta_1 > 0$ such that $be^{-\lambda^*(c)c\tau} \le 1 - r$ for $c \in (c^*, c^* + \delta_1)$. Let $\delta = \min\{\delta_1, 1\}$. Choose a sequence $\{c_n\}_{n=1}^{\infty}$ with $c_n \in (c^*, c^* + \delta)$ and $c_n \to c^*$ as $n \to \infty$. Then Theorem 2.1 implies that (1.6) and (1.7) have a monotone solution $(\phi_n(\xi), \psi_n(\xi))$ with $c = c_n$, so $(\phi_n(\xi), \psi_n(\xi))$ is a fixed point of the operator F with $\gamma_i(c) = \gamma_i(c_n)$. Since such a travelling wavefront is invariant under the sense of phase shift, then we can assume that $\phi_n(0) = \frac{1}{2}$ for all n. Furthermore, Lemma 2.2 indicates that $(\phi_n(\xi), \psi_n(\xi))$ is equicontinuous.

By Ascoli–Arzela lemma and a nested subsequence argument, there exists a subsequence of $(\phi_n(\xi), \psi_n(\xi))$ which converges uniformly on every compact subset of $\mathbb{R}$, and hence pointwise on $\mathbb{R}$ to a vector function $(\phi(\xi), \psi(\xi)) \in X$. According to the Lebesgue's dominant convergence theorem, $(\phi(\xi), \psi(\xi))$ is a fixed point of the operator F with $\gamma_i(c) = \gamma_i(c^*)$. Hence, $(\phi(\xi), \psi(\xi))$ satisfies (1.6) with $c = c^*$. Moreover, the monotonicity of $(\phi_n(\xi), \psi_n(\xi))$ and $\phi_n(0) = \frac{1}{2}$ implies that $(\phi(\xi), \psi(\xi))$ is nondecreasing and $\phi(0) = \frac{1}{2}$.

We now consider the asymptotic behavior of $(\phi(\xi), \psi(\xi))$. In fact, since $(\phi(\xi), \psi(\xi))$ is nondecreasing and bounded, then

$$\lim_{\xi \to \pm\infty} (\phi''(\xi), \psi''(\xi)) = \lim_{\xi \to \pm\infty} (\phi'(\xi), \psi'(\xi)) = (0, 0).$$

Combining this with (1.6), then there exists constants $\phi_\pm, \psi_\pm$ such that

$$\lim_{\xi \to \pm\infty} (\phi(\xi), \psi(\xi)) = (\phi_\pm, \psi_\pm), \qquad (0, 0) \le (\phi_\pm, \psi_\pm) \le (1, 1).$$

Then $\phi(0) = \frac{1}{2}$ implies that $\phi_- \in [0, \frac{1}{2}]$, $\phi_+ \in [\frac{1}{2}, 1]$ and the following equalities

$$\phi_\pm(1 - r - \phi_\pm + r\psi_\pm) = 0, \qquad \phi_\pm(1 - \psi_\pm) = 0. \tag{2.2}$$

We now prove (2.1) in three cases.

(i) Since $\phi_+ \in [\frac{1}{2}, 1]$, then $\psi_+ = 1$ and $\phi_+ = 1$ are obvious by (2.2).
(ii) If $\phi_- = 0$, then there exists some $\alpha \in [0, 1]$ such that $\psi_- = \alpha$.
(iii) If $\phi_- \in (0, \frac{1}{2}]$, then $\phi_-(1 - \psi_-) = 0$ indicates that $\psi_- = 1$ while $\phi_-(1 - r - \phi_- + r\psi_-) = 0$ means that $\phi_- = 0$ or 1, which is a contradiction.

It is clear that (i)–(iii) imply (2.1). The proof is complete. □

Remark 2.4. In Theorem 2.3, we can only prove a weaker asymptotic boundary condition (2.1) than that of (1.7) since $(0, \alpha)$ is the equilibrium of (1.6) for any $\alpha \in \mathbb{R}$. We conjecture $\alpha = 0$ in (2.1), and we shall further investigate the problem in our forthcoming research.

Remark 2.5. Although we prove Theorem 2.3 by a method similar to that of [7], their results cannot be applied directly since (1.6) has infinite constant equilibrium states.

3. Nonexistence and minimal wave speed

In this section, we will prove that (1.6) and (1.7) have no positive solution (do not require the monotonicity) in the sense of functional if $c < c^*$ holds.

We first consider the classical Fisher equation

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = \Delta w(x,t) + \mathrm{d}w(x,t)\left[1 - \dfrac{w(x,t)}{K}\right], \\ w(x,0) = w(x), \end{cases} \tag{3.1}$$

where all the constants are positive and $w(x) \in Y$ which is defined by

$$Y = C(\mathbb{R}, \mathbb{R}) = \{u(x) : u \text{ is uniformly continuous and bounded for all } x \in \mathbb{R}\},$$

then it is clear that Y is a Banach space with the super norm. By the theory of asymptotic spreading of reaction-diffusion equation, which was earlier proposed in Aronson and Weinberger [1] and recently developed by [7,16,18], the following result is true.

Lemma 3.1. *Assume that $w(x) \geq 0$ for $x \in \mathbb{R}$, $w(x) = 0$ outside a bounded interval of $\mathbb{R}$ and $0 < w(x) < K$ on a nonempty subset of $\mathbb{R}$. Let $w(x,t)$ be defined by (3.1), then*

(i) $\lim_{t\to\infty} \sup_{|x|>ct} w(x,t) = 0$ *holds for any given* $c > 2\sqrt{d}$;

(ii) $\lim_{t\to\infty} \inf_{|x|<ct} w(x,t) = K$ *holds for any given* $c \in \left(0, 2\sqrt{d}\right)$.

Moreover, for $t \geq 0$ and $w(x) \in Y$, define $T(t)$ as follows

$$T(t)w(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{(x-y)^2}{4t}} w(y)\mathrm{d}y,$$

then $T(t) : Y \to Y$ is an analytic semigroup for $t \geq 0$ [13,15]. Thus the following result is obvious by the theory of abstract functional differential equations [9] (we also refer to Smith and Zhao [14] for the delayed reaction-diffusion equation on $\mathbb{R}$).

Lemma 3.2. *Assume that $\overline{w}(x,t) \in Y$ for all $t \in [0,t')$. If $\overline{w}(x,0) \geq w(x)$ and*

$$\overline{w}(x,t) \geq T(t)\overline{w}(x,s) + \int_s^t T(t-\theta)\left\{\mathrm{d}\overline{w}(x,\theta)\left[1 - \frac{\overline{w}(x,\theta)}{K}\right]\right\}\mathrm{d}\theta, \quad x \in \mathbb{R},$$

for any $0 \leq s < t < t'$. Then $\overline{w}(x,t) \geq w(x,t)$ holds for all $(x,t) \in \mathbb{R} \times [0,t')$.

We now consider the initial value problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t) + u(x,t)\left[1 - r - u(x,t) + rv(x,t-\tau)\right], \quad x \in \mathbb{R},\ t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = \Delta v(x,t) + bu(x,t)\left[1 - v(x,t)\right], \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = u(x),\ v(x,s) = z(x,s), \quad x \in \mathbb{R},\ s \in [-\tau, 0], \end{cases} \tag{3.2}$$

with $(u(\cdot), z(\cdot,s)) \in X$ for all $s \in [-\tau, 0]$. By the theory of the abstract functional differential equation [9], the existence of *mild solution* of (3.2) in the space X is formulated as follows.

Lemma 3.3. *Assume that $0 \leq u(x), z(x,s) \leq 1$ for any $x \in \mathbb{R}, s \in [-\tau, 0]$. Then (3.2) has a unique mild solution $(u(x,t), v(x,t))$ defined for all $(x,t) \in \mathbb{R} \times (0,\infty)$. Moreover, $0 \leq u(x,t), v(x,t) \leq 1$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$ and takes the form as follows*

$$\begin{cases} u(x,t) = T(t)u(x) + \displaystyle\int_0^t T(t-s)\left\{u(x,s)\left[1 - r - u(x,s) + rv(x,s-\tau)\right]\right\}\mathrm{d}s, \\ v(x,t) = T(t)z(x,0) + \displaystyle\int_0^t T(t-s)\left\{bu(x,s)\left[1 - v(x,s)\right]\right\}\mathrm{d}s. \end{cases} \tag{3.3}$$

Lemma 3.4. *Assume that the initial value of (3.2) satisfies*

(I) $0 < u(x), z(x,s) < 1$ *if* $x \in (-1,1), s \in [-\tau, 0]$,

(II) $u(x) = z(x,s) = 0$ *if* $|x| \geq 1, s \in [-\tau, 0]$.

Then $\lim_{t\to\infty} \inf_{|x|<ct} u(x,t) \geq 1 - r$ *holds for any given* $c \in (0, c^*)$.

Note that $T(t)u(x) \geq 0$ if $u(x) \geq 0$, then (3.3) and Lemma 3.3 imply that

$$u(x,t) \geq T(t)u(x) + \int_0^t T(t-s)\{u(x,s)[1-r-u(x,s)]\}\,ds$$

for all $(x,t) \in \mathbb{R} \times (0,\infty)$. Therefore, Lemma 3.4 is a direct consequence of Lemmas 3.1–3.3, so we omit its proof here (we can also refer to Smith and Zhao [14]).

Lemma 3.5. *Assume that* $(u_1(\cdot), z_1(\cdot, s)) \in X$ *for* $s \in [-\tau, 0]$ *and*

$$0 \leq u_1(x) \leq u(x) \leq 1, \qquad 0 \leq z_1(x,s) \leq z(x,s) \leq 1$$

for all $x \in \mathbb{R}$, $s \in [-\tau, 0]$. *Let* $(u_1(x,t), v_1(x,t))$ *and* $(u(x,t), v(x,t))$ *be the mild solutions of* (3.2) *with initial values* (u_1, z_1) *and* (u, z), *respectively. Then*

$$0 \leq u_1(x,t) \leq u(x,t) \leq 1, \qquad 0 \leq v_1(x,t) \leq v(x,t) \leq 1, \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

Lemma 3.5 is clear by Martin and Smith [9], and the proof is omitted here.

Theorem 3.6. *Assume that* $c < c^*$ *holds. Then* (1.6) *and* (1.7) *have no monotone solution.*

Proof. We argue it by contradiction, were the stated conclusion false, then there exists a constant $c_1 \in (0, c^*)$ such that there exists $(\phi(\xi), \psi(\xi))$ satisfying (1.6) and (1.7) with $c = c_1$ and $\xi = x + c_1 t$. Note that the travelling wavefront is invariant in the sense of phase shift, so $(\phi(\xi + h), \psi(\xi + h))$ also satisfies (1.6) and (1.7) with $c = c_1$ and $\xi = x + c_1 t$. Assume that the initial value of (3.2) satisfies the conditions (I)–(II) in Lemma 3.4, then we can always choose $h > 0$ sufficiently large such that

$$0 \leq u(x) \leq \phi(x+h) \leq 1, \qquad 0 \leq z(x,s) \leq \psi(x + c_1 s + h) \leq 1$$

hold for all $x \in \mathbb{R}$, $s \in [-\tau, 0]$. Let $c = \frac{c_1 + c^*}{2}$ in Lemma 3.4. Then the comparison principle (Lemma 3.5) indicates that

$$0 \leq u(x,t) \leq \phi(x + c_1 t + h) \leq 1, 0 \leq v(x,t) \leq \psi(x + c_1 t + h) \leq 1$$

for all $(x,t) \in \mathbb{R} \times [0,\infty)$, which further implies a contradiction between Lemma 3.4 and (1.7) as $x + ct \to -\infty$. The proof is complete. □

Theorem 3.7. *Assume that* $c < c^*$ *holds. Then* (1.6) *and* (1.7) *have no positive solution.*

The proof of Theorem 3.7 is similar to that of Theorem 3.6, so we omit it here.

Remark 3.8. The results in Liang and Zhao [7] cannot be applied to consider the nonexistence of travelling wavefront directly, the reason is similar to that of Remark 2.5. However, the proof in this paper is motivated by Liang and Zhao [7] and Thieme and Zhao [16].

Remark 3.9. What we have done implies that c^* is the *minimal wave speed* of system (1.4), which is under the sense that (1.6) and (1.7) have no nontrivial positive solution if $c < c^*$ while they have a nontrivial monotone solution if $c \geq c^*$.

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