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Consistency of Monomial and Difference Representations of Functions Arising from Empirical Phenomena*

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Choice probabilities in the behavioral sciences are often analyzed from the standpoint of a *difference representation* such as $P(\mathbf{x}, x, \mathbf{y}) = F[u(\mathbf{x}, x) - g(\mathbf{y})]$. Here, $\mathbf{x}$ and $\mathbf{y}$ are real, positive vector variables, x is a positive real variable, $P(\mathbf{x}, x, \mathbf{y})$ is the probability of choosing alternative $(\mathbf{x}, x)$ over alternative $\mathbf{y}$, and u, g and F are real valued, continuous functions, strictly increasing in all arguments. In some situations (e.g. in psychophysics), the researchers are more interested in the functions u and g than in the function F. In such cases, they investigate the choice phenomenon by estimating empirically the value x such that $P(\mathbf{x}, x, \mathbf{y}) = \rho$, for some values of ρ, and for many values of the variables involved in $\mathbf{x}$ and $\mathbf{y}$. In other words, they study the function ξ satisfying $\xi(\mathbf{x}, \mathbf{y}; \rho) = x \Leftrightarrow P(\mathbf{x}, x, \mathbf{y}) = \rho$. A reasonable model to consider for the function ξ is offered by the *monomial representation*

$$\xi(\mathbf{x}, \mathbf{y}; \rho) = \prod_{i=1}^{n-1} x_i^{-\eta_i(\rho)} \prod_{j=1}^{m} y_j^{\zeta_j(\rho)} C(\rho),$$

in which the η_i's, the ζ_j's and C are functions of ρ. In this paper we investigate the consistency of these difference and monomial representations. The main result is that, under some background conditions, if both the difference and the monomial representations are assumed, then: (i) all functions η_i $(1 \leq i \leq n-1)$ must be

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constant; (ii) if one of the functions ζ_j is nonconstant, then all of them must be of the form $\zeta_j(\rho) = \theta_j \exp[\delta F^{-1}(\rho)]$, for some constants $\theta_j > 0$ $(1 \le j \le m)$ and $\delta \neq 0$, where F^{-1} is the inverse of the function F of the difference representation. Surprisingly, F can be chosen rather arbitrarily.

1. INTRODUCTION

Choice or detection probabilities are often represented by a difference, as in the equation

$$P(X,Y) = F[u(X) - g(Y)], \tag{1}$$

where $P(X,Y)$ denotes either the probability of choosing alternative X over alternative Y or the probability of detecting a stimulus X over a background Y, and u, g, and F are real valued functions, with F strictly increasing and continuous.

Such a representation may arise, for instance, as a special case of a 'random utility model' [3, 15] in which random variables $\mathbf{U}_X$ and $\mathbf{G}_Y$ are attached to X and Y, respectively, and $P(X,Y)$ measures the probability that $\mathbf{U}_X$ exceeds $\mathbf{G}_Y$. If $\mathbf{U}_X$ and $\mathbf{G}_Y$ are independent Gaussian random variables with expectations $u(X)$ and $g(Y)$ and with the same variance equal to $\frac{1}{2}$, then a special case of (1) is obtained through

$$P(X,Y) = \mathbb{P}(\mathbf{U}_X > \mathbf{G}_Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u(X)-g(Y)} e^{-z^2/2}\, dz, \tag{2}$$

where $\mathbb{P}$ denotes the probability measure. (Thus, the function F in (1) is the distribution function of a Gaussian random variable with an expectation equal to zero and a variance equal to one.) When $u = g$ in (2), we get the celebrated 'Law-of-Comparative-Judgements (Case V)' [4, 18, 19]. Sometimes, X and Y are vectors, as in [5, 8, 9]. With $u = g$ and X, Y positive real numbers, this equation offers the theoretical support for Fechner's method for constructing a psychophysical scale [6, 10, 14]. Assuming that u and g are different functions is justified when the choice situation is asymmetrical (for instance, the two alternatives are presented successively, X appearing before Y), or when X is a stimulus to be detected over some background denoted by Y. In general, Eq. (1) plays a fundamental role as a model for choice or detection behavior, either explicitly or implicitly. General references can be found in [6 or 17].

Here, X and Y denote real positive vectors. To stress this fact and for another reason that will soon be apparent, we switch notation from now on and write

$$X = (\mathbf{x}, x_n), \qquad \text{with } \mathbf{x} = (x_1, \ldots, x_{n-1})$$

$$Y = \mathbf{y} = (y_1, \ldots, y_m),$$

thus singling out the last component of X. The quantities represented by the positive real numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ may be evaluated on ratio scales measuring aspects of the stimuli, for instance. To avoid multiplying the parentheses, we write $P(\mathbf{x}, x_n, \mathbf{y})$ to denote the probability of choosing stimulus $(\mathbf{x}, x_n)$ over stimulus $\mathbf{y}$.

In principle, the difference representation of Eq. (1) can be tested experimentally without making specific assumptions regarding the form of the function F (see [6, 11]). Such a test is difficult in practice, however. Moreover, at least in some scientific fields, researchers are reluctant to make assumptions about the function F because they are typically much more interested in the forms of u and g than in that of F (and making an erroneous assumption on F might lead to mistaken conclusions about u and g). For that reason, they routinely study the phenomenon represented by (1) by estimating empirically x such that $P(\mathbf{x}, x, \mathbf{y}) = \rho$, for some values of ρ, and for many values of the variables involved in $\mathbf{x}$ and $\mathbf{y}$. In other words, they study the function $(\mathbf{x}, \mathbf{y}, \rho) \mapsto \xi(\mathbf{x}, \mathbf{y}; \rho)$ satisfying

$$\xi(\mathbf{x}, \mathbf{y}; \rho) = x \Leftrightarrow P(\mathbf{x}, x, \mathbf{y}) = \rho.$$

Special experimental methods have been designed to construct—or at least approximate—the function ξ empirically, and are used routinely in sensory psychology [13]. A simple model for the function ξ is offered by the product

$$\xi(\mathbf{x}, \mathbf{y}; \rho) = \prod_{i=1}^{n-1} x_i^{-\eta_i(\rho)} \prod_{j=1}^{m} y_j^{\zeta_j(\rho)} C(\rho). \tag{3}$$

Such a model, which is linking a collection of ratio scales through a monomial representation, has the form of the laws of classical physics and is a natural one to consider. For examples of applications in psychophysics, see among many: (Case $m = n = 1$) [7, 12, 16]; (Case $m = n = 2$) [8, 9].

Studying the compatibility of the representations (1) and (3) is the subject of this paper. We shall see that, under some reasonable background assumptions concerning the domains of variation of the variables x_i and y_j, the representations (1) and (3) forces all the functions η_i in (3) to be constant. Moreover, either all the functions ζ_j must also be constant and $C = \exp \circ F^{-1}$, where F^{-1} is the inverse of the function F in (1), or if at least one of the ζ_j's is nonconstant, then all of them must have the form $\zeta_j(\rho) = \theta_j \exp[\delta F^{-1}(\rho)]$, for some constants $\theta_j > 0$ $(1 \leq j \leq m)$ and $\delta \neq 0$. None of these results hinges on the assumption that the function P is measuring a probability, i.e. is bounded above by 1 and below by 0. This can be achieved just by choosing the otherwise arbitrary continuous and strictly increasing function F so that its value lies between 0 and 1.

Section 2 is devoted to definitions and preparatory material. The case $n = m = 1$ of Eq. (3) is treated in Section 3, paving the way for our main results in Section 4. A couple of examples are given in Section 5.

2. BASIC CONCEPT AND PRELIMINARY RESULTS

We first consider the function P in (1) and examine critical properties of its domain and its range. (We call *range* of a function the set of its values. This set is frequently called the "co-domain" of the function.)

2.1. DEFINITION. We write $\mathbb{R}$ for the set of real numbers. For any positive integer k, we use the abbreviation

$$\mathbf{1}_k = (\underbrace{1, \ldots, 1}_{k \text{ times}}),$$

a vector in $\mathbb{R}^k$. We will sometimes write

$$(\mathbf{1}_k, x) \quad \text{for} \quad (\underbrace{1, \ldots, 1}_{k \text{ times}}, x)$$

and use other similar improper but convenient notation. For $1 \le i \le n$ and $1 \le j \le m$, let $]a_i, a'_i[$ and $]b_j, b'_j[$ be $n + m$ real open intervals, with $0 < a_i < 1 < a'_i$ and $0 < b_j < 1 < b'_j$. Singling out the interval $]a_n, a'_n[$, we define the Cartesian products

$$\begin{aligned} A_{n-1} &=]a_1, a'_1[\times \cdots \times]a_{n-1}, a'_{n-1}[\qquad (n > 1), \\ B_m &=]b_1, b'_1[\times \cdots \times]b_m, b'_m[, \qquad (m \ge 1), \end{aligned}$$

with

$$\begin{aligned} \mathbf{x} &= (x_1, \ldots, x_{n-1}) \in A_{n-1}, \\ \mathbf{y} &= (y_1, \ldots, y_m) \in B_m, \end{aligned}$$

denoting variable vectors. By convention $A_0 = \varnothing$. A central concept is a real valued function P defined for all $(\mathbf{x}, x_n, \mathbf{y})$ in $A_{n-1} \times]a_n, a'_n[\times B_m$, and with range

$$J = P(A_{n-1} \times]a_n, a'_n[\times B_m).$$

By hypothesis, J *contains the point* $\rho_0 = P(\mathbf{1}_{n-1}, 1, \mathbf{1}_m)$. We suppose that P is continuous in all $n + m$ arguments, strictly increasing in x_i for $1 \le i \le n$ and strictly decreasing in y_j for $1 \le j \le m$. For any fixed $\mathbf{x}$ in A_{n-1} and $\mathbf{y}$ in B_m, the function $x_n \mapsto P(\mathbf{x}, x_n, \mathbf{y})$ is strictly increasing and continuous on $]a_n, a'_n[$. Thus, its range

$$S_{\mathbf{x},\mathbf{y}} = P(\mathbf{x},]a_n, a'_n[, \mathbf{y})$$

must be an open interval.

2.2. LEMMA. (i) *The collection* $(S_{\mathbf{x},\mathbf{y}})_{(\mathbf{x},\mathbf{y}) \in A_{n-1} \times B_m}$ *is an open covering of the range* J *of the function* P.

(ii) *The function* P *is continuous.*

(iii) *The set* J *is an open interval.*

Property (i) of this Lemma will be used repeatedly to extend functions and their properties from the open intervals $S_{\mathbf{x},\mathbf{y}}$ to J.

Proof. (i) For any $\rho \in J$ we have $P(\mathbf{x}, x_n, \mathbf{y}) = \rho$ for some $\mathbf{x} \in A_{n-1}$, $x_n \in]a_n, a'_n[$ and $\mathbf{y} \in B_m$, which implies $\rho \in S_{\mathbf{x},\mathbf{y}}$. Because each $S_{\mathbf{x},\mathbf{y}}$ is an open interval, $(S_{\mathbf{x},\mathbf{y}})_{(\mathbf{x},\mathbf{y}) \in A_{n-1} \times B_m}$ is an open covering of J, which therefore must be an open set.

(ii) The continuity of P follows by a standard argument from the facts that P is strictly monotonic and continuous in each of its variables.

(iii) Because P is continuous on the connected set $A_{n-1} \times]a_n, a'_n[\times B_m$, its range $J = P(A_{n-1} \times]a_n, a'_n[\times B_m)$ is also connected, that is, J is an interval. By (i), J is an open set. Thus, J is an open interval. ∎

2.3. DEFINITION. For any $(\mathbf{x}, \mathbf{y})$ in $A_{n-1} \times B_m$ and ρ in $S_{\mathbf{x},\mathbf{y}}$, there exists a unique x_n in $]a_n, a'_n[$ such that $P(\mathbf{x}, x_n, \mathbf{y}) = \rho$. Denote this x_n by $\xi(\mathbf{x}, \mathbf{y}; \rho)$. Accordingly, the equivalence

$$\xi(\mathbf{x}, \mathbf{y}; \rho) = x_n \Leftrightarrow P(\mathbf{x}, x_n, \mathbf{y}) = \rho \tag{4}$$

defines a function ξ for all $(\mathbf{x}, \mathbf{y})$ in $A_{n-1} \times B_m$ and ρ in $S_{\mathbf{x},\mathbf{y}}$. This function is continuous in all variables, strictly increasing in ρ and in y_j for $1 \le j \le m$, and strictly decreasing in x_i for $1 \le i \le n-1$.

2.4. DEFINITION. We say that the function ξ has a *monomial representation* if

$$\text{[M]} \qquad \xi(\mathbf{x}, \mathbf{y}; \rho) = \prod_{i=1}^{n-1} x_i^{-\eta_i(\rho)} \prod_{j=1}^{m} y_j^{\zeta_j(\rho)} C(\rho), \qquad (\mathbf{x} \in A_{n-1}, \mathbf{y} \in B_m, \rho \in S_{\mathbf{x},\mathbf{y}})$$

holds for some positive valued functions η_i $(1 \le i \le n-1)$, ζ_j $(1 \le j \le m)$, and C defined on the open interval J (cf. Lemma 2.2(iii)).

2.5. *Remark.* In the case $n = m = 1$, we have $A_{n-1} = A_0 = \varnothing$ and $\mathbf{x}$ vanishes from [M]. We then simplify the notation, writing $]a, a'[=]a_1, a_1'[$ and $]b, b'[=]b_1, b_1'[$, with a function $(\rho, y) \mapsto \xi(y; \rho)$ defined by

$$\xi(y; \rho) = x \Leftrightarrow P(x, y) = \rho, \tag{5}$$

which specializes the equivalence (4). The function ξ is defined for all y in $]b, b'[$ and for all ρ in an open interval $S_y = P(]a, a'[, y)$. Notice that, by Lemma 2.2(i) (applied to the case $n = m = 1$), the collection $(S_y)_{y \in]b, b'[}$, is an open covering of $J = P(]a, a'[,]b, b'[)$. Equation [M] becomes

$$\text{[M1]} \qquad \xi(y; \rho) = y^{\zeta(\rho)} C(\rho), \qquad \left(y \in]b, b'[, \rho \in S_y\right)$$

with $\zeta, C > 0$ and defined on J.

The results of this paper concern the pair of functions P and ξ linked by the equivalence (4) and with ξ satisfying [M], with all the side conditions holding. Another representation will also play a central role, which constrains the function P.

2.6. Definition. The function P has a *difference representation* if there exist real valued functions u, g, and F, continuous and strictly increasing in all their variables, such that

$$\text{[D]} \qquad P(\mathbf{x}, x, \mathbf{y}) = F[u(\mathbf{x}, x) - g(\mathbf{y})],$$

for all $(\mathbf{x}, x, \mathbf{y})$ in $A_{n-1} \times]a_n, a_n'[\times B_m$. Thus, u and g are defined on $A_{n-1} \times]a_n, a_n'[$ and B_m, respectively.

In the case where $n = m = 1$, [D] simplifies to

$$\text{[D1]} \qquad P(x, y) = F[u(x) - g(y)]$$

for all $x \in]a, a'[$ and $y \in]b, b'[$; or equivalently, cf. (5),

$$\xi(y; \rho) = u^{-1}[g(y) + F^{-1}(\rho)] = y^{\zeta(y)} C(\rho), \tag{6}$$

if [M1] holds. The following two results will be useful.

2.7. Lemma. *Let P have a difference representation. The domain D of F is an open interval. The real valued functions u_0, g_0, and F_0 defined on $A_{n-1} \times]a_n, a_n'[$, B_m, and $D_0 = \{t \in \mathbb{R} | t + t_0 \in D\}$, respectively, where*

$$t_0 = u(\mathbf{1}_{n-1}, 1) - g(\mathbf{1}_m), \tag{7}$$

by the equations

$$u_0(\mathbf{x}, x) = u(\mathbf{x}, x) - u(\mathbf{1}_{n-1}, 1), \tag{8}$$

$$g_0(\mathbf{y}) = g(\mathbf{y}) - g(\mathbf{1}_m), \tag{9}$$

$$F_0(t) = F(t + t_0), \tag{10}$$

are continuous and strictly increasing in all arguments. Moreover,

(i) D_0 *contains* 0 *and is also an open interval*;

(ii) $P(\mathbf{x}, x, \mathbf{y}) = F_0[u_0(\mathbf{x}, x) - g_0(\mathbf{y})]$
for all $(\mathbf{x}, x, \mathbf{y}) \in A_{n-1} \times]a_n, a'_n[\times B_m$;

(iii) $u_0(\mathbf{1}_{n-1}, 1) = g_0(\mathbf{1}_m) = 0$.

When $m = n = 1$, we have as a consequence of (ii):

$$\xi(y; \rho) = u_0^{-1}\left[g_0(y) + F_0^{-1}(\rho)\right]. \tag{11}$$

Proof. We prove that D is an open interval by an argument similar to that used in the proof for Lemma 2.2 for the open interval J. The function $(\mathbf{x}, x, \mathbf{y}) \mapsto [u(\mathbf{x}, x) - g(\mathbf{y})]$ is strictly monotonic and continuous in all variables, and maps the connected subset $A_{n-1} \times]a_n, a'_n[\times B_m$ of $\mathbb{R}^{n+m}$ onto D. This function is necessarily continuous. Accordingly, D is connected, thus an interval. This interval must be open because it is the union of all the ranges of the functions $x \mapsto u(\mathbf{x}, x) - g(\mathbf{y})$ for all $(\mathbf{x}, \mathbf{y}) \in A_{n-1} \times B_m$, all of which are open intervals.

Clearly, (i) and (ii) in the Lemma are satisfied by definition, and (iii) by substitution. ∎

2.8. PROPOSITION. *The general solution for the functional equation*

$$f(s + t) = r(s)w(t) + f(t), \tag{12}$$

defined for all (s, t) *in an open connected subset* R *of* $\mathbb{R}^2$ *containing* $(0, 0)$, *and where the three functions* f, w, *and* r *are real valued*, f *is continuous and strictly increasing*, *while* w *is nonconstant*, *is given by*

(i) $f(t) = K(1 - e^{\delta t}) + M$

(ii) $r(s) = \frac{K}{L}(1 - e^{\delta s})$

(iii) $w(t) = Le^{\delta t}$,

where $L \neq 0$, $\delta K < 0$, *but otherwise* δ, K, L, *and* M *are arbitrary constants.*

2.9. *Remark.* Note that if r is strictly increasing then $\delta K/L < 0$, and thus $L > O$. Also, if $f(0) = 0$ is assumed, then $M = 0$.

Proof of Proposition 2.8. We first note that if (12) is defined on R, then r is defined on the open interval $R_r = \{s | \exists t$, such that $(s, t) \in R\}$, w is defined on the open interval $R_w = \{t | \exists s$, such that $(s, t) \in R\}$, and f is defined on the union of R_r and the open interval $\{s + t | (s, t) \in R\}$.

Since $0 \in R_w$, we can set $t = 0$ in (12), yielding

$$f(s) = r(s)w(0) + f(0). \tag{13}$$

We cannot have $w(0) = 0$ because that would imply the constancy of f, contrary to our hypothesis. We define $L = w(0) \neq 0$, and from (13) derive

$$r(s) = \frac{1}{L}[f(s) - f(0)]. \tag{14}$$

We also define

$$f_0(t) = f(t) - f(0), \tag{15}$$

$$w_0(t) = \frac{w(t)}{L}, \tag{16}$$

and get, from substituting (14), (15), and (16) into (12),

$$f_0(s + t) = f_0(s)w_0(t) + f_0(t). \tag{17}$$

Because R is open and contains $(0, 0)$, there is a nonempty subset R' of R which is symmetric, that is, $(s, t) \in R'$ implies $(t, s) \in R'$. Thus, for any (s, t) in R'

$$f_0(s)w_0(t) + f_0(t) = f_0(t)w_0(s) + f_0(s). \tag{18}$$

Since w is nonconstant, so is w_0 and there exists some s_0 such that $w_0(s_0) \neq 1$. Choosing $s = s_0$ in (18), we get

$$f_0(t) = K[1 - w_0(t)], \tag{19}$$

where $K = f_0(s_0)/(1 - w_0(s_0)) \neq 0$ because f, and thus f_0, are nonconstant. Note that, as f_0 is continuous, so is w_0. Substituting (19) into (17), we obtain after simplification

$$w_0(s + t) = w_0(s)w_0(t). \tag{20}$$

One can extend (20) from R' to $\mathbb{R}^2$ (see [2, pp. 74–82]) and get as continuous nonconstant general solution

$$w_0(t) = e^{\delta t} \qquad (\delta \neq 0, 1). \tag{21}$$

From (16), we get (iii) in the conclusion of Proposition 2.8. Next, using successively (19), (15) with $M = f(0)$, and then (14), we obtain (i) and (ii) (cf. also [1]). Finally, notice that $L \neq 0$ because w is nonconstant and $\delta K < 0$ because f is strictly increasing. ▮

3. THE CASE $n = m = 1$

The two representations [D1] and [M1] form a consistent system of functional equations. In other words, there exist functions u, g, F, ζ, and C which jointly satisfy [D1] and [M1]. While the forms of u, g, ζ, and C are quite limited, the form of F turns out to be pretty much arbitrary. We suppose here that the function ζ of [M1] is nonconstant. The situation where ζ is constant is a special case of both Theorem 4.3 and Corollary 4.4.

3.1. THEOREM. *Suppose that the functions P and ξ are linked by the equivalence* (5), *and that representations* [D1] *and* [M1] *jointly hold for P and ξ, respectively, with a nonconstant function ζ. Then the general solution for continuous strictly increasing functions u, g, and F, and positive functions ζ and C with ζ nonconstant is given by*

$$u(x) = \frac{1}{\delta} \ln \ln \left(\frac{\tilde{a}}{x} \right)^{\gamma}, \qquad g(y) = \frac{1}{\delta} \ln \ln \left(\frac{\tilde{b}}{y} \right)^{\theta\gamma}, \tag{22}$$

$$\zeta(\rho) = \theta e^{\delta F^{-1}(\rho)}, \qquad C(\rho) = \tilde{b}^{-\theta \exp[\delta F^{-1}(\rho)]} \tilde{a},$$

where $\theta > 0$, $\tilde{a}$, $\tilde{b}$, δ, and γ are constants satisfying either $\tilde{a} \geq a'$, $\tilde{b} \geq b'$ and $\delta < 0 < \gamma$, or $0 < \tilde{a} \leq a, 0 < \tilde{b} \leq b$ and $\delta > 0 > \gamma$, but otherwise arbitrary. There are no additional constraints on the function F.

Actually, we shall prove the following result:

3.2. THEOREM. *Suppose that the functions P and ξ are linked by the equivalence* (5). *The three statements below are then equivalent.*

(i) *The function P satisfies* [D1] *for some continuous and strictly increasing functions u, g, and F. Moreover, the function ξ satisfies* [M1] *for some positive functions ζ and C defined on J, with ζ nonconstant.*

(ii) *The functions P satisfies* [D1] *for some functions u, g, and F, with u and g specified as follows. There exist five constants $\theta > 0$, $\tilde{a}$, $\tilde{b}$, δ, and γ satisfying either $\tilde{a} \geq a'$, $\tilde{b} \geq b'$, and $\gamma > 0 > \delta$ (Case* [a1]), *or $0 < \tilde{a} \leq a$, $0 < \tilde{b} \leq b$ and $\gamma < 0 < \delta$ (Case* [b1]), *such that*

$$u(x) = \frac{1}{\delta} \ln \ln \left(\frac{\tilde{a}}{x} \right)^{\gamma} \tag{23}$$

$$g(y) = \frac{1}{\delta} \ln \ln \left(\frac{\tilde{b}}{y} \right)^{\theta\gamma}. \tag{24}$$

Consequently, we have

$$P(x, y) = F\left(\frac{1}{\delta} \ln \ln \left(\frac{\tilde{a}}{x} \right)^{\gamma} - \frac{1}{\delta} \ln \ln \left(\frac{\tilde{b}}{y} \right)^{\theta\gamma} \right) \tag{25}$$

with F strictly increasing and continuous, but otherwise arbitrary.

(iii) *The function ξ satisfies* [M1]. *More specifically, there exists four constants $\theta > 0$, $\tilde{a}$, $\tilde{b}$, and δ satisfying either $\tilde{a} \geq a'$, $\tilde{b} \geq b'$, and $\delta < 0$ (Case* [a1]), *or $0 < \tilde{a} \leq a$, $0 < \tilde{b} \leq b$, and $\delta > 0$ (Case* [b1]), *such that*

$$\zeta(\rho) = \theta e^{\delta G(\rho)} \tag{26}$$

$$C(\rho) = \tilde{b}^{-\theta \exp[\delta G(\rho)]} \tilde{a}, \tag{27}$$

where G is a strictly increasing continuous function on J. Consequently, [M1] *takes the form*

$$\xi(y; \rho) = y^{\theta \exp[\delta G(\rho)]} \tilde{b}^{-\theta \exp[\delta G(\rho)]} \tilde{a}. \tag{28}$$

For the functions F in (i) *and* (ii) *and G in* (iii), *we have $G = F^{-1}$.*

Note in passing that Eq. (28) can be written under the simpler form

$$\frac{\xi(y; \rho)}{\tilde{a}} = \left(\frac{y}{\tilde{b}} \right)^{\theta \exp[\delta G(\rho)]}.$$

In the proof below, we follow the scheme: (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

Proof of Theorem 3.2. (i) ⇒ (ii). Condition [M1] implies that $\xi(y; \rho)$ is homogeneous of degree $\zeta(\rho)$ in y, that is, for any y in $]b, b'[$ and $\lambda > 0$ such that $\lambda y \in]b, b'[$,

$$\xi(\lambda y; \rho) = (\lambda y)^{\zeta(\rho)} C(\rho) = \lambda^{\zeta(\rho)} \xi(y; \rho).$$

Using [D1] and (6), (11) with u_0, g_0 and F_0 as in Lemma 2.7, we obtain the functional equation

$$u_0^{-1}\left[g_0(\lambda y) + F_0^{-1}(\rho)\right] = \lambda^{\zeta(\rho)} u_0^{-1}\left[g_0(y) + F_0^{-1}(\rho)\right]. \tag{29}$$

Taking logarithms on both sides and setting $y = 1$ yields, with $g_0(1) = 0$ (cf. Lemma 2.7, Eq. (9))

$$\ln\left\{u_0^{-1}\left[g_0(\lambda) + F_0^{-1}(\rho)\right]\right\} = \zeta(\rho)\ln\lambda + \ln\left\{u_0^{-1}\left[F^{-1}(\rho)\right]\right\}. \tag{30}$$

By writing

$$s = g_0(\lambda), \qquad t = F_0^{-1}(\rho).$$

and

$$f_0 = \ln\circ u_0^{-1}, \qquad w = \zeta\circ F_0, \qquad r = \ln\circ g_0^{-1}, \tag{31}$$

Eq. (30) becomes

$$f_0(s+t) = r(s)w(t) + f_0(t). \tag{32}$$

From the assumptions of the Theorem and by Lemma 2.7 and (31), we deduce that the three functions in (32) are continuous, and defined on an open, connected subset of $\mathbb{R}\times\mathbb{R}$ containing the point $(0,0)$. Notice that $f_0(0) = \ln[u_0^{-1}(0)] = \ln 1 = 0$. Moreover, the function $w(t) = \zeta[F_0(t)]$ takes at least two distinct values. From Proposition 2.8 and Remark 2.9 ($\delta K/L < 0$; $f_0(0) = 0$ implies $M = 0$), we conclude that the only possible form for the three functions in (32) are the following:

$$f_0(t) = K(1 - e^{\delta t}) = \ln\left[u_0^{-1}(s)\right] \tag{33}$$

$$r(s) = \frac{K}{L}(1 - e^{\delta s}) = \ln\left[g_0^{-1}(t)\right] \tag{34}$$

$$w(t) = Le^{\delta t} = \zeta\left[F_0(t)\right], \tag{35}$$

where L, K, and δ are constants that satisfy $L > 0$ and $\delta K < 0$, but are otherwise arbitrary, and the last equality in each line recalls the definitions of f_0, r and w in (31). From these three equations, we can easily derive the forms of the functions u, g, and ζ, the latter in terms of the function F. With $F_0(t) = \rho$ we get from (35), using (10),

$$\zeta(\rho) = Le^{\delta F_0^{-1}(\rho)} = Le^{\delta[F^{-1}(\rho) - t_0]} = \frac{L}{e^{\delta t_0}}e^{\delta F^{-1}(\rho)} = \theta e^{\delta F^{-1}(\rho)},$$

where (cf. (7))

$$\theta = L/e^{\delta t_0} = L/e^{\delta[u(1)-g(1)]} > 0. \tag{36}$$

Thus, for all $\rho \in J$,

$$\zeta(\rho) = \theta e^{\delta F^{-1}(\rho)}. \tag{37}$$

We now define the two constants

$$\tilde{a} = e^K \tag{38}$$

$$\tilde{b} = e^{K/L}. \tag{39}$$

Note that we must have

$$\tilde{a} \notin]a, a'[\text{ and } \tilde{b} \notin]b, b'[. \tag{40}$$

Indeed, with $x = u_0^{-1}(s) \in]a, a'[$, Eq. (33) yields $\ln x = K[1 - e^{\delta u_0(x)}]$, that is

$$1 - \frac{1}{K}\ln x = e^{\delta u_0(x)} > 0, \tag{41}$$

yielding $1 > (1/K)\ln x$. Thus, either $\tilde{a} = e^K > x$ for all $x \in]a, a'[$ (if $K > 0$), so $\tilde{a} \geq a'$, or by a similar argument $\tilde{a} \leq a$ (if $K < 0$). The argument is similar for g_0 and $\tilde{b}$, with L/K and (34) replacing $1/K$ and (33), i.e. $\tilde{b} \notin]b, b'[$ must hold. In view of $\delta K < 0$, two cases arise:

Case [a1]: either $\tilde{a} \geq a' > 1$ and $\tilde{b} \geq b' > 1$, thus $K > 0 > \delta$;
Case [b1]: or $0 < \tilde{a} \leq a < 1$ and $0 < \tilde{b} \leq b < 1$, thus $K < 0 < \delta$.

From (41), we obtain by the definition of u_0 (see Eq. (8) in Lemma 2.7) and by (38),

$$\ln\left(\frac{\tilde{a}}{x}\right)^{1/K} = \ln\left(\frac{e^K}{x}\right)^{1/K} = 1 - \frac{1}{K}\ln x = e^{\delta u_0(x)} = \frac{e^{\delta u(x)}}{e^{\delta u(1)}}$$

or equivalently, with

$$\gamma = e^{\delta u(1)}/K, \tag{42}$$

we get

$$u(x) = \frac{1}{\delta}\ln\ln\left(\frac{\tilde{a}}{x}\right)^{\gamma}, \tag{43}$$

which proves (23). Notice that $\ln(\tilde{a}/x)^{\gamma}$ is positive for all $x \in]a, a'[$: either Case [a1] holds with $\tilde{a} \geq a' > x$ and $K, \gamma > 0$; or Case [b1] holds with $\tilde{a} \leq a < x$ and $K, \gamma < 0$. Comparing Eqs. (33) and (34), we see that $g(x)$ differs from $u(x)$ only in that we replace $1/K$ and $u(1)$ by L/K and $g(1)$, respectively. Noticing (cf. (36) and (42)) that

$$\frac{Le^{\delta g(1)}}{K} = \frac{L}{e^{\delta[u(1)-g(1)]}} \cdot \frac{e^{\delta u(1)}}{K} = \theta\gamma$$

we obtain

$$g(y) = \frac{1}{\delta} \ln \ln\left(\frac{\tilde{b}}{y}\right)^{\theta\gamma}, \tag{44}$$

which is (24), with $\ln(\tilde{b}/y)^{\theta\gamma}$ always positive whether Case [a1] holds with $\tilde{b} \geq b > y$ and $K, \gamma > 0$, or Case [b1] holds with $\tilde{b} \leq b < y$ and $K, \gamma < 0$. From [D1], (43) and (44), we get

$$P(x, y) = F\left(\frac{1}{\delta} \ln \ln\left(\frac{\tilde{a}}{x}\right)^{\gamma} - \frac{1}{\delta} \ln \ln\left(\frac{\tilde{b}}{y}\right)^{\gamma\theta}\right) \tag{45}$$

which is (25), establishing the implication "(i) ⇒ (ii)" of the theorem.

(ii) ⇒ (iii). Solving (25) for $x = \xi(y; \rho)$ with $\rho = P(x, y)$ and $G = F^{-1}$ yields (after some manipulation in which the γ's cancel out):

$$\begin{aligned} x = \xi(y; \rho) &= \left(\frac{y}{\tilde{b}}\right)^{\theta \exp[\delta G(\rho)]} \tilde{a} \\ &= \overbrace{y^{\theta \exp[\delta G(\rho)]}}^{\zeta(\rho)} \overbrace{\tilde{b}^{-\theta \exp[\delta G(\rho)]} \tilde{a}}^{C(\rho)}, \end{aligned} \tag{46}$$

that is, (28) holds with ζ and C specified by (26) and (27).

(iii) ⇒ (i). By hypothesis, the function ξ satisfies [M1], where ζ and C are defined on J by (26) and (27), respectively, G is some real valued, strictly increasing, continuous function, and with constants $\theta > 0$, $\tilde{a}$, $\tilde{b}$, and δ satisfying either $\tilde{a} \geq a'$, $\tilde{b} \geq b'$, and $\delta < 0$ (in Case [a1]), or $\tilde{a} \leq a$, $\tilde{b} \leq b$, and $\delta > 0$ (in Case [b1]). The function ξ satisfies thus (46). Solving (46) for $\rho = P(x, y)$ with $F = G^{-1}$ yields (45) for any constant γ positive in Case [a1] and negative in Case [b1]. Equation (45) has the form [D1] with continuous strictly increasing u and g defined on $]a, a'[$ and $]b, b'[$, and F strictly increasing and continuous. Thus, both [D1] and [M1] are verified, the latter with positive ζ and C, and ζ nonconstant; that is, statement (i) holds. This completes the proof of Theorem 3.2. ∎

4. THE GENERAL CASE

4.1 THEOREM. *Suppose that* (P, ξ) *is a pair of functions linked by the equivalence* (4). *The following three conditions are then equivalent.*

(i) *The function* P *satisfies* [D] *for some functions* u, g, *and* F *strictly increasing and continuous in all arguments. Moreover,* ξ *satisfies* [M] *for some positive functions* C, η_i $(1 \le i \le n-1)$, *and* ζ_j $(1 \le j \le m)$, *all defined on* J, *with at least one of the* ζ_j *nonconstant.*

(ii) *The function* P *satisfies* [D] *with* F *strictly increasing and continuous and with* u, g *specified by*

$$u(\mathbf{x}, x) = \frac{1}{\delta} \ln \ln \left(\frac{\tilde{A}}{x \prod_{i=1}^{n-1} x_i^{\alpha_i}} \right)^{\gamma} \tag{47}$$

$$g(\mathbf{y}) = \frac{1}{\delta} \ln \ln \left(\frac{\tilde{B}}{\prod_{j=1}^{m} y_j^{\theta_j}} \right)^{\gamma}. \tag{48}$$

for some constants $\alpha_i > 0$ $(1 \le i \le n-1)$, $\theta_j > 0$ $(1 \le j \le m)$, γ, δ, $\tilde{A}$, *and* $\tilde{B}$, *the latter four satisfying either Case* [a] *or Case* [b] *below*:

$$(\text{Case [a]}) \quad \delta < 0 < \gamma, \qquad \tilde{A} \ge a'_n \prod_{i=1}^{n-1} a_i'^{\alpha_i} > 1, \qquad \tilde{B} \ge \prod_{j=1}^{m} b_j'^{\theta_j},$$

$$(\text{Case [b]}) \quad \delta > 0 > \gamma, \qquad 0 < \tilde{A} \le a_n \prod_{i=1}^{n-1} a_i^{\alpha_i}, \qquad 0 < \tilde{B} \le \prod_{j=1}^{m} b_j^{\theta_j}.$$

Accordingly, the function P *takes the form*

$$P(\mathbf{x}, x, \mathbf{y}) = F\left[\frac{1}{\delta} \ln \ln \left(\frac{\tilde{A}}{x \prod_{i=1}^{n-1} x_i^{\alpha_i}} \right)^{\gamma} - \frac{1}{\delta} \ln \ln \left(\frac{\tilde{B}}{\prod_{j=1}^{m} y_j^{\theta_j}} \right)^{\gamma} \right] \tag{49}$$

$$= \begin{cases} F\left[\frac{1}{\delta} \ln \ln \left(\frac{\tilde{A}}{x \prod_{i=1}^{n-1} x_i^{\alpha_i}} \right) - \frac{1}{\delta} \ln \ln \left(\frac{\tilde{B}}{\prod_{j=1}^{m} y_j^{\theta_j}} \right) \right] & \text{if } \gamma > 0, \quad \text{(49a)} \\ F\left[\frac{1}{\delta} \ln \ln \left(\frac{x \prod_{i=1}^{n-1} x_i^{\alpha_i}}{\tilde{A}} \right) - \frac{1}{\delta} \ln \ln \left(\frac{\prod_{j=1}^{m} y_j^{\theta_j}}{\tilde{B}} \right) \right] & \text{if } \gamma < 0. \quad \text{(49b)} \end{cases}$$

(iii) *The function* ξ *satisfies* [M] *for some positive functions* C, η_i, *and* ζ_j, *all defined on* J, *with constant* $\eta_i = \alpha_i$ $(1 \le i \le n-1)$, *and nonconstant* ζ_j $(1 \le j \le m)$. *Moreover, there exist constants* $\theta_j > 0$ $(1 \le j \le m)$, $\delta \neq 0$, $\tilde{A}$, $\tilde{B}$ *satisfying either Case* [a] *or Case* [b] *above, such that for all* $\rho \in J$

$$\zeta_j(\rho) = \theta_j \exp[\delta G(\rho)], \qquad (1 \le j \le m), \tag{50}$$

$$C(\rho) = \tilde{B}^{-\exp[\delta G(\rho)]}\tilde{A}, \tag{51}$$

where G *is a strictly increasing and continuous* (*but otherwise arbitrary*) *function on* J. *Consequently*, [M] *takes the form*

$$\xi(\mathbf{x}, \mathbf{y}; \rho) = \tilde{A} \prod_{i=1}^{n-1} x_i^{-\alpha_i} \left(\frac{1}{\tilde{B}} \prod_{j=1}^{m} y_j^{\theta_j} \right)^{\exp[\delta G(\rho)]}. \tag{52}$$

Notice that Theorem 3.2 is the particular case of Theorem 4.1 where $n = m = 1$, $\tilde{a} = \tilde{A}$ and $\tilde{b} = \tilde{B}^{1/\theta}$. Because our proof is long, we first summarize it.

We begin by establishing the implication "(i) ⇒ (iii)," and prove that, when one of the ζ_j's in [M] is nonconstant, then they all must be nonconstant and of the form specified by (50), with $\theta_j > 0$, $\delta \neq 0$, and $G = F^{-1}$. We then prove (51). Finally, we show that all η_i must be constant if one of the ζ_j's is nonconstant. (The case where all ζ_j's are constant is treated in Theorem 4.2.) Equation (52) obtains. The representations (49) and (52) follow easily from each other, with $G = F^{-1}$ and γ arbitrarily positive or negative in Case [a] or [b], respectively. We have thus "(i) ⇒ (iii) ⇔ (ii)." It remains to establish "(ii) and (iii) ⇒ (i)," which is readily obtained by observing that (49) has the form [D], with u and g defined by (47) and (48), and that (52) has the form [M], with the η_i constant, and the ζ_j and C defined by (50) and (51), respectively.

In proving "(i) ⇒ (iii)" (the main difficulty), it will be convenient to advance according to the following plan.

Outline of (i) ⇒ (iii)

Step 1. We suppose that one of the exponent functions ζ_j in [M], say ζ_1, is nonconstant. Keeping $x_1, \ldots, x_{n-1}$ and $y_2, \ldots, y_m$ constant and using Theorem 3.2, it follows that $\zeta_1(\rho) = \theta_1 e^{\delta_1 F^{-1}(\rho)}$ with constants $\theta_1 > 0$ and $\delta_1 \neq 0$ (which, however, may depend upon $x_1, \ldots, x_{n-1}$ and $y_2, \ldots, y_m$) and for all ρ in J.

Step 2. We show that, if two exponent functions ζ_j are nonconstant, then the constant δ_j must be the same in both cases. Thus, if any exponent function ζ_j is nonconstant, then we must have $\zeta_j(\rho) = \theta_j e^{\delta F^{-1}(\rho)}$ for some constants $\theta_j > 0$ and $\delta \neq 0$.

Step 3. Using Theorem 3.2 again, we prove that, if one of the exponent functions ζ_j is nonconstant, then the function C of [M] must have the form

$$C(\rho) = \tilde{B}^{\exp[\delta F^{-1}(\rho)]}\tilde{A}$$

for some positive constants $\tilde{A}$ and $\tilde{B}$ with either: $\tilde{A} \geq a'_n$, $\tilde{B} \geq b_j'^{\theta_j}$ $(1 \leq j \leq m)$, and $\delta < 0$ (Case [a]); or $\tilde{A} \leq a_n$, $\tilde{B} \leq b_j^{\theta_j}$ $(1 \leq j \leq m)$, and $\delta > 0$ (Case [b]).

Step 4. We then consider the case where one of the functions ζ_j would be nonconstant, while some other function ζ_k would be constant, and we show that a contradiction arises.

Step 5. We then turn to the exponent functions η_i. We suppose that one of these functions is nonconstant, and that one of the functions ζ_j is also nonconstant. We prove that this hypothesis leads to a contradiction. Thus, all exponent functions η_i must be constant.

Step 6. Equation (52) follows by substituting in [M] the functions η_i, ζ_j and C by their expressions obtained in Steps 1–5.

4.2. CONVENTION. *In the proof, we shall use the functions* u_0, g_0, *and* F_0 *introduced in Lemma* 2.7 *but omit the subscript* 0. *The original functions* u, g, *and* F *are restored near the end of the proof of Theorem* 4.1.

Proof of Theorem 4.1. (i) ⇒ (iii).

Step 1. Suppose that $\zeta_1(\rho)$ is nonconstant. Fix $\mathbf{x}_0 = (x_1, \ldots, x_{n-1})$, and $\mathbf{y}_0 = (y_2, \ldots, y_m)$ arbitrarily. Then [M] reduces to

$$\mu(y; \rho) = \xi(\mathbf{x}_0, (y, \mathbf{y}_0); \rho) = y^{\zeta_1(\rho)} C^*(\rho), \tag{53}$$

where the first equality defines the function μ and with

$$C^*(\rho) = \prod_{i=1}^{n-1} x_i^{-\eta_i(\rho)} \prod_{j=2}^{m} y_j^{\zeta_j(\rho)} C(\rho).$$

Similarly, [D] reduces to

$$\begin{aligned} p(x, y) &= P(\mathbf{x}_0, x, (y, \mathbf{y}_0)) \\ &= F[u(\mathbf{x}_0, x) - g(y, \mathbf{y}_0)] \\ &= H[v(x) - h(y)], \end{aligned}$$

with obvious definitions of the functions v, h, and with H an appropriate restriction of the function F of statement (i). Note that the range of the function H is an open interval. This interval, as well as the functions v, h, and H, may depend upon the values chosen for $\mathbf{x}$ and $\mathbf{y}$. Accordingly, we denote the range of H by $I_{\mathbf{x},\mathbf{y}}$. Applying Theorem 3.2 to the pair of functions (p, μ), there exist $\theta_1(\mathbf{x},\mathbf{y}) > 0$ and $\delta_1(\mathbf{x},\mathbf{y}) \neq 0$ such that

$$\zeta_1(\rho) = \theta_1(\mathbf{x},\mathbf{y}) e^{\delta_1(\mathbf{x},\mathbf{y}) G_{1,\mathbf{x},\mathbf{y}}(\rho)},$$

for some strictly increasing, continuous function $G_{1,\mathbf{x},\mathbf{y}}$ defined for all ρ in $I_{\mathbf{x},\mathbf{y}}$. Note that $G_{1,\mathbf{x},\mathbf{y}}$ is the restriction of F^{-1} to $I_{\mathbf{x},\mathbf{y}}$. So we can write

$$\zeta_1(\rho) = \theta_1(\mathbf{x},\mathbf{y}) e^{\delta_1(\mathbf{x},\mathbf{y}) G(\rho)} \quad \text{for all } \mathbf{x},\mathbf{y}. \tag{54}$$

By Convention 4.2 and Lemma 2.7, 0 is in the domain of F. If $F(0) = \rho_0$, then $G(\rho_0) = 0$. Setting $\rho = \rho_0$ in (54) yields $\theta_1(\mathbf{x},\mathbf{y}) = \zeta_1(\rho_0)$, a constant. We denote that constant by θ_1. We have now $\zeta_1(\rho) = \theta_1 e^{\delta_1(\mathbf{x},\mathbf{y}) G(\rho)}$. The left hand side of the last equation is independent of $\mathbf{x}$ and $\mathbf{y}$, so also the right side; thus $\delta_1(\mathbf{x},\mathbf{y})$ is constant. Denoting this constant by δ_1, we get $\zeta_1(\rho) = \theta_1 e^{\delta_1 G(\rho)}$ for all ρ in J with constant θ_1, δ_1, and with $G = F^{-1}$. This argument can be used to prove that every nonconstant function ζ_j has the form

$$\zeta_j(\rho) = \theta_j e^{\delta_j G(\rho)},$$

where $\theta_j > 0$ and $\delta_j \neq 0$ are constant.

Step 2. Suppose thus that ζ_1 and ζ_2 are nonconstant. This implies

$$\zeta_1(\rho) = \theta_1 e^{\delta_1 G(\rho)},$$

$$\zeta_1(\rho) = \theta_2 e^{\delta_2 G(\rho)},$$

with $G = F^{-1}$ and constant $\theta_1, \theta_2 > 0$ and $\delta_1, \delta_2 \neq 0$, cf. Step 1. Putting these ζ_1, ζ_2 into [M] with $y_1 = y_2 = y$ and fixing $\mathbf{x} = (x_1, \dots, x_{n-1})$ and $\mathbf{y} = (y_3, \dots, y_m)$, we get for all y in $]a_1, a_1'[\cap]a_2, a_2'[$

$$\overline{\mu}_\rho(y) = \xi(\mathbf{x}, (y, y, \mathbf{y}); \rho) = y^{\zeta_{1,2}(\rho)} \overline{C}(\rho) \tag{55}$$

with

$$\zeta_{1,2}(\rho) = \theta_1 \exp[\delta_1 G(\rho)] + \theta_2 \exp[\delta_2 G(\rho)] \tag{56}$$

and

$$\overline{C}(\rho) = \prod_{i=1}^{n-1} x_i^{-\eta_i(\rho)} \prod_{j=2}^{m} y_j^{\zeta_j(\rho)} C(\rho).$$

But a functional form for the nonconstant function $\zeta_{1,2}$ of (55)–(56) can also be obtained directly from Theorem 3.2, applied to the function $\overline{\mu}$ in (55) and to the function $\bar{p}$ defined by

$$\bar{p}(x, y) = \overline{H}\left[\bar{v}(x) - \bar{h}(y)\right]$$
$$= P(\mathbf{x}, (y, y, \mathbf{y})) = F\left[u(\mathbf{x}, x) - g(y, y, \mathbf{y})\right].$$

This yields

$$\zeta_{1,2}(\rho) = \theta_{1,2} \exp\left[\delta_{1,2} G(\rho)\right]$$

for all ρ in some open interval J' of J, where $\theta_{1,2} > 0$ and $\delta_{1,2} \neq 0$ are constant, by a similar argument as in Step 1. This gives

$$\theta_1 e^{\delta_1 G(\rho)} + \theta_2 e^{\delta_2 G(\rho)} = \theta_{1,2} e^{\delta_{1,2} G(\rho)} \tag{57}$$

for all ρ in J. Equation (57) states that the functions $e^{\delta_1 s}$, $e^{\delta_2 s}$, and $e^{\delta_{1,2} s}$ are linearly dependent, which holds only if $\delta_{1,2} = \delta_1 = \delta_2 = \delta$, where the last equality defines the constant δ. It shows also that $\delta_1 < 0 < \delta_2$ is not possible. This applies obviously to all $s = G(\rho)$ and thus to all ρ in J. The above argument can be used for any pair of subscripts i, j for which ζ_j and ζ_i are nonconstant. So, $\delta j = \delta$, that is, (50) holds for any nonconstant ζ_j in [M]; we have thus

$$\zeta_j(\rho) = \theta_j e^{\delta G(\rho)}. \tag{58}$$

From here on, to avoid lengthy formulas in our calculations, we occasionally adopt the abbreviation

$$\Delta(\rho) = e^{\delta G(\rho)}. \tag{59}$$

Step 3. We turn to the function C of [M]. If ζ_1 is nonconstant, we have by Theorem 3.2, for some constants $\tilde{b}_1$ and $\tilde{A}$,

$$C(\rho) = \tilde{b}_1^{-\theta_1 \Delta(\rho)} \tilde{A},$$

with either $\tilde{b}_1 \geq b'_1$, $\tilde{A} \geq a_n$, and $\delta < 0$ (Case [a]), or $\tilde{b}_1 \leq b_1$, $\tilde{A} \leq a_n$, and $\delta > 0$ (Case [b]).

Again, the same argument can be used for any subscript j for which ζ_j is nonconstant. In particular, if both ζ_j and ζ_k are nonconstant, we would thus have

$$C(\rho) = \tilde{b}_j^{-\theta_j \Delta(\rho)} \tilde{A} = \tilde{b}_k^{-\theta_k \Delta(\rho)} \tilde{A},$$

leading to $\tilde{b}_j^{\theta_j} = \tilde{b}_k^{\theta_k}$. Thus $\tilde{B} := \tilde{b}_j^{\theta_j}$ is independent of the subscript. Since δ does not depend upon the subscript and can be positive or negative, we have either $\delta < 0$ and $\tilde{B} \geq b_j'^{\theta_j}$, $\tilde{B} \geq b_k'^{\theta_k}$, or $\delta > 0$ and $\tilde{B} \leq b_j^{\theta_j}$, $\tilde{B} \leq b_k^{\theta_k}$, a dichotomy which generalizes to all subscripts $1 \leq j \leq m$ (see Cases [a] and [b] below). We obtain, using (59)

$$C(\rho) = \tilde{B}^{-\exp[\delta F^{-1}(\rho)]}\tilde{A}. \tag{60}$$

The two possibilities

$$(\text{Case [a]})\ \delta < 0, \qquad \tilde{A} \geq a_n', \quad \text{and} \quad \tilde{B} \geq b_j'^{\theta_j}\ (1 \leq j \leq m) \tag{61}$$

$$(\text{Case [b]})\ \delta > 0, \qquad \tilde{A} \leq a_n', \quad \text{and} \quad \tilde{B} \leq b_j'^{\theta_j}\ (1 \leq j \leq m) \tag{62}$$

will be elaborated later in this proof.

Step 4. We now consider the case where one of the functions ζ_j in [M], say ζ_1, would vary with ρ, while another, say ζ_2, would remain constant: $\zeta_2(\rho) = \theta_2 > 0$ for all ρ. Setting $x_1 = \cdots = x_{n-1} = y_3 = \cdots = y_m = 1$ in [M], we obtain by (58) and (60)

$$\begin{aligned} \xi[\mathbf{1}_{n-1}, (y_1, y_2, \mathbf{1}_{m-2}); \rho] &= y_1^{\zeta_1(\rho)} y_2^{\theta_2} C(\rho) \\ &= y_1^{\theta_1 \exp[\delta G(\rho)]} y_2^{\theta_2} \tilde{B}^{-\exp[\delta G(\rho)]}\tilde{A} = x. \end{aligned}$$

Solving for $G(\rho) = F^{-1}(\rho) = u(\mathbf{1}_{n-1}, x) - g(y_1, y_2, \mathbf{1}_{m-2})$ (cf. [D] and (4)) leads to

$$u_1(x) - k(y_1, y_2) = \frac{1}{\delta}\ln\left(\frac{\ln y_2^{\theta_2} + \ln \tilde{A} - \ln x}{\ln \tilde{B} - \ln y_1^{\theta_1}}\right), \tag{63}$$

where

$$u_1(x) = u(\mathbf{1}_{n-1}, x), \qquad k(y_1, y_2) = g(y_1, y_2, \mathbf{1}_{m-2}).$$

Thus u_1 is strictly increasing, k is strictly increasing in both variables, and $u_1(1) = 0$ by Lemma 2.7(iii) and Convention 4.2. This leads to a contradiction because, on the right hand side of (63), x and (y_1, y_2) cannot be additively separated. To show this, we put $x = 1$ in (63) and get

$$-k(y_1, y_2) = \frac{1}{\delta}\ln\left(\frac{\ln y_2^{\theta_2} + \ln \tilde{A}}{\ln \tilde{B} - \ln y_1^{\theta_1}}\right).$$

Substituting this $-k(y_1, y_2)$ in (63) yields

$$u_1(x) + \frac{1}{\delta}\ln\left(\frac{\ln y_2^{\theta_2} + \ln \tilde{A}}{\ln \tilde{B} - \ln y_1^{\theta_1}}\right) = \frac{1}{\delta}\ln\left(\frac{\ln y_2^{\theta_2} + \ln \tilde{A} - \ln x}{\ln \tilde{B} - \ln y_1^{\theta_1}}\right),$$

or equivalently,

$$u_1(x) = \frac{1}{\delta}\ln\left(1 - \frac{\ln x}{\ln y_2^{\theta_2} + \ln \tilde{A}}\right),$$

an equation whose right side varies with both x and y_2 (because $\theta_2 \neq 0$), while the left side varies with x, an absurdity. We conclude that if, as hypothesized in statement (i), one of the exponent functions ζ_j is nonconstant, then none of them is constant. Moreover, there are constants $\theta_j > 0$ $(1 \leq j \leq m)$, δ, $\tilde{A}$, and $\tilde{B}$, with the latter three satisfying either (61) (Case [a]) or (62) (Case [b]), such that $G = F^{-1}$

$$\zeta_j(\rho) = \theta_j \exp[\delta G(\rho)] \tag{64}$$

$$C(\rho) = \tilde{B}^{-\exp[\delta G(\rho)]}\tilde{A}. \tag{65}$$

We still have to show that all exponent functions η_i, $1 \leq i \leq n-1$ are constant if one of the ζ_j is nonconstant. We proceed by contradiction and suppose that η_1 (for example) varies with ρ. Since, as we have just seen, all ζ_j's are nonconstant if one of them is, we may as well consider ζ_m.

Step 5. Suppose by contradiction that η_1 and ζ_m are nonconstant. In [M], set $x_2 = \cdots = x_{n-1} = y_1 = \cdots = y_{m-1} = 1$. By the argument in Step 4, using (64) with $j = 1$ and (65), we obtain

$$\begin{aligned} x_n &= \xi(x_1, \mathbf{1}_{n-2}, (\mathbf{1}_{m-1}, y_m); \rho) \\ &= x_1^{-\eta_1(\rho)} y_m^{\theta_m \Delta(\rho)} \tilde{B}^{-\Delta(\rho)} \tilde{A}. \end{aligned} \tag{66}$$

Solving this equation for x_1 yields

$$\begin{aligned} x_1 &= \kappa(x_n, y_m; \rho) \\ &= y_m^{\theta_m \Delta(\rho)/\eta_1(\rho)} \tilde{B}^{-\Delta(\rho)/\eta_1(\rho)} \left(\tilde{A}/x_n\right)^{1/\eta_1(\rho)}, \end{aligned} \tag{67}$$

in which the second equality defines the function κ.

Case 1. Suppose first that $\phi(\rho) = \theta_m \Delta(\rho)/\eta_1(\rho)$ is nonconstant in ρ. This will lead to a contradiction. Indeed, notice that, for a fixed x_n, κ has the form [M1]. Fixing temporarily x_n, we apply Theorem 3.2 to κ and the appropriate special case of [D] to get, with ζ, $\tilde{a}$ and $\tilde{b}$ as in Theorem 3.2,

$$\frac{\theta_m \Delta(\rho)}{\eta_1(\rho)} = \zeta(\rho) = \theta'(x_n)\exp[\delta'(x_n)G(\rho)] \tag{68}$$

$$\tilde{B}^{-\Delta(\rho)/\eta_1(\rho)}\left(\tilde{A}/x_n\right)^{1/\eta_1(\rho)} = \tilde{b}^{-\theta'(x_n)\exp[\delta'(x_n)G(\rho)]}\tilde{a} \tag{69}$$

for some "constants" $\theta'(x_n) > 0$ and $\delta'(x_n) \neq 0$, which may *a priori* depend upon x_n. (Thus, the left hand side of (69) plays the role of $C(\rho)$ in Theorem 3.2, cf. Eq. (27).) By Lemma 2.7(i) (cf. Convention 4.2), 0 is in the domain of F, say $F(0) = \rho_0$, that is $G(\rho_0) = 0$ (cf. Step 1). Setting $\rho = \rho_0$ in (68) yields $\zeta(\rho) = \theta'(x_n)$. Thus, θ' does not depend upon the value of x_n, and because $\exp[\delta'(x_n)G(\rho)] = \zeta(\rho)/\theta'(x_n)$ does not depend on x_n, neither does δ'. We obtain thus from (68) and (59)

$$\eta_1(\rho) = \frac{\theta_m}{\theta'}\exp[(\delta - \delta')G(\rho)] \tag{70}$$

with

$$\delta \neq \delta' \tag{71}$$

because by hypothesis η_1 is nonconstant with ρ. Raising both sides of (69) to the power of $\eta_1(\rho)$ and using (70), we can thus, after some manipulation, rewrite this equation as

$$\tilde{b}^{-\theta_m \exp[\delta G(\rho)]}\tilde{a}^{(\theta_m/\theta')\exp[(\delta-\delta')G(\rho)]} = \tilde{B}^{-\exp[\delta G(\rho)]}\left(\tilde{A}/x_n\right).$$

Taking logarithms on both sides and rearranging yields

$$e^{\delta G(\rho)}\left(\ln \tilde{B} - \theta_m \ln \tilde{b}\right) + \frac{\theta_m}{\theta'}e^{G(\rho)(\delta-\delta')}\ln \tilde{a} = \ln\left(\tilde{A}/x_n\right).$$

Thus, the exponential functions $e^{\delta G(\rho)}$, $e^{[G(\rho)(\delta-\delta')]}$, and e^0 are linearly dependent. By an argument already used earlier (in Step 2), this can happen only if $\delta = \delta' = 0$, contradicting (71).

Case 2. Thus, if $\eta_1(\rho)$ is nonconstant with ρ, then $\phi(\rho) = \theta_m \Delta(\rho)/\eta_1(\rho)$ must be constant, that is, we must have

$$\eta_1(\rho) = \beta_1 \Delta(\rho)$$

for some positive constant β_1. This too will lead to a contradiction. Substituting into (66) yields

$$\begin{aligned} x_n &= \xi((x_1, \mathbf{1}_{n-2}), (\mathbf{1}_{m-1}, y_m); \rho) \\ &= x_1^{-\beta_1\Delta(\rho)} y_m^{\theta_m\Delta(\rho)} \tilde{B}^{-\Delta(\rho)} \tilde{A}. \end{aligned} \tag{72}$$

Grouping factors, and solving for $G(\rho) = (1/\delta)\ln \Delta(\rho)$ (cf. Eq. (59)) gives, in view of [D],

$$G(\rho) = \frac{1}{\delta}\ln\frac{\ln\left(x_n/\tilde{A}\right)}{\ln\left(x_1^{-\beta_1}y_m^{\theta_m}\tilde{B}^{-1}\right)} = w^*(x_1, x_n) - k^*(y_m) \tag{73}$$

with

$$w^*(x_1, x_n) = u(x_1, \mathbf{1}_{n-2}, x_n), \quad \text{and} \quad k^*(y_1) = g(\mathbf{1}_{m-1}, y_m).$$

It is easy to verify that the functional equation in the last equality of (73) cannot be solved for the functions w^* and k^*. The argument is similar to that used in the case of Eq. (63). We have $k^*(1) = 0$ (by Lemma 2.7(iii) and Convention 4.2). With $y_m = 1$, (73) gives

$$w^*(x_1, x_n) = \frac{1}{\delta}\ln\frac{\ln\left(x_n/\tilde{A}\right)}{\ln\left(x_1^{-\beta_1}\tilde{B}^{-1}\right)}.$$

Substituting into (73), we get

$$\frac{1}{\delta}\ln\frac{\ln\left(x_n/\tilde{A}\right)}{\ln\left(x_1^{-\beta_1}y_1^{\theta_1}\tilde{B}^{-1}\right)} = \frac{1}{\delta}\ln\frac{\ln\left(x_n/\tilde{A}\right)}{\ln\left(x_1^{-\beta_1}\tilde{B}^{-1}\right)} - k^*(y_1),$$

leading, after simplification, to

$$k^*(y_1) = \frac{1}{\delta}\ln\left(\frac{\ln\left(x_1^{-\beta_1}\tilde{B}^{-1}\right)}{\ln\left(x_1^{-\beta_1}y_1^{\theta_1}\tilde{B}^{-1}\right)}\right),$$

with the left side varying only with y_1 while the right side varies with y_1 and x_1. Again, we obtain a contradiction.

Thus, both ϕ nonconstant and ϕ constant lead to contradiction. So, η_1 and ζ_m cannot be both nonconstant, and neither can η_i and ζ_m for any choice of the subscript i be simultaneously nonconstant.

Step 6. Thus, under the hypotheses of statement (i), all the η_i's are necessarily constant, $\eta_i = \alpha_i$, $(1 \le i \le n-1)$, all the ζ_j $(1 \le j \le m)$ take the form (64), and the function C takes the form (65). This means that [M] can be specified as (52):

$$\xi(\mathbf{x}, \mathbf{y}; \rho) = \tilde{A}\prod_{i=1}^{n-1} x_i^{-\alpha_i}\left(\frac{1}{\tilde{B}}\prod_{j=1}^{m} y_j^{\theta_j}\right)^{\exp[\delta G(\rho)]}. \tag{74}$$

Setting $x_n = \xi(\mathbf{x}, \mathbf{y}; \rho)$, we can rewrite (74) as

$$\frac{\ln \tilde{A} - \ln\left(x_n \prod_{i=1}^{n-1} x_i^{\alpha_i}\right)}{\ln \tilde{B} - \ln\left(\prod_{j=1}^{m} y_j^{\theta_j}\right)} = \exp[\delta G(\rho)]. \tag{75}$$

Since the right hand side is positive, the numerator and the denominator in the left hand side must have the same sign for all values of the variables.

Suppose that $\delta < 0$. We show that both the numerator and the denominator in (75) are then necessarily positive and we must have

$$\tilde{A} \geq a'_n \prod_{i=1}^{n-1} a_i'^{\alpha_i} \quad \text{and} \quad \tilde{B} \geq \prod_{j=1}^{m} b_j'^{\theta_j}, \tag{76}$$

that is, all the conditions of Case [a] in statement (iii) of the Theorem must hold. (The constant γ is irrelevant in statement (iii).) Indeed, if $\delta < 0$, then by (61) $\tilde{A} \geq a'_n$ and $\tilde{B} \geq b_j'^{\theta_j}$ for $1 \leq j \leq m$. Fix $x_i = 1$ for $1 \leq i \leq n-1$. The numerator in (75) is then positive for all values of $x_n \in]a_n, a'_n[$. This implies that the denominator of (75) is also positive for all values of y_j, $1 \leq j \leq m$, establishing the second inequality in (76). The first inequality in (76) follows from the fact that the numerator and the denominator in (75) must have the same sign.

A similar argument is used to show that if $\delta > 0$, then both the numerator and the denominator in (75) must be negative and we must have

$$0 < \tilde{A} \leq a'_n \prod_{i=1}^{n-1} a_i'^{\alpha_1} \quad \text{and} \quad 0 < \tilde{B} \leq \prod_{j=1}^{m} b_j'^{\theta_j}, \tag{77}$$

that is, Case [b] in statement (iii) of the Theorem hold.

Thus, (i) $\Rightarrow$ (iii) is proved.

(iii) $\Leftrightarrow$ (ii).

Solving (49) with respect to $x = \xi(\mathbf{x}, \mathbf{y}; \rho)$ yields (52), which is of the form [M] with constant $\eta_i = \alpha_i > 0$ $(1 \leq i \leq n-1)$ and C, ζ_j $(1 \leq j \leq m)$ given by (51), (50). This proves (ii) $\Rightarrow$ (iii).

We have seen in Step 6 of (i) $\Rightarrow$ (iii) that Eq. (52) readily leads to (75). If Case [a] of (iii) is satisfied ($\delta < 0$), we take logarithms on both sides of (74) and solve for $\rho = P(\mathbf{x}, x_n, \mathbf{y})$. This yields

$$P(\mathbf{x}, x, \mathbf{y}) = F\left[\frac{1}{\delta} \ln \ln\left(\frac{\tilde{A}}{x \prod_{i=1}^{n-1} x_i^{\alpha_i}}\right) - \frac{1}{\delta} \ln \ln\left(\frac{\tilde{B}}{\prod_{j=1}^{m} y_j^{\theta_j}}\right)\right], \tag{78}$$

which is (49a), and we have u and g given by (47) and (48) but with $\gamma = 1$.

At this point, we recall that by Convention 4.2, the functions u, g, and F appearing in our proof had an implicit subscript 0. Reestablishing the subscript, we see that, instead of (78), we actually got

$$P(\mathbf{x}, x, \mathbf{y}) = F_0\left[\frac{1}{\delta}\ln\ln\left(\frac{\tilde{A}}{x\prod_{i=1}^{n-1} x_i^{\alpha_i}}\right) - \frac{1}{\delta}\ln\ln\left(\frac{\tilde{B}}{\prod_{j=1}^{m} y_j^{\theta_j}}\right)\right],$$

with (see Lemma 2.7) $F_0(t) = F(t + t_0)$. We can write

$$t_0 = -\frac{1}{\delta}\ln\theta \tag{79}$$

for some positive θ, and with the notation

$$\theta_j^* = \theta\theta_j > 0\ (1 \le j \le m), \qquad \tilde{B}^* = \theta\tilde{B}, \tag{80}$$

we obtain

$$P(\mathbf{x}, x, \mathbf{y}) = F\left[\frac{1}{\delta}\ln\ln\left(\frac{\tilde{A}}{x\prod_{i=1}^{n-1} x_i^{\alpha_i}}\right) - \frac{1}{\delta}\ln\ln\left(\frac{\tilde{B}^*}{\prod_{j=1}^{m} y_j^{\theta_j^*}}\right)\right]. \tag{81}$$

This gives a pair u, g as in (47) and (48) but again with $\gamma = 1$. The general forms of u and g in [D] follow by noting that, given F, the functions u and g are clearly determined up to a common additive constant. We can write this constant as $(1/\delta)\ln\gamma$ $(\gamma > 0)$. So we obtain (47) and (48) as asserted. Thus $\delta < 0 < \gamma$ and Case [a] of the statement (ii) holds. Case [b] of (iii) is dealt with similarly, and we get statement (ii) with $\gamma < 0 < \delta$. We conclude that (iii) $\Rightarrow$ (ii).

We have to reinstall the implicit subscript 0 also in (74) and get, in view of (10) and (79), $F_0^{-1}(\rho) = F^{-1}(\rho) - t_0 = G(\rho) + (1/\delta)\ln\theta$ and, using also (80),

$$\begin{aligned}\xi(\mathbf{x}, \mathbf{y}; \rho) &= \tilde{A}\prod_{i=1}^{n-1} x_i^{-\alpha_i}\left(\frac{1}{\tilde{B}}\prod_{j=1}^{m} y_j^{\theta_j}\right)^{\exp[\delta G(\rho) + \ln\theta]} \\ &= \tilde{A}\prod_{i=1}^{n-1} x_i^{-\alpha_i}\left(\frac{1}{\tilde{B}^*}\prod_{j=1}^{m} y_j^{\theta_j^*}\right)^{\exp[\delta G(\rho)]},\end{aligned}$$

that is, removing the stars from $\tilde{B}^*$ and θ_j^*, we obtain again (52). It is of the form [M] and determines $\eta_i(\rho) = \alpha_i$ $(1 \le i \le n-1)$ and $C(\rho)$, $\zeta_j(\rho)$ $(1 \le j \le m)$ uniquely as (51) and (50) (with G, not G_0 in the exponent).

(ii) and (iii) ⇒ (i). Examining (49) and (52), we see that [D] and [M] jointly hold for the functions P and ξ, with u and g defined by (47) and (48), respectively, F arbitrarily continuous and strictly increasing, $\eta_i = \alpha_i$ constant $(1 \le i \le n-1)$, ζ_j $(1 \le j \le m)$, and C defined by (50) and (51) respectively, establishing (i). This concludes the proof of Theorem 4.1. ∎

We turn to the case of constant functions ζ_j.

4.3. THEOREM. *Suppose that* [M] *holds with* ξ *strictly increasing in* ρ *and* y_j $(1 \le j \le m)$, *strictly decreasing in* x_i $(1 \le i \le n-1)$, *and continuous in all variables. Then the following two conditions are equivalent.*

(i) *At least one of the* ζ_j *is constant. Moreover, the function P linked to* ξ *by Eq.* (4) *satisfies representation* [D] *for some functions u, g, and F strictly increasing and continuous in all variables.*

(ii) *All* η_i *and* ζ_j *are constant*: $\eta_i(\rho) = \alpha_i$ $(1 \le i \le n-1)$, $\zeta_j(\rho) = \beta_j$ $(1 \le j \le m)$.

If either of these conditions holds, the function C in [M] *is continuous and strictly increasing, thus has a continuous and strictly increasing inverse* $H = C^{-1}$, *and we have*

$$P(\mathbf{x}, x_n, \mathbf{y}) = H\left(\frac{x_n \prod_{i=1}^{n-1} x_i^{\alpha_i}}{\prod_{j=1}^{m} y_j^{\beta_j}}\right). \tag{82}$$

In particular, if in [M1], $\zeta(\rho) = \beta$, a constant, then $P(x, y) = G(x/y^{\beta})$ (cf. Falmagne, 1985, p. 203).

Proof. (i) ⇒ (ii). Suppose, for example, that in [M] $\zeta_m = \theta_m$ constant. Thus, all the other functions ζ_j $(1 \le j < m)$ must also be constant (Theorem 4.1, (i) ⇒ (iii)). For contradiction, suppose also that η_1 is nonconstant. Setting $x_2 = \cdots = x_{n-1} = y_1 = \cdots = y_{m-1} = 1$ simplifies [M] to $x_n = \xi((x_1, \mathbf{1}_{n-2}), (\mathbf{1}_{m-1}, y_m); \rho) = y_m^{\theta_m} x_1^{-\eta_1(\rho)} C(\rho)$. Solving for x_1 yields

$$\begin{aligned} x_1 &= x_n^{-1/\eta_1(\rho)} y_m^{\theta_m/\eta_1(\rho)} C(\rho)^{1/\eta_1(\rho)} \\ &= x_n^{-\eta_1'(\rho)} y_m^{\zeta_m'(\rho)} C'(\rho) = \hat{\xi}(x_n, y_m; \rho) \end{aligned}$$

with obvious definitions of η_1', ζ_m', C', and $\hat{\xi}$. Similarly, [D] simplifies to

$$\begin{aligned} \hat{P}(x_1, x_n, y_m) &= P[(x_1, \mathbf{1}_{n-2}, x_n), (\mathbf{1}_{m-1}, y_m)] \\ &= F[u(x_1, \mathbf{1}_{n-2}, x_n) - g(\mathbf{1}_{m-1}, y_m)] \\ &= F[\hat{u}(x_n, y_m) - \hat{g}(y_m)], \end{aligned}$$

with obvious definitions of $\hat{u}$, $\hat{g}$, and $\hat{P}$. Note that the pair of functions $(\hat{P}, \hat{\xi})$ satisfies [D] and [M], with nonconstant ζ_m'. Applying the implication (i) ⇒ (iii) of Theorem 4.1, we obtain that the exponent function η_1' should be constant. From this contradiction, we conclude that (i) ⇒ (ii).

(ii) ⇒ (i). Since the functions η_i and ζ_j in [M] are constant (equal to α_i and β_j, respectively), and ξ is strictly increasing and continuous in ρ, the function C must be strictly increasing and continuous. Thus, C has a continuous and strictly increasing inverse $H = C^{-1}$. Solving [M] for $C(\rho)$ with $\xi(\mathbf{x}, \mathbf{y}; \rho) = x_n$ and applying H to both sides yields Eq. (82). Representation [D] follows by rewriting (82) in the form

$$P(\mathbf{x}, x_n, \mathbf{y}) = (H \circ \exp)\left[\ln\left(x_n \prod_{i=1}^{n-1} x_i^{\alpha_i}\right) - \ln\left(\prod_{j=1}^{m} y_j^{\beta_j}\right)\right],$$

with $u(\mathbf{x}, x_n) = \ln(x_n \prod_{i=1}^{n-1} x_i^{\alpha_i})$, $g(\mathbf{y}) = \ln(\prod_{j=1}^{m} y_j^{\beta_j})$ and $F = H \circ \exp$. ∎

4.4. COROLLARY. *Suppose that* (P, ξ) *is a pair of functions linked by the equivalence* (4) *with all the side conditions holding. If* [D] *and* [M] *jointly hold for P and* ξ *respectively, then all the functions* η_i *in* [M] *are necessarily constant.*

This follows immediately from Theorems 4.1 and 4.3.

5. EXAMPLES

5.1. *A model satisfying the conditions of Theorem* 4.1

Take

$$x_3 = \xi((x_1, x_2), (y_1, y_2); \rho) = \tilde{A} x_1^{-\alpha_1} x_2^{-\alpha_2} \left(\frac{1}{\tilde{B}} y_1^{\theta_1} y_2^{\theta_2}\right)^{(\rho/(1-\rho))^{\delta}},$$

with α_1, α_2, θ_1, θ_2 positive and δ negative. Solving for ρ yields

$$\begin{aligned}
\rho &= P(x_1, x_2, x_3, y_1, y_2) \\
&= \left[1 + \exp\left(-\frac{1}{\delta}\ln\frac{\ln \tilde{A} - \ln(x_1^{\alpha_1} x_2^{\alpha_2} x_3)}{\ln \tilde{B} - \ln(y_1^{\theta_1} y_2^{\theta_2})}\right)\right]^{-1} \\
&= F\left[\frac{1}{\delta}\ln\frac{\ln \tilde{A} - \ln(x_1^{\alpha_1} x_2^{\alpha_2} x_3)}{\ln \tilde{B} - \ln(y_1^{\theta_1} y_2^{\theta_2})}\right] \\
&= F\left[\frac{1}{\delta}\ln\ln\left(\frac{\tilde{A}}{x_1^{\alpha_1} x_2^{\alpha_2} x_3}\right) - \frac{1}{\delta}\ln\ln\left(\frac{\tilde{B}}{y_q^{\theta_1} y_2^{\theta_2}}\right)\right],
\end{aligned}$$

of the form [D] with $F(s) = (1 + e^{-s})^{-1}$, the logistic function.

5.2. *A model failing* [D] *but satisfying* [M]

Take

$$y_1 = \mu((x_1, x_2), (y_2, y_3); \rho) = x_1^{\theta_1(\rho/(1-\rho))^\delta} x_2^{\theta_2} y_2^{-\alpha_2(\rho/(1-\rho))^\delta} B^{-1} A^{(\rho/(1-\rho))^\delta}.$$

As we see, μ is of the form [M]. Solving for ρ yields

$$\rho = P_{x_1, x_2; y_1, y_2, y_3} = H\left(\frac{1}{\delta} \ln \frac{\ln\left(B y_1 x_2^{-\theta_2}\right)}{\ln\left(y_2^{-\alpha_2} A x_1^{\theta_1}\right)} \right),$$

where H is the logistic function $H(s) = (1 + e^{-s})^{-1}$.

The difference representation [D] cannot hold for this model; that is, we cannot have F, v, and h continuous and strictly increasing in all arguments such that

$$H\left(\frac{1}{\delta} \ln \frac{\ln\left(B y_1 x_2^{-\theta_2}\right)}{\ln\left(y_2^{-\alpha_2} A x_2^{\theta_1}\right)} \right) = F[v(y_1, y_2) - h(x_1, x_2)]. \qquad (83)$$

Indeed, since F is strictly increasing, we have

$$F[v(y_1, y_2) - h(x_1, x_2)] \geq F[v(y_1', y_2') - h(x_1, x_2)]$$
$$\Leftrightarrow v(y_1, y_2) \geq v(y_1', y_2').$$

Assuming (83) would lead to

$$\frac{\ln\left(B y_1 x_2^{-\theta_2}\right)}{\ln\left(y_2^{-\alpha_2} A x_1^{\theta_1}\right)} \geq \frac{\ln\left(B y_1' x_2^{-\theta_2}\right)}{\ln\left(y_2'^{-\alpha_2} A x_1^{\theta_1}\right)} \Leftrightarrow \frac{\ln\left(B y_1 x'^{-\theta_2}\right)}{\ln\left(y_2^{-\alpha_2} A x_1'^{\theta_1}\right)} \geq \frac{\ln\left(B y_1' x_2'^{-\theta_2}\right)}{\ln\left(y_2'^{-\alpha_2} A x_1'^{\theta_1}\right)}.$$

The above equivalence is of the form

$$\frac{s + t}{w + m} \geq \frac{s' + t}{w' + m} \Leftrightarrow \frac{s + t'}{w + m'} \geq \frac{s' + t'}{w' + m'},$$

which is an absurdity. (Take for instance $s = 1$, $w = 2$, $s' = 30$, $w' = 40$, $t = m = 10$, $t' = m' = 1$. The left side of the above "equivalence" gives $\frac{11}{12} > \frac{4}{5}$, and the right side $\frac{2}{3} < \frac{31}{41}$.

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