

Partitions of the Set of Finite Sequences

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We consider partitions of the set of finite sequences of a cardinal and investigate the existence of different kinds of homogeneous objects for them. We prove some properties of the first cardinal satisfying a natural partition relation and study how different types of homogeneities are interrelated. © 1995 Academic Press, Inc.

INTRODUCTION

Our starting point is the following question. Given $F : \omega^{<\omega} \rightarrow 2$, a partition of the finite sequences of natural numbers into two pieces, is there a sequence of pairs of numbers, H_0, H_1, \dots , such that for every $n \in \omega$ the product $\prod_{i=0}^n H_i$ is contained in one of the pieces of the partition? It is easy to find a partition for which no such homogenous sequence of pairs exist (see [CDP]), but it can be shown that if the homogeneity is only required for products of length n for infinitely many $n \in \omega$, then the property holds for every partition [H]. We will consider several versions of this partition property and study the way they are related to each other. In particular we will consider partitions into more than two pieces and also other ways of weakening the homogeneity condition.

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The partition relations considered in this paper are of the type usually called polarized, since the partitions are defined on sets of ordered sequences. For a survey of results on polarized partition relations, the reader can consult [W] or the papers [ER, EHR]. We will adopt the usual set theoretical notation, if κ is a cardinal, $\kappa^{<\omega}$ denotes the set of finite sequences of elements of κ . We will use extensively the “arrow” notation which is so convenient to express properties of partitions of sets.

DEFINITION. The partition symbol

$$(*) \begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

means that $\forall F : \kappa^{<\omega} \rightarrow 2$, there is a sequence H_0, H_1, \dots such that $\forall i |H_i| = 2$ and $\forall n F$ is constant on $\prod_{i=0}^n H_i$.

Note that increasing the numbers in the right hand column of the partition symbol gives a (possibly) stronger partition relation.

If we do not require all the set of the sequence H_0, H_1, \dots to have at least two elements, we obtain another type of property.

DEFINITION.

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ \vdots \\ \overbrace{\quad}^n \\ 2 \\ \underbrace{\quad\quad\quad} \\ \vdots \end{pmatrix}$$

means that $\forall F : \kappa^{<\omega} \rightarrow 2$, there is a sequence H_0, H_1, \dots such that $\exists i_0, \dots, i_{n-1} \forall j < n |H_{i_j}| = 2$ and $\forall n F$ is constant on $\prod_{i=0}^n H_i$. In other words, at least n of the sets in the homogeneous sequence have two elements. As in [DPH], we say in this case that there are n floating 2's.

We can also weaken the partition relation by requiring the function F to be constant not on all products, but only on infinitely many of them.

DEFINITION.

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

means that for every $F : \kappa^{<\omega} \rightarrow 2$, there is a sequence H_0, H_1, \dots such that for each i $|H_i| = 2$ and the set $\{k : F \text{ is constant on } \prod_{i=0}^k H_i\}$ is infinite. This type of homogeneous sequences will be called ∞ -homogeneous.

We will also consider the case of ∞ -homogeneous sequences with floating 2's.

There is another type of homogeneity which could be considered. Say that the sequence H_0, H_1, \dots is eventually homogeneous for F if there is a $k \in \omega$ such that F is constant on all products $\prod_{i=0}^n H_i$ for $n \geq k$.

The fact that every partition of $\kappa^{<\omega}$ admits an eventually homogeneous sequence of pairs of elements of κ will be expressed by

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[E]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}.$$

Facts. (a) Eventual homogeneity is equivalent to full homogeneity. To show, for example, that

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[E]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

implies

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix},$$

one can proceed as follows. Given $F : \kappa^{<\omega} \rightarrow 2$, define another function $G : \kappa^{<\omega} \rightarrow 2$ by

$$G(\alpha_0, \alpha_1, \dots, \alpha_{2^k 3^n}) = F(\alpha_0, \alpha_1, \dots, \alpha_k) \quad \text{for all } n, k \in \omega,$$

and, say, $G(\alpha_0, \alpha_1, \dots, \alpha_k) = 0$ if k is not of the form $2^i 3^j$. If H_0, H_1, \dots is an eventually homogeneous sequence for G , then G is constant on products of initial segments of this sequence of length at least a certain k . To see that F is constant on any product $\prod_{i=0}^n H_i$, let (ξ_0, \dots, ξ_m) and (η_0, \dots, η_m) be two elements of this product. Let $n \geq 1$ be such that $2^m 3^n > k$, and for all $i, m < i < 2^m 3^n$, let α_i be the first element of H_i . Then we have, $F(\xi_0, \dots, \xi_m) = G(\xi_0, \dots, \xi_m, \alpha_{m+1}, \dots, \alpha_{2^m 3^n}) = G(\eta_0, \dots, \eta_m, \alpha_{m+1}, \dots, \alpha_{2^m 3^n}) = F(\eta_0, \dots, \eta_m)$.

(b) None of the properties with full homogeneity hold for $\kappa = \omega$. For example, to show that

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \not\rightarrow^{<\omega} \begin{pmatrix} 2 \\ \text{---} \\ \vdots \end{pmatrix},$$

let $F : \omega^{<\omega} \rightarrow 2$ be defined as

$$F(a_0, a_1, \dots, a_{n-1}) = 0 \text{ if and only if } |\{i : a_i = n - i\}| \text{ is even.}$$

Suppose that H_0, H_1, \dots is homogeneous for F and, say, $|H_i| = 2$. Put $H_i = \{a_i, b_i\}$. Then, if $(a_0, a_1, \dots, a_i, \dots, a_{i+a_i}) \in \prod_{j=0}^{i+a_i+1} H_j$, we have that

$$F(a_0, \dots, a_i, \dots, a_{i+a_i}) \neq F(a_0, \dots, b_i, \dots, a_{i+a_i}),$$

since $a_i \neq b_i$.

Given fact (b), we can ask if there are cardinals satisfying the full homogeneity properties. Assuming large cardinals, the answer is positive. For example, if $\kappa \rightarrow (\omega)^{<\omega}$ then κ satisfies (*). So, assuming that there are cardinals satisfying these properties, we might ask which is the first one to satisfy them. We devote the next section to this question.

1. A CARDINAL DEFINED BY A PARTITION PROPERTY

PROPOSITION 1.1. *The first cardinal satisfying property (*) is \prod_1^1 -describable (and therefore it is not weakly compact).*

Proof. Let κ be the first cardinal satisfying property (*). If κ has cofinality ω , then it is describable by a first order formula. If $\text{cof } \kappa > \omega$, then κ can be described by the following \prod_1^1 formula, $\forall F : \kappa^{<\omega} \rightarrow 2 \exists H$, where H is an ω -sequence of pairs of ordinals smaller than κ which is homogeneous for F .

Note that the first quantifier is the only second-order quantifier. The property of being a homogeneous sequence is first order. ■

PROPOSITION 1.2. *The first cardinal satisfying property (*) cannot be a successor cardinal.*

Proof. Suppose that κ is an infinite cardinal and $F : \kappa^{<\omega} \rightarrow 2$ does not have a homogeneous sequence of pairs. We will show that there is a partition $G : (\kappa^+)^{<\omega} \rightarrow 2$ without a homogeneous sequence of pairs. To define G , fix for each $\alpha < \kappa^+$ a one-one map $i_\alpha : \alpha + 1 \rightarrow \kappa$, and define $G(\alpha_0, \alpha_1, \dots, \alpha_{2n-1}) = F(\xi_0, \dots, \xi_n)$, where $\xi_i = i_{\max\{\alpha_{2i}, \alpha_{2i+1}\}}(\min\{\alpha_{2i}, \alpha_{2i+1}\})$

(and define $G(s)=0$ if s is a sequence of odd length). If H_0, H_1, \dots is a homogeneous sequence of pairs for G , then we obtain a sequence of pairs I_0, I_1, \dots homogeneous for F as follows. For every $i \in \omega$, let $\eta_i = \max(H_{2i} \cup H_{2i+1})$, and $I_i = i''_{\eta_i} H$, where $H = H_{2i+1}$ if $\eta_i \in H_{2i}$, and $H = H_{2i}$ otherwise. ■

LEMMA 1.3.

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ - & - & - \\ \vdots \end{pmatrix}$$

implies that

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ - & - & - \\ \vdots \end{pmatrix}_{2^\omega}.$$

Proof. We use a standard trick. Given $F : \kappa^{<\omega} \rightarrow 2^\omega$, define a partition $G : \kappa^{<\omega} \rightarrow 2$ by $G(a_0, a_1, \dots, a_{2^i/3}) = F(a_0, \dots, a_i)(j)$ and put $G(a_0, \dots, a_n) = 0$ for all other sequences. Suppose that H_0, H_1, \dots is homogeneous for F , and, say, H_i has two elements. We show that the same sequence is homogeneous for G . Given $(a_0, a_1, \dots, a_i) \in \prod_{k=0}^i H_k$, for every $j \in \omega$ let $a_{i+1} \in H_{i+1}, \dots, a_{2^i/3} \in H_{2^i/3}$, then

$$F(a_0, \dots, a_i)(j) = G(a_0, \dots, a_i, \dots, a_{2^i/3}). \quad \blacksquare$$

Note that using the same idea, analogous implications can be proved for the case of partition relations defined with any number of 2's (fixed or floating in the right-hand side column. In particular, (*) implies that

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}_{2^\omega}.$$

The next result shows that once we have subindex ω , we obtain the partition relation with arbitrarily large subindices below κ .

THEOREM 1.4. *Let κ be the first cardinal such that*

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

and let $\gamma < \kappa$, then

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}.$$

Proof. We argue by contradiction. Suppose that the theorem does not hold, and let $f : \kappa^{<\omega} \rightarrow \gamma$ be a partition without homogeneous sequences of pairs. Since $\gamma < \kappa$,

$$\begin{pmatrix} \gamma \\ \gamma \\ \vdots \end{pmatrix} \not\xrightarrow{<\omega} \begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix}.$$

Let $G : \gamma^{<\omega} \rightarrow 2$ be a partition witnessing this fact. We will now build a partition $F : \kappa^{<\omega} \rightarrow \omega$ with no homogeneous sequence of pairs, contradicting that

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}.$$

The idea is to define the partition F by combining f and G . First, we need a uniform way to code subsequences of (finite) sequences of ordinals. For this we will use the natural enumeration of the prime numbers $\{p_0, p_1, \dots\}$.

For any sequence $\alpha = (\alpha_0, \dots, \alpha_n) \in \kappa^{<\omega}$ with $n \geq 2$, consider the subsequences $\beta_i = (\alpha_{p_i}, \alpha_{p_i^2}, \dots, \alpha_{p_i^{k_i}})$, for every $i = 0, 1, \dots, u$, where u is the largest natural number such that $p_u \leq n$ and, for each $i \leq u$, k_i is the largest natural number such that $p_i^{k_i} \leq n$.

Now, for each $i \leq u$, let $\beta_{i,r} = (\alpha_{p_i}, \alpha_{p_i^2}, \dots, \alpha_{p_i^r})$, for every $r = 1, \dots, k_i$, be the initial segments of β_i determined by r . Each (a_0, \dots, a_v) with $v \leq u$, $a_i \leq k_i$ determines a sequence of ordinals below γ in the following way, $f(\beta_{0,a_0}), f(\beta_{1,a_1}), \dots, f(\beta_{v,a_v})$. We list all such sequences in the order induced by the lexicographical order of the sequences (a_0, \dots, a_v) with $v \leq u$, $a_i \leq k_i$. Let $\bar{\alpha}_0, \dots, \bar{\alpha}_{m_n}$ be this enumeration. Note that the order here only depends on the lexicographical ordering of finite sequences of natural numbers and not on the actual values $\alpha_0, \dots, \alpha_n$.

Fix an enumeration of the finite sequences of zeroes and ones $\langle \rangle : 2^{<\omega} \rightarrow \omega$.

We are now ready to define $F : \kappa^{<\omega} \rightarrow \omega$ by

$$F(\alpha_0, \dots, \alpha_n) = \langle G(\bar{\alpha}_0), \dots, G(\bar{\alpha}_{m_n}) \rangle \quad (n \geq 2),$$

$$F(\alpha_0) = F(\alpha_0, \alpha_1) = 0.$$

It is routine now to check that F does not have homogeneous sequences of pairs. Let $\{H_i : i \in \omega\}$ be a sequence with H_i a pair of ordinals below κ for every $i \in \omega$. By hypothesis, none of the sequences $\{H_{p_v^k} : 0 < v < \omega\}$ is homogeneous for f , so for each $k \in \omega$ there is $n_k \in \omega$ such that f is not constant on $H_{p_k} \times H_{p_k^2} \times \dots \times H_{p_k^{n_k}}$. Let $\xi^k, \eta^k \in H_{p_k} \times H_{p_k^2} \times \dots \times H_{p_k^{n_k}}$ be such that $f(\xi^k) \neq f(\eta^k)$, and put $\bar{H}_k = \{f(\xi^k), f(\eta^k)\}$.

Since, by hypothesis, the sequence $\{\bar{H}_k : k \in \omega\}$ is not homogeneous for G , there is $l \in \omega$ and $s, t \in \bar{H}_0 \times \dots \times \bar{H}_l$ such that $G(s) \neq G(t)$.

Let $s = (s_0, \dots, s_l)$ and $t = (t_0, \dots, t_l)$. For each $k \leq l$, $s_k, t_k \in f^n H_{p_k} \times \dots \times H_{p_k^{n_k}}$. Let, for each $k \leq l$, $(e_{p_k}^k, \dots, e_{p_k^{n_k}}^k)$ and $(\sigma_{p_k}^k, \dots, \sigma_{p_k^{n_k}}^k)$ be such that $s_k = f(e_{p_k}^k, \dots, e_{p_k^{n_k}}^k)$ and $t_k = f(\sigma_{p_k}^k, \dots, \sigma_{p_k^{n_k}}^k)$.

Let $n \geq \max\{p_0^{n_0}, \dots, p_l^{n_l}\}$, we will see that F is not constant on $\prod_{i=0}^n H_i$. Let $(\alpha_0, \dots, \alpha_n), (\beta_0, \dots, \beta_n) \in \prod_{i=0}^n H_i$ be defined as follows.

For $k \neq p_0, \dots, p_0^{n_0}, p_1, \dots, p_1^{n_1}, \dots, p_l, \dots, p_l^{n_l}$, then $\alpha_k = \beta_k = \min H_k$. And for each $k \leq l$, $0 < j \leq n_k$, $\alpha_{p_k^j} = e_{p_k^j}^k$ and $\beta_{p_k^j} = \sigma_{p_k^j}^k$. The way F was defined ensures that there is an $m \leq m_n$ such that $s = \bar{\alpha}_m$ and $t = \bar{\beta}_m$; since $G(s) \neq G(t)$, we get that $F(\alpha) \neq F(\beta)$. ■

Of course, the partition property is false with subindex κ and it is interesting to note that the same is the case for ∞ -homogeneity.

PROPOSITION 1.5. For any infinite κ ,

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\infty]{< \omega} \begin{pmatrix} 2 \\ \dots \\ \vdots \end{pmatrix}_\kappa$$

Proof. Fix a bijection from the finite sequences of ordinals in κ onto κ , and let $\langle \alpha_0, \dots, \alpha_{k-1} \rangle$ be the ordinal assigned to the sequence $(\alpha_0, \dots, \alpha_{k-1})$. Define $F : \kappa^{< \omega} \rightarrow \kappa$ by $F(\alpha_0, \dots, \alpha_{n-1}) = \langle \alpha_0, \dots, \alpha_{n-1} \rangle$. Now, given any sequence H_0, H_1, \dots , of subsets of κ , with, say, $H_i = \alpha_i, \beta_i$ ($\alpha_i \neq \beta_i$), we have $\langle \alpha_0, \dots, \alpha_{i-1}, \alpha_i \rangle \neq \langle \alpha_0, \dots, \alpha_{i-1}, \beta_i \rangle$, and so, the sequence H_0, H_1, \dots is not ∞ -homogeneous for F . ■

LEMMA 1.6. Let κ be the first cardinal satisfying (*) and let $\gamma \leq \kappa$ be the first cardinal such that $2^\gamma \geq \kappa$. If $\text{cof}(\kappa) > \omega$, $\text{cof}(\kappa) \geq \gamma$.

Proof. We will show that if $\omega < \text{cof}(\kappa) < 2^{\text{cof}(\kappa)} < \kappa$, then κ is not the first cardinal satisfying (*). Assume that the inequalities mentioned hold and that κ is the first such that (*). Define $F : \kappa^{< \omega} \rightarrow 2^{\text{cof}(\kappa)}$ as follows.

Let $\eta_0, \eta_1, \dots, \eta_\xi, \dots$ ($\xi < \text{cof}(\kappa)$) be a cofinal sequence in κ . For each $\xi < \text{cof}(\kappa)$, let $F_\xi : (\eta_\xi)^{< \omega} \rightarrow 2$ be a partition with no homogeneous sequence of pairs.

Define $F : \kappa^{<\omega} \rightarrow 2^{\text{cof}(\kappa)}$ as follows $F(\alpha_0, \alpha_1, \dots, \alpha_n) = \langle \varepsilon_\xi : \xi < \text{cof}(\kappa) \rangle$, where, for each $\xi < \text{cof}(\kappa)$, $\varepsilon_\xi = F_\xi(\alpha_0, \alpha_1, \dots, \alpha_n)$ if $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \text{dom}(F_\xi)$ and, say, $\varepsilon_\xi = 0$ otherwise.

If $\text{cof}(\kappa) > \omega$, this partition does not have a homogeneous sequence of pairs, since in this case, for every sequence of pairs of elements of κ , $H_0, H_1, \dots, H_n, \dots$, there is a $\xi < \text{cof}(\kappa)$ such that each $H_n \subseteq \eta_\xi$, and therefore this would be a homogeneous sequence for F_ξ , a contradiction.

This lemma indicates that if the first κ with (*) is a strong limit and $\text{cof}(\kappa) > \omega$ then κ is strongly inaccessible. Therefore, with the GCH we obtain the following result.

THEOREM 1.7 (GCH). *The first κ such that (*) holds has cofinality ω or it is inaccessible.*

Proof. The result follows from the previous lemma and the fact that κ is a limit cardinal. ■

2. VARIATIONS ON HOMOGENEITY

In this section we will present several results interrelating different types of homogeneity.

PROPOSITION 2.1. *The following are equivalent:*

$$\begin{aligned}
 \text{(a)} \quad & \binom{\kappa}{\kappa} \xrightarrow{<\omega} \binom{2}{2} \binom{\vdots}{\dots} \\
 \text{(b)} \quad & \binom{\kappa}{\kappa} \xrightarrow{<\omega} \binom{2}{2}.
 \end{aligned}$$

Proof. Clearly, (b) implies (a), so we prove that (a) implies (b). Given a sequence $s = (\alpha_0, \dots, \alpha_n) \in \kappa^n$, let $s^* = (s_0^*, s_1^*, \dots)$ be the set of its subsequences ordered as follows: first, the one-element subsequences, then the two-element subsequences, etc. And for each n , the n -element subsequences are ordered lexicographically (where the α 's are taken in the order in which they appear in the original sequence, α_0 precedes α_1 , α_1 precedes α_2 , etc.). Note that since there might be repetitions, we could have, say, $\alpha_1 = \alpha_3 = \alpha$,

in which case the subsequence (α) appears in the second place and in the fourth place of s^* .

Fix now an enumeration of all the finite sequences of 0's and 1's; as usual, denote by $\langle \varepsilon_0, \dots, \varepsilon_n \rangle$ the number assigned to the sequence $(\varepsilon_0, \dots, \varepsilon_n)$.

Given that $F : \kappa^{<\omega} \rightarrow 2$, define $G : \kappa^{<\omega} \rightarrow \omega$ as follows, for $s = (\alpha_0, \dots, \alpha_n) \in \kappa^n$. $G(s) = \langle F(s_0^*), F(s_1^*), \dots \rangle$. By hypothesis, and the lemma, there is a sequence H_0, H_1, \dots that is homogeneous for G with infinitely many two-element sets. Let H_{i_0}, H_{i_1}, \dots be the subsequence formed by the two-element sets. We claim that this subsequence is homogeneous for F . To see this, let $(\alpha_0, \dots, \alpha_n)$ and $(\beta_0, \dots, \beta_n)$ be two sequences in $\prod_{j=0}^n H_{i_j}$. Let $(\xi_0, \dots, \xi_{j_n}), (\eta_0, \dots, \eta_{j_n})$ be sequences in $\prod_{i=0}^{j_n} H_{i_j}$ such that for every $j \leq n$, $\xi_{i_j} = \alpha_j$ and $\eta_{i_j} = \beta_j$ ($(\xi_0, \dots, \xi_{j_n})$ and $(\eta_0, \dots, \eta_{j_n})$ coincide in all other coordinates). By homogeneity of H_0, H_1, \dots for G , $G((\xi_0, \dots, \xi_{j_n})) = G((\eta_0, \dots, \eta_{j_n}))$, and by definition of G , $F((\alpha_0, \dots, \alpha_n))$ occupies the same place in the sequence coded by $G((\xi_0, \dots, \xi_{j_n}))$ as $F((\beta_0, \dots, \beta_n))$ in the sequence coded by $G((\eta_0, \dots, \eta_{j_n}))$ and, therefore, they are equal. ■

The analogous equivalence for the case of n floating 2's is obtained by a slight variation of this proof. The same technique also gives the following proposition.

PROPOSITION 2.2.

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}_\omega \quad \text{implies } (*).$$

Proof. Given a function $F : \kappa^{<\omega} \rightarrow 2$, we define a function $G : \kappa^{<\omega} \rightarrow \omega$ by $G(s) = \langle F(s_0^*), F(s_1^*), \dots \rangle$ (here we are using the same notation as in the previous proof). A homogeneous sequence for G is also homogeneous for F . ■

Coding finite sequences of natural numbers by natural numbers we obtain that

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}_\omega$$

is equivalent to

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}_\omega.$$

Nevertheless, the equivalence does not hold for finite subindices, since (see fact (b))

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \not\stackrel{<\omega}{\longrightarrow} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix},$$

but see below.

PROPOSITION 2.3 (Henle, [H]).

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \stackrel{<\omega}{\underset{\infty}{\longrightarrow}} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}.$$

Henle used this proposition to show that $ZF + DC + \omega \rightarrow (\omega)^\omega$ proves that

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \omega \\ 2 \\ 2 \\ \vdots \end{pmatrix};$$

in other words, for every $F : \omega^\omega \rightarrow 2$ (we are dealing with infinite sequences here), there is a sequence H_0, H_1, \dots of subsets of ω such that H_0 is infinite, $|H_i| = 2$ for every $i > 0$, and F is constant on the product $\prod_{i \geq 0} H_i$. This type of partition relation contradicts the Axiom of Choice (see [DPH] for related results).

PROPOSITION 2.4.

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \stackrel{<\omega}{\underset{\infty}{\longrightarrow}} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

implies that

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \stackrel{<\omega}{\underset{\infty}{\longrightarrow}} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}_n$$

for any $n \in \omega$.

Proof. The proof is similar to that of 1.4 of [DPH]. It is shown, by induction, that the partition relation with subindex k implies the

partition relation with subindex $k + 1$. Given a partition $F : \kappa^{<\omega} \rightarrow k + 1$, define an auxiliary partition $G : \kappa^{<\omega} \rightarrow k$ as follows. Given a sequence $s = \langle \alpha_0, \alpha_1, \dots, \alpha_{2i-1} \rangle$ of ordinals smaller than κ , call s_e and s_o the two sequences of length i obtained by splitting s in its even and odd parts, respectively. Now, put

$$\begin{aligned} G(s) = 0 & \quad \text{if } F(s_e) - F(s_o) \in \{-1, 1, 2\}, \\ G(s) = 1 & \quad \text{if } F(s_e) - F(s_o) \in \{0, -2\}, \\ G(s) = 2 & \quad \text{if } F(s_e) - F(s_o) \in \{-3, 3\}, \\ & \quad \vdots \\ G(s) = k - 1 & \quad \text{if } F(s_e) - F(s_o) \in \{-k, k\}. \end{aligned}$$

If H_0, H_1, \dots is ∞ -homogeneous for G , then either H_0, H_2, H_4, \dots or H_1, H_3, H_5, \dots is ∞ -homogeneous for F . ■

PROPOSITION 2.5. *For every k there is a natural number n such that*

$$\binom{n}{n} \xrightarrow[\infty]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \\ k \\ \vdots \\ 2 \\ 1 \\ 1 \\ \vdots \end{pmatrix}.$$

This proposition is an immediate consequence of the following implication. Note that the antecedent holds for some n is a consequence of finite Ramsey's theorem.

LEMMA 2.6.

$$\binom{n}{n} \longrightarrow \begin{pmatrix} 2 \\ 2 \\ \vdots \\ k \\ \vdots \\ 2 \end{pmatrix}$$

implies that

$$\begin{pmatrix} n \\ n \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \\ k \\ \vdots \\ 2 \\ 1 \\ 1 \\ \vdots \end{pmatrix}.$$

Proof. Suppose that $F : n^{<\omega} \rightarrow 2$ does not have an ∞ -homogeneous sequence $H_0, H_1, \dots, H_{k-1}, H_k, \dots$ with $|H_0| = \dots = |H_{k-1}| = 2$.

Fix a sequence $\alpha_0, \alpha_1, \dots$ in n^ω , and for each $j \in \omega$ define $G_j : n^k \rightarrow 2$ by $G_j(a_0, \dots, a_{k-1}) = F(a_0, \dots, a_{k-1}, \alpha_0, \dots, \alpha_j)$. By hypothesis, let, for each $j \in \omega$, I'_0, \dots, I'_{k-1} be a sequence of pairs of elements of n homogeneous for G_j . Infinitely many of these sequences of pairs must coincide, so there is a sequence I_0, \dots, I_{k-1} such that $\{j : (I'_0, \dots, I'_{k-1}) = (I_0, \dots, I_{k-1})\}$ is infinite. Then $I_0, \dots, I_{k-1}, \alpha_0, \alpha_1, \dots$ is ∞ -homogeneous for F . ■

PROPOSITION 2.7. For all $n \in \omega$,

$$\begin{pmatrix} n \\ n \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}.$$

The proposition follows from the following.

LEMMA 2.8. For all $n \in \omega$,

$$\begin{pmatrix} n \\ n \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \\ (n(n-1)/2) + 1 \\ \vdots \\ 2 \\ \vdots \\ \vdots \end{pmatrix}.$$

Proof. We can rearrange the elements of any sequence $s \in n^{<\omega}$ to put them in increasing order by means of a permutation. Since there might be repetitions in the sequence, the increasing sequence obtained is not necessarily strictly increasing. Let $t(s)$ be the number of transpositions of adjacent elements necessary to obtain this reordering of s .

Define $F(s) = 0$ if and only if $t(s)$ is even. The number of two-element subsets of n is $(n(n-1)/2) + 1$, so if H_0, H_1, \dots is a sequence of subsets of n and $\lceil (n(n-1)/2) + 1 \rceil$ -many of them have at least two elements, one of these pairs must be repeated, say, $H_k = H_j = \{a, b\}$ (with $k < j$ and $a < b$). We will see that F is not constant on any product of the form $\prod_{i=0}^m H_i$ with $m > k$. Let $m > k$, for any pair $(u, v) \in \{a, b\}^2$ consider the sequence $s(u, v) \in \prod_{i=0}^m H_i$ given by setting $s(u, v)(i) = \min H_i$ if $i \neq k, j$, $s(u, v)(k) = u$ and $s(u, v)(j) = v$. We will find two of these $s(u, v)$ with different values under F . By changing the coordinates k or j , the value of F might change, and this depends on the number of elements of the sequence between the old and the new values.

Let $x = |\{i : k < i < j \text{ and } \min H_i = a\}|$,

$$y = |\{i \leq m : k < i < j \text{ and } a < \min H_i < b\}| \text{ and}$$

$$z = |\{i \leq m : k < i < j \text{ and } \min H_i = b\}|.$$

These are, respectively, the number of values in a sequence of the form $s(u, v)$ which are smaller than a , between a and b , and above b . We have $F(s(a, a)) = F(s(b, b)) + x + 2y + z$ and $F(s(a, b)) = F(s(b, a)) + x + y + z + 1$.

Given $u \in a, b$, $F(s(u, a)) = F(s(u, b)) + y + z$, and $F(s(a, u)) = F(s(b, u)) + x + y$. So, in any case, there are two sequences of the form $s(u, v)$ (with $u, v \in a, b$) with different images under F . ■

COROLLARY 2.9. *The following are equivalent:*

$$(a) \begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

$$(b) \begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}.$$

Proof. Clearly, (a) implies (b). Proposition 2.7 indicates that if κ is finite, then both properties are false. By Proposition 8, we know that (a) holds for any infinite κ and, therefore, (b) also holds. ■

LEMMA 2.10. *The partition property*

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}^{2^\gamma}$$

implies that if $\{F_\xi : \xi \in \gamma\}$ is a family of partitions, $F_\xi : \kappa^{<\omega} \rightarrow 2$ for each $\xi \in \gamma$, there is a sequence $\{H_n : n \in \omega\}$ which is homogeneous for every F_ξ .

Proof. Let $\{F_\xi : \xi < \gamma\}$ be a sequence of partitions. Define $F : \kappa^{<\omega} \rightarrow 2^\gamma$ by $F(\alpha_0, \dots, \alpha_{n-1}) = \langle F_\xi(\alpha_0, \dots, \alpha_{n-1}) : \xi < \gamma \rangle$. Any homogeneous sequence for F is homogeneous for all the F_ξ . ■

DEFINITION. Given $F : \kappa^{<\omega} \rightarrow 2$, a sequence of subsets of κ $\{H_n : n \in \omega\}$ is superhomogeneous for F if for every one-one mapping $f : \omega \rightarrow \omega$ the sequence $\{H_{f(n)} : n \in \omega\}$ is homogeneous for F .

LEMMA 2.11.

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

implies that every $F : \kappa^{<\omega} \rightarrow 2$ has a superhomogeneous sequence of type $(2, 2, \dots)$.

Proof. For every $s \in \omega^{<\omega}$ let $m_s = \max\{s(i) : i \in \text{dom}(s)\}$. Let $F : \kappa^{<\omega} \rightarrow 2$ be a partition; for each one-one sequence $s \in \omega^{<\omega}$ if $lh(s) = n$, define $F_s(\alpha_0, \dots, \alpha_{m_s}) = F(\alpha_{s(0)}, \dots, \alpha_{s(n-1)})$ (and F identically 0 on all other sequences).

By 1.3 and 2.10 there is a sequence $\{H_n : n \in \omega\}$ of pairs of ordinals in κ which is homogeneous for all the F_s . We verify that it is also superhomogeneous for F . Let $f : \omega \rightarrow \omega$ be one-one, and let $s_n = f \upharpoonright n$. For each $n \in \omega$, F_{s_n} is constant on $\prod_{i=0}^{m_{s_n}} H_i$. If $(\alpha_0, \dots, \alpha_{n-1})$ and $(\beta_0, \dots, \beta_{n-1})$ are in $H_{f(0)} \times \dots \times H_{f(n-1)}$, there are sequences $(\delta_0, \dots, \delta_{m_{s_n}})$ and $(\gamma_0, \dots, \gamma_{m_{s_n}})$ in $H_0 \times \dots \times H_{m_{s_n}}$ such that $\alpha_i = \delta_{s_n(i)}$ and $\beta_i = \gamma_{s_n(i)}$ for each $i < n$. Since $F_{s_n}(\delta_0, \dots, \delta_{m_{s_n}}) = F_{s_n}(\gamma_0, \dots, \gamma_{m_{s_n}})$, we have that $F(\alpha_0, \dots, \alpha_{n-1}) = F(\beta_0, \dots, \beta_{n-1})$. ■

DEFINITION. We say that the sequence $\{H_n : n \in \omega\}$ is nonoverlapping if $\max H_i < \min H_{i+1}$ for every $i \in \omega$. The symbol

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\prec]{< \omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

means that every partition $F : \kappa^{< \omega} \rightarrow 2$ admits a nonoverlapping homogeneous sequence.

PROPOSITION 2.12.

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{< \omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}$$

implies

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\prec]{< \omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix}.$$

Proof. Let $F : \kappa^{< \omega} \rightarrow 2$ be given. Define the auxiliary partition $G : \kappa^{< \omega} \rightarrow 5$ defined as

$G(\alpha_0, \alpha_1, \dots, \alpha_k)$

$$= \begin{cases} F(\alpha_0) & \text{if } k = 0 \\ 0 & \text{if } k \geq 1, \alpha_0 < \alpha_1 \text{ and } F(\alpha_0, \alpha_1, \dots, \alpha_k) = 0 \\ 1 & \text{if } k \geq 1, \alpha_0 < \alpha_1 \text{ and } F(\alpha_0, \alpha_1, \dots, \alpha_k) = 1 \\ 2 & \text{if } k \geq 1, \alpha_0 > \alpha_1 \text{ and } F(\alpha_0, \alpha_1, \dots, \alpha_k) = 0 \\ 3 & \text{if } k \geq 1, \alpha_0 > \alpha_1 \text{ and } F(\alpha_0, \alpha_1, \dots, \alpha_k) = 1 \\ 4 & \text{if } k \geq 1 \text{ and } \alpha_0 = \alpha_1. \end{cases}$$

Let $\{H_n : n \in \omega\}$ be a superhomogeneous sequence of pairs for G . For any $i, j \in \omega$, let $\alpha \in H_i$ and $\beta \in H_j$ with $\alpha \neq \beta$. Since G is constant on $H_i \times H_j$, let m be the constant value. If $\alpha < \beta$, then $m \in \{0, 1\}$ and $(x, y) \in H_i \times H_j$ implies $x < y$. If $\alpha > \beta$, then $m \in \{2, 3\}$ and $(x, y) \in H_i \times H_j$ implies that $y < x$. In any case we have that $\max H_i < \min H_j$ or $\max H_j < \min H_i$.

Let $\{H_{f(0)}, H_{f(1)}, \dots\}$ be a reordering of the sequence so that for all $i \in \omega$, $H_{f(i)} < H_{f(i+1)}$. By superhomogeneity of $\{H_n : n \in \omega\}$, the sequence $\{H_{f(n)} : n \in \omega\}$ is homogeneous for G , and therefore also for F . ■

3. LARGER SETS IN THE HOMOGENEOUS SEQUENCE

We start this section by noting that

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix} \Rightarrow \begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} n \\ n \\ \vdots \end{pmatrix}$$

for any $n \in \omega$. This can be shown using the same idea as in Corollary 1.3 of [DPH]. For example, to obtain 4's in the right-hand side column, given a partition $F : \bigcup_n \prod_{i \leq n} \kappa^i \rightarrow 2$, define an auxiliary G by putting

$$G(\alpha_0, \dots, \alpha_{2i-1}) = F(\langle \alpha_0, \alpha_1 \rangle, \dots, \langle \alpha_{2i-2}, \alpha_{2i-1} \rangle)$$

(where $\langle \rangle : \kappa^2 \rightarrow \kappa$ is a bijection). From a homogeneous sequence of pairs for G we extract a homogeneous sequence for F formed by sets with four elements.

In [CDP] it is shown that $(2^\omega)^+$ is the first cardinal κ such that

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ \text{---} \\ \vdots \end{pmatrix},$$

in fact,

$$\begin{pmatrix} (2^\omega)^+ \\ (2^\omega)^+ \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} (2^\omega)^+ \\ \text{---} \\ \vdots \end{pmatrix}.$$

We will compute the first cardinal satisfying the partition relation with exactly n 2's in the right-hand side column (improving on results of [CDP]).

PROPOSITION 3.1. *Let $\{\kappa_i : i \in \omega\}$ be a sequence of infinite cardinals, and $A = \{\{H_i : i \in \omega\} : H_i \in [\kappa_i]^{\alpha_i}\}$. If κ is a cardinal such that $\text{cof}(\kappa) > |A|$. Then*

$$(a) \quad \begin{pmatrix} \kappa_0 \\ \kappa_1 \\ \kappa_2 \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} \quad \text{implies} \quad \begin{pmatrix} \kappa \\ \kappa_0 \\ \kappa_1 \\ \kappa_2 \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} \kappa \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix}.$$

$$(b) \begin{pmatrix} \kappa_0 \\ \kappa_1 \\ \kappa_2 \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} \quad \text{implies} \quad \begin{pmatrix} \kappa \\ \kappa_0 \\ \kappa_1 \\ \kappa_2 \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} \kappa \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix}.$$

Proof. We prove (b); the proof of (a) is almost identical.

Given a partition $F : \bigcup_{n \in \omega} (\kappa \times \prod_{i \in n} \kappa_i) \rightarrow 2$; for each $\xi \in \kappa$ define $F_\xi : \bigcup_{n \in \omega} \prod_{i \leq n} \kappa_i \rightarrow 2$ by $F_\xi(\beta_0, \beta_1, \dots, \beta_k) = F(\xi, \beta_0, \beta_1, \dots, \beta_k)$. For each of the partitions F_ξ there is $H^\xi = \{H_n^\xi : n \in \omega\}$, $A^\xi = \{n_k^\xi : k \in \omega\} \in [\omega]^\omega$, and $f^\xi \in 2^\omega$ such that, for every $k \in \omega$, $F_\xi^n \prod_{m \leq n_k} H_m^\xi = \{f^\xi(k)\}$. By the hypothesis on κ , there must be a ζ such that $K = \{\xi : \langle H^\xi, A^\xi, f^\xi \rangle = \langle H^\zeta, A^\zeta, f^\zeta \rangle\}$ has cardinality κ . Then K, H^ζ is a homogeneous sequence for F . ■

COROLLARY 3.2. *The first cardinal κ such that*

$$\begin{pmatrix} \kappa \\ \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ \vdots \\ \overset{n}{\dots} \\ 2 \\ 1 \\ \vdots \end{pmatrix}$$

is $(2^\omega)^{n+}$ (the n th successor of 2^ω). In consequence, $\sup\{(2^\omega)^{n+} : n \in \omega\}$ is the first cardinal satisfying

$$\begin{pmatrix} \kappa \\ \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ \vdots \\ \overset{n}{\dots} \\ 2 \\ 1 \\ \vdots \end{pmatrix} \quad \text{for all } n \in \omega.$$

Proof. By induction on n . We already have mentioned the result for $n = 1$. Suppose it holds for m ; then we have that if $\kappa = (2^\omega)^{m+}$,

$$\begin{pmatrix} \kappa \\ \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ \vdots \\ \overset{m}{\dots} \\ 2 \\ 1 \\ \vdots \end{pmatrix}.$$

The set $A = \{f : \omega \rightarrow [\kappa]^2\}$ has cardinality κ , so applying the previous proposition we get

$$\begin{pmatrix} \kappa^+ \\ \kappa^+ \\ \kappa^+ \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} \kappa^+ \\ 2 \\ \vdots \\ \overbrace{\quad}^m \\ 2 \\ 1 \\ \vdots \end{pmatrix},$$

and, therefore, $(2^\omega)^{(m+1)+}$ is not smaller than the least κ , such that

$$\begin{pmatrix} \kappa \\ \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ \vdots \\ \overbrace{\quad}^{m+1} \\ 2 \\ 1 \\ \vdots \end{pmatrix}.$$

By Lemmas 1.10 and 2.3 of [CDP], $(2^\omega)^{(m)+}$ does not satisfy this partition property, and from this follows the result. ■

COROLLARY 3.3. *For each $n \geq 1$,*

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ \vdots \\ \overbrace{\quad}^n \\ 2 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \Rightarrow \begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} \omega \\ \vdots \\ \overbrace{\quad}^n \\ \omega \\ 1 \\ 1 \\ \vdots \end{pmatrix}.$$

It is worth mentioning that, assuming the GCH, using the Erdős–Rado theorem and the results in Section 2 of [CDP], one can show that for every $n \geq 1$,

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} 2 \\ \vdots \\ \overbrace{\quad}^n \\ 2 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \Rightarrow \begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow{<\omega} \begin{pmatrix} \omega_1 \\ \vdots \\ \overbrace{\quad}^n \\ \omega_1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}.$$

It is easy to show that

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} \omega \\ \text{---} \\ 1 \\ 1 \\ \vdots \end{pmatrix},$$

the function defined by $F(n_0, n_1, \dots, n_k) = 0$, if and only if $|\{i \leq k : n_i > k\}|$ is even, is a counterexample. To see this, suppose that H_0, H_1, \dots is a sequence of subsets of ω with, say, H_k infinite. Then F cannot be constant on any product $\prod_{i=0}^m H_i$ with $m \geq k$, $\min(H_k)$.

As we see next, this partition relation is more interesting for uncountable cardinals.

LEMMA 3.4. *Let $\lambda = \text{cof}(\kappa)$. Then*

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} \kappa \\ \text{---} \\ 1 \\ 1 \\ \vdots \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda \\ \lambda \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} \lambda \\ \text{---} \\ 1 \\ 1 \\ \vdots \end{pmatrix}.$$

In particular, if the left-hand side of the implication holds, then $\text{cof}(\kappa) > \omega$.

Proof. Let $\{X_\beta : \beta < \lambda\}$ a family of subsets of κ such that

- (1) $\cup \{X_\beta : \beta < \lambda\} = \kappa$
- (2) $X_\alpha \cap X_\beta = \emptyset$ (for $\alpha < \beta < \lambda$).
- (3) $|X_\alpha| < \kappa$ for all $\alpha < \lambda$.

For every $\alpha < \kappa$, let β_α be the unique β such that $\alpha \in X_\beta$.

If $F : \lambda^{<\omega} \rightarrow 2$ is a partition, define $G : \kappa^{<\omega} \rightarrow 2$ by

$$G(\alpha_0, \dots, \alpha_k) = F(\beta_{\alpha_0}, \dots, \beta_{\alpha_k}).$$

Let $\{H_n : n \in \omega\}$ be such that for some n_0 , $|H_{n_0}| = \kappa$, and for some $A \in [\omega]^\omega$ there is $i \in 2$ such that for every $k \in A$, $G'' \prod_{n \leq k} H_n = \{i\}$. For each $n \in \omega$ define $I_n = \{\beta_\alpha : \alpha \in H_n\}$. Then $|I_{n_0}| = \lambda$ and $\{I_n : n \in \omega\}$ is the desired ∞ -homogeneous sequence for F .

PROPOSITION 3.5.

$$\begin{pmatrix} \kappa \\ \kappa \\ \vdots \end{pmatrix} \xrightarrow[\infty]{<\omega} \begin{pmatrix} \kappa \\ \text{---} \\ 1 \\ 1 \\ \vdots \end{pmatrix} \Leftrightarrow \begin{pmatrix} \kappa \\ \omega \end{pmatrix} \rightarrow \begin{pmatrix} \kappa \\ \omega \end{pmatrix}.$$

Proof. (\Rightarrow) Let $F : \kappa \times \omega \rightarrow 2$. Define $G : \kappa^{<\omega} \rightarrow 2$ by $G(\alpha_0, \dots, \alpha_{n-1}) = F(\max\{\alpha_0, \dots, \alpha_{n-1}\}, n)$. Let $\{H_n : n \in \omega\}$ be ∞ -homogeneous for G with, say, $|H_m| = \kappa$. Let, for every $i \in \omega$, $i \neq m$, $\alpha_i = \min H_i$, and $\alpha_m = \min\{\alpha \in H_m : \alpha > \sup\{\alpha_i : i \neq m\}\}$ (this is possible since $\text{cof}(\kappa) > \omega$). Let $H = \{\alpha \in H_m : \alpha \geq \alpha_m\}$. There is $A \in [\omega]^\omega$ and $\varepsilon \in \{0, 1\}$ such that for all $n \in A$, $n > m$, $G \upharpoonright \prod_{i=0}^n H_i = \{\varepsilon\}$. The pair H, A is homogeneous for F , since for $(\alpha, n), (\beta, k) \in H \times A$, we have $F(\alpha, n) = G(\alpha_0, \dots, \alpha_{m-1}, \alpha, \alpha_{m+1}, \dots, \alpha_n) = G(\alpha_0, \dots, \alpha_{m-1}, \beta, \alpha_{m+1}, \dots, \alpha_k) = F(\beta, k)$.

(\Leftarrow) Let $F : \kappa^{<\omega} \rightarrow 2$, and let $\alpha_1, \alpha_2, \dots$ be an arbitrary sequence of ordinals in κ . Define $G(\alpha, n) = F(\alpha, \alpha_1, \dots, \alpha_n)$. If $H \in [\kappa]^\kappa$ and $A \in [\omega]^\omega$ are such that G is constant on $H \times A$, then the sequence H_0, H_1, \dots , where $H_0 = H$ and $H_i = \{\alpha_i\}$ for $i > 0$, is ∞ -homogeneous for F . ■

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