JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 70, 72–84 (1979)

Structure of Quadratic Inequalities

T. ANDO*

Division of Applied Mathematics, Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan

Submitted by Ky Fan

When A and B are $n \times n$ positive semi-definite matrices, and C is an $n \times n$ Hermitian matrix, the validity of a quadratic inequality

$$(x^*Ax)^{1/2}(x^*Bx)^{1/2} \geqslant |x^*Cx|$$

is shown to be equivalent to the existence of an $n \times n$ unitary matrix W such that

$$A^{1/2}WB^{1/2} + B^{1/2}W^*A^{1/2} = 2C.$$

Some related inequalities are also discussed.

1. INTRODUCTION

We consider complex $n \times n$ matrices. The transpose and the complex conjugate of a matrix C are denoted by C^T and $\bar{C}$, respectively while C^* is the conjugate transpose, i.e. $C^* = \bar{C}^T$. I is the identity matrix. For Hermitian A and B the relation $A \geqslant B$ means that $A - B$ is positive semi-definite. For a positive semi-definite A its (positive semi-definite) square root is denoted by $A^{1/2}$. The space of $n \times 1$ matrices is denoted by $\mathbf{C}^n$ and its elements, i.e. (n-column) vectors, by x, y, and z.

Horn [6] and FitzGerald and Horn [4] studied the structure of a Hermitian inequality:

$$x^*Ax \geqslant |x^*Bx| \qquad \text{for all } x \in \mathbf{C}^n,$$

and of a Hermitian-symmetric inequality:

$$x^*Ax \geqslant |x^TCx| \qquad \text{for all } x \in \mathbf{C}^n,$$

where A is positive semi-definite, B is Hermitian, and C is symmetric.

Our first concern is an inequality of domination:

$$x^*Ax \geqslant |x^*Cx| \qquad \text{for all } x \in \mathbf{C}^n,$$

* Research supported in part by Kakenhi 234004.

0022-247X/79/070072-13$02.00/0

where A is positive semi-definite, but C is arbitrary. We characterize the validity of this inequality by the existence of a matrix W such that

$$W^*W \leqslant I \qquad \text{and} \qquad 2A^{1/2}(I - W^*W)^{1/2}\, WA^{1/2} = C.$$

Our next aim is to analyse structure of an inequality of Schwartz type:

$$(x^*Ax)^{1/2}\,(x^*Bx)^{1/2} \geqslant |\, x^*Cx \,| \qquad \text{for all } x \in \mathbf{C}^n,$$

where A and B are positive semi-definite and C is Hermitian. We characterize the validity of this inequality by the existence of a unitary matrix W such that

$$A^{1/2}WB^{1/2} + B^{1/2}W^*A^{1/2} = 2C.$$

In the final section we analyze difficulty in treating an inequality of Schwartz type without Hermitian condition on C.

2. Inequalities of Domination

If $n \times n$ matrices A and C are positive semi-definite and Hermitian, respectively, the inequality of domination

$$x^*Ax \geqslant |\, x^*Cx \,| \qquad \text{for all } x \in \mathbf{C}^n$$

can be written in the form

$$A \geqslant C \geqslant -A.$$

Therefore the inequality of domination is equivalent to the existence of an $n \times n$ Hermitian matrix W such that

$$W^*W \leqslant I \qquad \text{and} \qquad A^{1/2}WA^{1/2} = C.$$

If Hermitian condition on C in the above inequality is removed, the situation is much complicated. Indeed, it means

$$2A \geqslant e^{i\theta}C + e^{-i\theta}C^* \geqslant -2A \qquad \text{for all } \theta \in \mathbf{R}.$$

We shall eliminate the parameter θ from this inequality by introducing a matrix.

LEMMA 1. *Let C be a complex $n \times n$ matrix. The following four statements are mutually equivalent:*

(a) $x^*x \geqslant |x^*Cx|$ *for all* $x \in \mathbf{C}^n$,

(b) $I + \frac{1}{2}e^{i\theta}C + \frac{1}{2}e^{-i\theta}C^* \geqslant 0$ *for all* $\theta \in \mathbf{R}$,

(c) *there is an* $n \times n$ *matrix* W *such that*

$$W^*W \leqslant I \qquad \text{and} \qquad 2(I - W^*W)^{1/2}\, W = C$$

(d) *there are* $n \times n$ *matrices* U *and* W *such that*

$$\{U + e^{i\theta}W\}^* \{U + e^{i\theta}W\} = I + \tfrac{1}{2}e^{i\theta}C + \tfrac{1}{2}e^{-i\theta}C^* \qquad \text{for all } \theta \in \mathbf{R}.$$

Proof. Equivalence of (a) and (b) are obvious. Equivalence of (a) and (c) was proved in [1] while equivalence of (b) and (d) is just a special case of [7, Theorem 3.2].

We apply this lemma to our inequality of domination.

THEOREM 2. *Let* A *and* C *be* $n \times n$ *matrices, and assume that* A *is positive semi-definite. Then the following statements are mutually equivalent:*

(i) $x^*Ax \geqslant |x^*Cx|$ *for all* $x \in \mathbf{C}^n$,

(ii) *there is an* $n \times n$ *matrix* W *such that*

$$W^*W \leqslant I \qquad \text{and} \qquad 2A^{1/2}(I - W^*W)^{1/2}\, WA^{1/2} = C,$$

(iii) *there are vectors* $x_i \in \mathbf{C}^n$ $i = 1,..., 2n$ *such that*

$$\sum_{i=1}^{2n} x_i x_i^* = A \qquad \text{and} \qquad 2\sum_{i=1}^{n} x_{2i-1}x_{2i}^* = C.$$

Proof. (i) $\Rightarrow$ (ii). Assume first that A is positive definite, and consider $S := A^{-1/2}CA^{-1/2}$. Then (i) implies that S satisfies (a) of Lemma 1 in place of C. Therefore there is W satisfying (c), which meets the requirement of (ii). When A is merely positive semi-definite, apply the above arguments to positive definite $A_\epsilon := A + \epsilon I$ for $\epsilon > 0$. Let W_ϵ be a matrix which satisfies

$$W_\epsilon^* W_\epsilon \leqslant I \qquad \text{and} \qquad 2A_\epsilon^{1/2}\{I - W_\epsilon^* W_\epsilon\}^{1/2}\, W_\epsilon A_\epsilon^{1/2} = C.$$

Since the set of matrices U for which $U^*U \leqslant I$ is compact with respect to the usual topology, W_ϵ can be assumed to converge to some W as $\epsilon \to 0$. Finally since $A_\epsilon^{1/2}$ and $\{I - W_\epsilon^* W_\epsilon\}^{1/2}$ converge to $A^{1/2}$ and $\{I - W^*W\}^{1/2}$, respectively, as $\epsilon \to 0$, the above relations imply that W meets the requirement of (ii).

(ii) $\Rightarrow$ (iii). If W satisfies (ii), then the $2n \times 2n$ matrix

$$\Delta := \begin{pmatrix} I - W^*W & (I - W^*W)^{1/2}\, W \\ W^*(I - W^*W)^{1/2} & W^*W \end{pmatrix}$$

is positive semi-definite with 0 as its eigenvalue of multiplicity $\geqslant n$, because

$$\Delta \cdot \begin{pmatrix} -(W^*W)^{1/2} & 0 \\ (I - W^*W)^{1/2} & 0 \end{pmatrix} = 0.$$

By the spectral theorem (cf. [5], p. 67) there are vectors $z_i \in \mathbf{C}^{2n}$ $i = 1,\ldots, n$ such that

$$\sum_{i=1}^{n} z_i z_i^* = \Delta.$$

Let

$$z_i^T := (u_i^T, v_i^T)$$

with u_i, $v_i \in \mathbf{C}^n$ $i = 1, 2,\ldots, n$. Then the above representation implies that

$$\sum_{i=1}^{n} u_i u_i^* + \sum_{i=1}^{n} v_i v_i^* = I - W^*W + W^*W = I$$

and

$$\sum_{i=1}^{n} u_i v_i^* = (I - W^*W)^{1/2}\, W.$$

Therefore by (ii) the vectors

$$x_{2i-1} := A^{1/2} u_i \qquad \text{and} \qquad x_{2i} := A^{1/2} v_i \qquad i = 1,\ldots, n$$

meet the requirement of (iii).

(iii) $\Rightarrow$ (i). If (x_i) satisfies (iii), for any vector $x \in \mathbf{C}^n$, the arithmetic-geometric mean inequality shows

$$\begin{aligned} |\, x^*Cx \,| &\leqslant 2 \sum_{i=1}^{n} |\, x^* x_{2i-1} \,| \cdot |\, x_{2i}^* x \,| \\ &\leqslant \sum_{i=1}^{n} |\, x^* x_{2i-1} \,|^2 + \sum_{i=1}^{n} |\, x_{2i}^* x \,|^2 \\ &= x^* \left(\sum_{i=1}^{2n} x_i x_i^* \right) x = x^*Ax. \end{aligned}$$

This completes the proof.

If x^*Cx in Theorem 2 (i) is replaced by x^TCx, there appears a Hermitian-symmetric inequality:

$$x^*Ax \geqslant |\, x^T(\tfrac{1}{2}C + \tfrac{1}{2}C^T)\, x \,|$$

because

$$x^TCx = x^TC^Tx.$$

Analysis of an inequality of this type has been done by FitzGerald and Horn [4, Theorems 2.1 and 2.3].

3. Inequalities of Schwartz Type

In order to analyse an inequality of Schwartz type:

$$(x^*Ax)^{1/2}(x^*Bx)^{1/2} \geqslant |x^*Cx| \qquad \text{for all } x \in \mathbf{C}^n,$$

where A and B are positive semi-definite and C is Hermitian, we need the following variant of Lemma 1.

Lemma 3. *Let A, B, and C be $n \times n$ matrices, and assume that A and B are positive semi-definite and C is Hermitian. Then the following statements are mutually equivalent:*

(b′) $\lambda^2 A + 2\lambda C + B \geqslant 0$ *for all* $\lambda \in \mathbf{R}$,

(c′) *there is an $n \times n$ unitary matrix W such that*

$$A^{1/2}WB^{1/2} + B^{1/2}W^*A^{1/2} = 2C,$$

(d′) *there are $n \times n$ matrices F and G such that*

$$\{\lambda F + G\}^*\{\lambda F + G\} = \lambda^2 A + 2\lambda C + B \qquad \textit{for all } \lambda \in \mathbf{R}.$$

Proof. Equivalence of (b′) and (d′) is just a special case of [7, Theorem 3.3], whose proof is accomplished by reduction to Lemma 1 via change of variable

$$\lambda = e^{-i\pi/2}\{e^{i\theta} - 1\}\{e^{i\theta} + 1\}^{-1}.$$

(c′) implies (d′). In fact,

$$\{\lambda A^{1/2} + WB^{1/2}\}^*\{\lambda A^{1/2} + WB^{1/2}\} = \lambda^2 A + 2\lambda C + B.$$

Finally suppose that F and G satisfy (d′). Comparision of coefficients shows that

$$F^*F = A, \qquad G^*G = B \qquad \text{and} \qquad F^*G + G^*F = 2C.$$

Take unitary matrices U and V such that

$$F = UA^{1/2} \qquad \text{and} \qquad G = VB^{1/2}.$$

Then the unitary matrix

$$W := U^*V$$

meets the requirement of (d′). This completes the proof.

THEOREM 4. *Let A, B, and C be $n \times n$ matrices and assume that A and B are positive semi-definite while C is Hermitian. Then the following statements are mutually equivalent:*

(I) $(x^*Ax)^{1/2}(x^*Bx)^{1/2} \geqslant |x^*Cx|$ *for all* $x \in \mathbf{C}^n$,

(II) *there is an $n \times n$ unitary matrix W such that*

$$A^{1/2}WB^{1/2} + B^{1/2}W^*A^{1/2} = 2C,$$

(III) *there are vectors $x_i \in \mathbf{C}^n$ $i = 1, 2,..., 2n$ such that*

$$\sum_{i=1}^{n} x_{2i-1}x_{2i-1}^* = A, \qquad \sum_{i=1}^{n} x_{2i}x_{2i}^* = B$$

and

$$\sum_{i=1}^{n} \{x_{2i-1}x_{2i}^* + x_{2i}x_{2i-1}^*\} = 2C.$$

Proof. (I) $\Rightarrow$ (II). By the arithmetic-geometric mean inequality (I) implies

$$\lambda^2 x^*Ax - 2\lambda\,|x^*Cx| + x^*Bx \geqslant 0 \qquad \text{for all } x \in \mathbf{C}^n \quad \text{and} \quad \lambda \in \mathbf{R}^+.$$

Since C is Hermitian, this inequality implies

$$\lambda^2 A + 2\lambda C + B \geqslant 0 \qquad \text{for all } \lambda \in \mathbf{R}.$$

Therefore by Lemma 3 there is an $n \times n$ unitary matrix W which meets the requirement of (II).

(II) $\Rightarrow$ (III). Let an $n \times n$ unitary matrix W satisfies (II). Unitarity implies that the $2n \times 2n$ matrix

$$\begin{pmatrix} I & W \\ W^* & I \end{pmatrix}$$

is positive semi-definite with 0 as its eigenvalue of multiplicity $\geqslant n$, because

$$\begin{pmatrix} I & W \\ W^* & I \end{pmatrix} \cdot \begin{pmatrix} -W & 0 \\ I & 0 \end{pmatrix} = 0.$$

Then by the spectral theorem (cf. [5], p. 67) there are vectors $z_i \in \mathbf{C}^{2n}$ $i = 1, 2,..., n$ such that

$$\begin{pmatrix} I & W \\ W^* & I \end{pmatrix} = \sum_{i=1}^{n} z_i z_i^*.$$

Let

$$z_i^T := (u_i^T, v_i^T)$$

with u_i, $v_i \in \mathbf{C}^n$. Then this representation shows

$$\sum_{i=1}^{n} u_i u_i^* = \sum_{i=1}^{n} v_i v_i^* = I \qquad \text{and} \qquad \sum_{i=1}^{n} u_i v_i^* = W.$$

Therefore by (II) the vectors

$$x_{2i-1} := A^{1/2} u_i \qquad \text{and} \qquad x_{2i} := B^{1/2} v_i \qquad (i = 1,\ldots, n)$$

meet the requirement of (III).

(III) ⇒ (I). Suppose that vectors (x_i) satisfy (III). Then for any $x \in \mathbf{C}^n$ by the Schwartz inequality

$$\begin{aligned} | x^*Cx | &\leqslant \tfrac{1}{2} \sum_{i=1}^{n} \{| x^*x_{2i-1} | \cdot | x_{2i}^*x | + | x^*x_{2i} | \cdot | x_{2i-1}^*x |\} \\ &= \sum_{i=1}^{2n} | x^*x_{2i-1} | \cdot | x_{2i}^*x | \leqslant \left\{\sum_{i=1}^{n} | x^*x_{2i-1} |^2\right\}^{1/2} \left\{\sum_{i=1}^{n} | x^*x_{2i} |^2\right\}^{1/2} \\ &= (x^*Ax)^{1/2} (x^*Bx)^{1/2}. \end{aligned}$$

This completes the proof.

Of course, implication (II) ⇒ (III) is true even if the unitarity of W is replaced by $W^*W \leqslant I$.

Now a Hermitian-symmetric inequality of Schwartz type can be derived from Theorem 4.

THEOREM 5. *Let A, B, and C be $n \times n$ matrices, and assume that A and B are positive semi-definite. Then the following statements are mutually equivalent:*

(I′) $(x^*Ax)^{1/2} (x^*Bx)^{1/2} \geqslant | x^TCx |$ *for all* $x \in \mathbf{C}^n$,

(II′) *there is an $n \times n$ matrix W such that*

$$W^*W \leqslant I \qquad \text{and} \qquad \bar{A}^{1/2} W B^{1/2} + \bar{B}^{1/2} W^T A^{1/2} = C + C^T,$$

(III′) *there are vectors $x_i \in \mathbf{C}^n$ $i = 1, 2,\ldots, 4n$ such that*

$$\sum_{i=1}^{2n} x_{2i-1} x_{2i-1}^* = A, \qquad \sum_{i=1}^{2n} x_{2i} x_{2i}^* = B$$

and

$$\sum_{i=1}^{2n} \{\bar{x}_{2i-1} x_{2i}^* + \bar{x}_{2i} x_{2i-1}^*\} = C + C^T.$$

Proof. (I′) ⇒ (II′). Let

$$D := \tfrac{1}{2}\{C + C^T\}.$$

Then D is symmetric and

$$x^T C x = x^T D x \qquad \text{for all } x \in \mathbf{C}^n.$$

Therefore (I′) implies

$$(x^* A x)^{1/2} (x^* B x)^{1/2} \geqslant | x^T D x | \qquad \text{for all } x \in \mathbf{C}^n.$$

Then, as in the proof of Theorem 4, this implies

$$x^* \left(\frac{\lambda}{2} A + \frac{1}{2\lambda} B\right) x \geqslant | x^T D x | \qquad \text{for all } x \in \mathbf{C}^n \quad \text{and} \quad \lambda \in \mathbf{R}^+.$$

These Hermitian-symmetric inequalities are shown in [4] to be equivalent to

$$\begin{pmatrix} \frac{\lambda}{2} \bar{A} + \frac{1}{2\lambda} \bar{B} & D \\ \bar{D} & \frac{\lambda}{2} A + \frac{1}{2\lambda} B \end{pmatrix} \geqslant 0 \qquad \text{for all } \lambda \in \mathbf{R}^+.$$

In other words,

$$\frac{\lambda}{2} \mathscr{A} + \frac{1}{2\lambda} \mathscr{B} \geqslant \mathscr{C} \qquad \text{for all } \lambda \in \mathbf{R}^+,$$

where

$$\mathscr{A} := \begin{pmatrix} \bar{A} & 0 \\ 0 & A \end{pmatrix}, \qquad \mathscr{B} := \begin{pmatrix} \bar{B} & 0 \\ 0 & B \end{pmatrix} \qquad \text{and} \qquad \mathscr{C} := \begin{pmatrix} 0 & D \\ \bar{D} & 0 \end{pmatrix}$$

or equivalently

$$(x^* \mathscr{A} x)^{1/2} (x^* \mathscr{B} x)^{1/2} \geqslant | x^* \mathscr{C} x | \qquad \text{for all } x \in \mathbf{C}^{2n}.$$

Apply Theorem 4 to the triple $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$ of $2n \times 2n$ matrices to see that there is a $2n \times 2n$ unitary matrix

$$\mathscr{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

such that

$$\mathscr{A}^{1/2} \mathscr{W} \mathscr{B}^{1/2} + \mathscr{B}^{1/2} \mathscr{W}^* \mathscr{A}^{1/2} = 2\mathscr{C}.$$

From this it follows that

$$\bar{A}^{1/2}W_{12}B^{1/2} + \bar{B}^{1/2}W_{21}^{*}A^{1/2} = 2D$$

and

$$A^{1/2}W_{21}\bar{B}^{1/2} + B^{1/2}W_{12}^{*}\bar{A}^{1/2} = 2\bar{D},$$

hence the $n \times n$ matrix

$$W := \tfrac{1}{2}(W_{12} + \overline{W_{21}})$$

satisfies

$$\bar{A}^{1/2}WB^{1/2} + \bar{B}^{1/2}W^{T}A^{1/2} = 2D = C + C^{T}.$$

Finally since $\mathscr{W}^{*}\mathscr{W} = I$ implies

$$W_{12}^{*}W_{12} \leqslant I \qquad \text{and} \qquad \bar{W}_{21}^{*}\bar{W}_{21} \leqslant I,$$

the matrix W also satisfies

$$W^{*}W \leqslant I.$$

This concludes that W meets the requirement of (II′).

Implications (II′) ⇒ (III′) and (III′) ⇒ (I′) can be proved in quite a similar way to the proof of (II) ⇒ (III) and (III) ⇒ (I) in Theorem 4. But since W in (II′) is not unitary, to appeal to the spectral theorem $4n$ number of vectors (x_i) are necessary.

We have only a partial result concerning an inequality of Schwartz type without Hermitian condition on C.

THEOREM 6. *Let A, B, and C be $n \times n$ matrices, and assume that A and B are positive semi-definite. If there are $n \times n$ matrices U and V such that*

$$\{e^{i\theta}U + V\}^{*}\{e^{i\theta}U + V\} \leqslant I \qquad \text{for all } \theta \in \mathbf{R} \tag{1}$$

and

$$A^{1/2}UB^{1/2} + B^{1/2}V^{*}A^{1/2} = C, \tag{2}$$

then

$$(x^{*}Ax)^{1/2}(x^{*}Bx)^{1/2} \geqslant |x^{*}Cx| \qquad \text{for all } x \in \mathbf{C}^{n}. \tag{3}$$

Conversely if the quadratic inequality (3) *is valid then there are $n \times n$ matrices U and V which satisfy* (2) *and*

$$\{e^{i\theta}U + V\}^{*}\{e^{i\theta}U + V\} \leqslant (4\pi^{-1})^{2}I, \qquad \text{for all } \theta \in \mathbf{R}. \tag{4}$$

Proof. Suppose that U and V satisfy (1) and (2). Let

$$W_{\theta} := e^{i\theta}U + e^{-i\theta}V.$$

Then assumption (1) implies

$$W_\theta^* W_\theta \leqslant I \qquad \text{for all } \theta \in \mathbf{R}$$

while (2) shows

$$A^{1/2} W_\theta B^{1/2} + B^{1/2} W_\theta^* A^{1/2} = e^{i\theta} C + e^{-i\theta} C^*. \tag{5}$$

Then Theorem 4 can be applied to positive semi-definite A, B and Hermitian $\frac{1}{2}\{e^{i\theta}C + e^{-i\theta}C^*\}$ to yield

$$(x^*Ax)^{1/2}(x^*Bx)^{1/2} \geqslant |\operatorname{Re}\{e^{i\theta}x^*Cx\}| \qquad \text{for all } \theta \in \mathbf{R}$$

which implies (3).

Suppose conversely that (3) is valid. By Theorem 4, applied A, B and Hermitian $\frac{1}{2}\{e^{i\theta}C + e^{-i\theta}C^*\}$ there is an $n \times n$ unitary matrix W_θ which satisfies (5). Inspection of the proof for Lemma 1 in [1] will show that W_θ is a measurable (matrix) function of θ. Consider the Fourier expansion

$$W_\theta = \sum_{k=-\infty}^{\infty} e^{ik\theta} W_k$$

where each W_k is an $n \times n$ matrix. Multiply the both sides of (5) by $\pi^{-1}\cos(\phi - \theta)$ and integrate from 0 to 2π to get

$$A^{1/2}\{e^{-i\phi}W_{-1} + e^{i\phi}W_1\} B^{1/2} + A^{1/2}\{e^{-i\phi}W_{-1} + e^{i\phi}W_1\}^* B^{1/2} = e^{i\phi}C + e^{-i\phi}C^*.$$

Since $\phi \in \mathbf{R}$ is arbitrary, comparison of coefficients will show that

$$A^{1/2} W_1 B^{1/2} + B^{1/2} W_{-1}^* A^{1/2} = C$$

which means that the matrices

$$U := W_1 \qquad \text{and} \qquad V := W_{-1}$$

satisfy (2). Since by definition

$$e^{i\phi}U + e^{-i\phi}V = \pi^{-1}\int_0^{2\pi} \cos(\phi - \theta)\, W_\theta \, d\theta$$

and

$$\pi^{-1}\int_0^{2\pi} |\cos(\phi - \theta)|\, d\theta = 4\pi^{-1} \qquad \text{for all } \theta \in \mathbf{R}$$

the unitarity of W_θ implies

$$\{e^{i\phi}U + e^{-i\phi}V\}^* \{e^{i\phi}U + e^{-i\phi}V\} \leqslant (4\pi^{-1})^2 I \qquad \text{for all } \phi \in \mathbf{R},$$

which is equivalent to (4).

Use of a modification of Lemma 1 will make it possible to eliminate the parameter θ from (1) and (4).

COROLLARY 7. *Let A and B be $n \times n$ positive semi-definite matrices. If there are $n \times n$ matrices U_1, U_2, V_1, and V_2 such that*

$$U_1^*U_1 + V_1^*V_1 \leqslant I \qquad \text{and} \qquad U_2U_2^* + V_2V_2^* \leqslant I,$$

then the matrix

$$C := A^{1/2}U_2U_1B^{1/2} + B^{1/2}V_1^*V_2^*A^{1/2}$$

satisfies the inequality

$$(x^*Ax)^{1/2}(x^*Bx)^{1/2} \geqslant |x^*Cx| \qquad \text{for all } x \in \mathbf{C}^n.$$

Proof. For any $x, y \in \mathbf{C}^n$ and $\theta \in \mathbf{R}$ the Schwartz inequality and the assumption show

$$\begin{aligned} &|y^*(e^{i\theta}U_2U_1 + V_2V_1)x| \\ &\leqslant (y^*U_2U_2^*y)^{1/2}(x^*U_1^*U_1x)^{1/2} + (y^*V_2V_2^*y)^{1/2}(x^*V_1^*V_1x)^{1/2} \\ &\leqslant \{y^*(U_2U_2^* + V_2V_2^*)y\}^{1/2}\{x^*(U_1^*U_1 + V_1^*V_1)x\}^{1/2} \\ &\leqslant (y^*y)^{1/2} \cdot (x^*x)^{1/2}, \end{aligned}$$

which implies that

$$\{e^{i\theta}U_2U_1 + V_2V_1\}^*\{e^{i\theta}U_2U_1 + V_2V_1\} \leqslant I.$$

Now the assertion follows from Theorem 6.

4. POSITIVE LINEAR MAPS

Let us denote by $\mathbf{M}_2$ (resp. $\mathbf{M}_n$) the complex linear space of all 2×2 (resp. $n \times n$) matrices. Given A, B, C, and D in $\mathbf{M}_n$, let us define a linear map Φ from $\mathbf{M}_2$ to $\mathbf{M}_n$ by

$$\Phi(X) := \xi_{11}A + \xi_{22}B + \xi_{21}C + \xi_{12}D \qquad \text{for } X = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}.$$

Then Φ preserves Hermitian property, i.e.

$$\Phi(X^*) = \Phi(X)^* \qquad \text{for all } X \in \mathbf{M}_2,$$

if and only if A and B are Hermitian, and $C^* = D$. Every linear map from $\mathbf{M}_2$ to $\mathbf{M}_n$ that preserves Hermitian property is obtained in this way.

Suppose now that A and B are Hermitian and $C^* = D$. The map Φ is said to be *positive* if

$$\Phi(X) \geqslant 0 \qquad \text{whenever} \qquad X \geqslant 0.$$

Since by the spectral theorem each positive semi-definite matrix is the sum of two matrices of the form

$$\begin{pmatrix} \xi\bar{\xi}, & \xi\bar{\eta} \\ \eta\bar{\xi}, & \eta\bar{\eta} \end{pmatrix},$$

Φ is positive if and only if

$$\xi\bar{\xi}(x^*Ax) + \eta\bar{\xi}(x^*Cx) + \bar{\eta}\xi(\overline{x^*Cx}) + \eta\bar{\eta}(x^*Bx) \geqslant 0$$

for all $\xi, \eta \in \mathbf{C}$ and $x \in \mathbf{C}^n$, which is equivalent to that A and B are positive semi-definite and

$$(x^*Ax)^{1/2}\,(x^*Bx)^{1/2} \geqslant | x^*Cx | \qquad \text{for all } x \in \mathbf{C}^n.$$

Therefore to determine the structure of this inequality means to describe all positive linear maps from $\mathbf{M}_2$ to $\mathbf{M}_n$; the problem in the latter form is still very difficult (cf. [8]).

The linear map Φ is said to be *completely positive* if there is a finite number of $2 \times n$ matrices (V_i) such that

$$\sum V_i^* X V_i = \Phi(X) \qquad \text{for all } X \in \mathbf{M}_2 .$$

This definition is different from, but equivalent to the usual one of complete positivity, as shown by Choi [3]. Complete positivity implies positivity, but not conversely. Indeed, Φ is completely positive if and only if the $2n \times 2n$ matrix

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix}$$

is positive, or equivalently

$$(x^*Ax)^{1/2}\,(y^*By)^{1/2} \geqslant | y^*Cx | \qquad \text{for all } x, y \in \mathbf{C}^n$$

(see [3, Theorem 2]). This last inequality is quite familiar and is studied in [6].

It should be mentioned that success in Theorem 4 with Hermitian C is a variant of the fact that, when restricted on the subspace of 2×2 symmetric matrices, each positive linear map coincides with a completely positive map (cf. [3, Theorem 7]). Also Theorem 2 can be derived from a result of Arveson [2, p. 302] on completely positive maps.

REFERENCES

1. T. ANDO, Structure of operators with numerical radius one, *Acta Sci. Math.* **34** (1973), 11–15.
2. W. B. ARVESON, Subalgebras of C^*-algebras, II, *Acta Math.* **128** (1972), 271–308.
3. M.-D. CHOI, Completely positive linear maps on complex matrices, *Linear Algebra Appl.* **10** (1975), 189–194.
4. C. H. FITZGERALD AND R. A. HORN, On the structure of Hermitian symmetric inequalities, *J. London Math. Soc.* (2) **15** (1977), 419–430.
5. M. MARCUS AND H. MINC, "A Survey of Matrix Theory and Matrix Inequalities," Prindle, Weber & Schmidt, Boston, 1964.
6. R. A. HORN, On inequalities between Hermitian and symmetric forms, *Linear Algebra Appl.* **11** (1975), 189–218.
7. M. ROSENBLUM AND J. ROVNYAK, The factorization problem for non-negative operator valued functions, *Bull. Amer. Math. Soc.* **77** (1971), 287–318.
8. E. STØRMER, Positive linear maps of operator algebras, *Acta Math.* **110** (1963), 233–278.