JOURNAL OF ALGEBRA **203,** 470–490 (1998)
ARTICLE NO. JA977328

Tame Quasi-tilted Algebras

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Received May 5, 1997

INTRODUCTION

Throughout the paper K denotes a fixed algebraically closed field. By an algebra we mean a finite dimensional K-algebra (associative, with an identity) and by a module a finite dimensional right module.

Tilting theory, initiated by Brenner and Butler [6] and Happel and Ringel [11], as a generalization of the Coxeter functors of Bernstein, Gelfand, and Ponomarev [3] and Auslander, Platzeck, and Reiten [1], has left traces everywhere in recent representation theory of algebras. A prominent role in this theory is played by the tilted algebras. Following [4, 11] an algebra A is called a tilted algebra if $A = \operatorname{End}_H(T)$, where H is a hereditary algebra and T is a tilting H-module ($\operatorname{Ext}^1_H(T, T) = 0$ and T is a direct sum of $n =$ rank of $K_0(H)$ pairwise non-isomorphic indecomposable modules). Presently, an extensive representation theory of tilted algebras is developed. In particular, the structure of all connected components of the Auslander–Reiten quivers of tilted algebras is known (see [13, 14, 18, 23–25, 33]). In the tame case, a detailed structure of the category of modules over tilted algebras is also known [13, 23]. An important theoretical development of tilting theory was the connection with the derived categories established by Happel [9]. Motivated by this connection, Happel, Reiten, and Smalø introduced in [10] a generalization of tilted algebras, called quasi-tilted algebras. It is the class of algebras of the form

* Supported by the Polish Scientific Grant KBN 2P03A 020 08.

0021-8693/98 $25.00

$A = \mathrm{End}_{\mathscr{H}}(T)$, where T is a tilting object in a hereditary abelian K-category $\mathscr{H}$. It was shown in [10] that an algebra A is quasi-tilted if and only if A is of global dimension at most two and each indecomposable A-module has projective dimension at most one or injective dimension at most one. Besides the tilted algebras, important classes of quasi-tilted algebras are provided by all tubular algebras and canonical algebras introduced in [23], and their relatives concealed-canonical algebras and almost concealed-canonical algebras investigated in [16, 19]. Recently, Lenzing and the author investigated in [17] a more general class of algebras, called quasi-tilted algebras of canonical type. Following [17] an algebra A is called quasi-tilted of canonical type provided $A = \mathrm{End}_{\mathscr{H}}(T)$ for a tilting object T in a hereditary abelian K-category $\mathscr{H}$ whose derived category $D^b(\mathscr{H})$ (of bounded complexes over $\mathscr{H}$) is equivalent, as a triangulated category, to the derived category D^b (mod Λ) of modules over a canonical algebra Λ. It is shown in [17] that an algebra A is quasi-tilted of canonical type if and only if A is a semiregular branch enlargement of a concealed canonical algebra. This determines the ring structure of such algebras. Moreover, in [17] (see also [19]) a rather complete account on the structure of the module category of quasi-tilted algebras of canonical type is given.

One of the interesting open problems in the representation theory of algebras is to decide whether every quasi-tilted algebra is tilted or of canonical type. We note that all representation-finite quasi-tilted algebras are tilted [10]. In general, the main known features concern the existence of a preprojective (respectively, preinjective) component [7] and the semiregularity of Auslander−Reiten components if the algebra is quasi-tilted but not tilted [8].

The main aim of this paper is to prove several characterizations of quasi-tilted algebras which are of tame representation-type, that is, for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. We characterize this class of algebras by the ring structure, weak non-negativity of the Euler quadratic form, the shape of the connected components of the Auslander−Reiten quiver, and their behaviour in the module category. In particular, we prove that every tame quasi-tilted algebra is either tilted or of canonical type.

The paper is organized as follows. In Section 1 we present our main results and recall the related background. Section 2 contains some known results on quasi-tilted algebras. In Section 3 we prove some new technical facts applied in the proof of our main result. The final Section 4 contains the proof of the main result of the paper.

The results of this paper were presented during the Conferences in Oberwolfach (1995), Marseille-Luminy (1996), and Geiranger (1996).

1. THE MAIN RESULTS AND THE RELATED BACKGROUND

Throughout this paper K will denote a fixed algebraically closed field. By an algebra is meant an associative finite dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic. For such an algebra A, there exists an isomorphism $A \simeq KQ/I$, where KQ is the path algebra of the Gabriel quiver $Q = Q_A$ of A and I is an admissible ideal in KQ. Equivalently, $A = KQ/I$ may be considered as a K-category whose object class is the set Q_0 of vertices of Q, and the set of morphisms $A(x, y)$ from x to y is the quotient of the K-space $KQ(x, y)$, formed by the linear combinations of paths in Q from x to y, by the subspace $I(x, y) = KQ(x, y) \cap I$. An algebra A with Q_A having no oriented cycle is said to be triangular. A full subcategory C of A is said to be convex if any path in Q_A with source and target in Q_C lies entirely in Q_C.

For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\operatorname{ind} A$ its full subcategory consisting of indecomposable modules. By an A-module is meant an object of $\operatorname{mod} A$. For an A-module X, we denote by $\underline{\dim} X$ the dimension-vector of X, being the image of X in the Grothendieck group $K_0(A) = \mathbb{Z}^n$, $n = |Q_0|$, of A. We denote by D the standard duality $\operatorname{Hom}_K(-, K)$: $\operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$, where A^{op} denotes the opposite algebra of A. Moreover, we denote by Γ_A the Auslander−Reiten quiver of A, and by τ_A and τ_A^- the Auslander−Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We shall not distinguish between an object of $\operatorname{ind} A$ and the vertex of Γ_A corresponding to it. A component $\mathscr{C}$ of Γ_A is said to be standard if the full subcategory of $\operatorname{ind} A$ formed by the modules from $\mathscr{C}$ is equivalent to the mesh-category $K(\mathscr{C})$ of $\mathscr{C}$ [23]. Examples of standard components are provided by preprojective components, preinjective components, connecting components of tilted algebras, and tubular families of canonical algebras [23]. Further, we denote by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$ and by $\operatorname{rad}^\infty(\operatorname{mod} A)$ the intersection of all finite powers $\operatorname{rad}^i(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. The component quiver Σ_A of A [27] is a quiver whose vertices are the connected components of Γ_A, and two components $\mathscr{C}$ and $\mathscr{D}$ are connected in Σ_A by an arrow $\mathscr{C} \to \mathscr{D}$ if $\operatorname{rad}^\infty(X, Y) \neq 0$ for some modules X in $\mathscr{C}$ and Y in $\mathscr{D}$. If Σ_A is directed then A is said to be component-directed. Recall also that a component $\mathscr{C}$ of Γ_A is called standard if the full subcategory of $\operatorname{mod} A$ formed by the modules from $\mathscr{C}$ is equivalent to the mesh-category $K(\mathscr{C})$ of $\mathscr{C}$ [23].

Let $K[x]$ be the polynomial algebra in one variable. Then A is said to be tame if, for any dimension d, there exists a finite number of $K[x]-A$-bimodules M_i, $1 \leq i \leq n_d$, which are free of finite rank as left $K[x]$-modules and all but finitely many isomorphism classes of indecomposable

A-modules of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some i. Let $\mu_A(d)$ be the least number of $K[x]-A$-bimodules satisfying the above condition for d. Then A is said to be of linear growth (respectively, domestic) if there exists a positive integer m such that $\mu_A(d) \le md$ (respectively, $\mu_A(d) \le m$). Examples of domestic algebras are provided by the tilted algebras of Euclidean type. The tubular algebras (in the sense of [23]) are non-domestic of linear growth (see [26, (3.6)]).

Assume that $A = KQ/I$ is an algebra of finite global dimension. Then the Euler characteristic χ_A of A is the integral quadratic form on $K_0(A)$ such that

$$\chi_A(\underline{\dim} X) = \sum_{i=0}^{\infty} (-1)^i \dim_K \operatorname{Ext}_A^i(X, X)$$

for any A-module X (see [23, (2.4)]). We say that χ_A is weakly non-negative if $\chi_A(x) \ge 0$ for any vector x in $K_0(A)$ with non-negative coordinates. If A is triangular and gl.dim $A \le 2$, then χ_A coincides with the Tits form q_A of A, defined in general for $x = (x_i)_{i \in Q_0} \in K_0(A)$ as

$$q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{(i \to j) \in Q_1} x_i x_j + \sum_{i, j \in Q_0} r_{ij} x_i x_j,$$

where Q_0 and Q_1 are the sets of vertices and arrows of Q, respectively, and r_{ij} is the cardinality of $L \cap I(i, j)$, for a minimal set of generators $L \subset \bigcup_{i, j \in Q_0} I(i, j)$ of the ideal I (see [5]). It is well known that if A is triangular and tame then q_A is weakly non-negative (see [20]).

Let C be a tame concealed algebra, that is, an algebra of the form $\operatorname{End}_H(T)$, where T is a preprojective tilting module over a hereditary algebra H of Euclidean type. Then Γ_C consists of a preprojective component $\mathscr{P}$, a preinjective component $\mathscr{Q}$, and a family $\mathscr{T} = (\mathscr{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes separating $\mathscr{P}$ from $\mathscr{Q}$ [23, (4.3)]. By a semiregular branch enlargement of C we mean an algebra of the form

$$\Lambda = \begin{bmatrix} F & M & 0 \\ 0 & C & D(N) \\ 0 & 0 & B \end{bmatrix},$$

where

$$\Lambda^+ = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix} \qquad \left(\text{respectively, } \Lambda^- = \begin{bmatrix} C & D(N) \\ 0 & B \end{bmatrix}\right)$$

is a tubular extension (respectively, tubular coextension) of C in the sense of [23, (4.7)], and no tube in $\mathscr{T}$ admits both a direct summand of M and a direct summand of N. It is known that such an algebra Λ is quasi-tilted. Moreover, Λ is tame if and only if both Λ^+ and Λ^- are tame (see [23]), or equivalently, are tubular algebras or tilted algebras of Euclidean type. Further, if Λ is tame, then Λ is of linear growth and every component of Γ_A is standard. Finally, Λ is domestic if and only if both Λ^+ and Λ^- are tilted algebras of Euclidean type.

We may state now the main result of the paper.

THEOREM A. *Let A be a connected quasi-tilted algebra. The following conditions are equivalent*:

(i) *A is tame.*

(ii) *A is of linear growth.*

(iii) *A is tame tilted or a tame semiregular branch enlargement of a tame concealed algebra.*

(iv) *χ_A is weakly non-negative.*

(v) $\dim_K \mathrm{Ext}^1_A(X, X) \leq \dim_K \mathrm{End}_A(X)$ *for any module X in* ind A.

(vi) *Every component of Γ_A is standard.*

(vii) *A is component-directed.*

Observe that each of the conditions (iii) or (iv) is rather easy to check. As a direct consequence of the above theorem and [15, (1.5); 26, (3.6); 29, (2.8)] we get the following characterization of domestic quasi-tilted algebras.

COROLLARY B. *Let A be a quasi-tilted algebra. The following conditions are equivalent*:

(i) *A is domestic.*

(ii) *A is tame and does not contain a convex subcategory which is tubular.*

(iii) $\bigcap_{i \geq 1}(\mathrm{rad}^{\infty}(\mathrm{mod}\, A))^i = 0$.

(iv) $(\mathrm{rad}^{\infty}(\mathrm{mod}\, A))^5 = 0$.

The following fact follows from Theorem A and [23, (4.9) and (5.2)].

COROLLARY C. *Let A be a tame* (*respectively, domestic*) *quasi-tilted algebra. Then all but a finite number of connected components in Γ_A are stable tubes* (*respectively, stable tubes of rank one*).

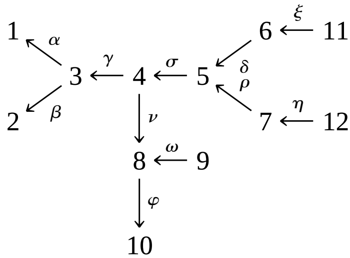
It would be interesting to know whether, for a quasi-tilted algebra, the converse implications are also true.

Recall that two algebras Λ and Γ are called triangle equivalent if their derived categories $D^b(\mathrm{mod}\,\Lambda)$ and $D^b(\mathrm{mod}\,\Gamma)$ are equivalent as triangulated categories (see [9]). We get the following consequence of Theorem A and [9; 17, (3.4)].

COROLLARY D. *Let A be a connected tame quasi-tilted algebra. Then A is triangle-equivalent to a hereditary algebra or a canonical algebra.*

For basic background on the representation theory applied here we refer to [2, 10, 23, 27].

We end this section with an example of a tame quasi-tilted algebra which is neither tilted nor tubular. Let Λ be the algebra KQ/I where Q is the quiver



and I is the ideal in KQ generated by the paths $\sigma\nu$, $\omega\varphi$, $\xi\delta\sigma\gamma\beta$, and $\eta\rho\sigma\gamma\alpha$. Observe that the convex subcategory C of Λ given by the vertices $1, 2, 3, 4, 5, 6, 7$ is a tame hereditary algebra of type $\tilde{\mathbb{D}}_6$, and hence of tubular type $(2, 2, 4)$. Denote by Λ^- the convex subcategory of Λ given by the vertices $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, and by Λ^+ the convex subcategory Λ given by the vertices $1, 2, 3, 4, 5, 6, 7, 11, 12$. Then Λ^- is a tubular coextension of C of tubular type $(2, 2, 7)$, and hence a representation-infinite tilted algebra of type $\tilde{\mathbb{D}}_9$ having a complete slice in its preprojective component. On the other hand, Λ^+ is a tubular extension of C of tubular type $(2, 4, 4)$, and hence a tubular algebra. Moreover, the tubular coextension Λ^- is obtained from C by using one simple regular module in the unique tube of Γ_C of rank 4 and rooting a branch of length 3 (given by the vertices $8, 9, 10$) in the coextension vertex 8, whereas the tubular extension Λ^+ of C is obtained from C by two one-point extensions, with the extension vertices 11 and 12, using two simple regular modules lying in one of tubes of rank 2 in Γ_C. Therefore, Λ is a tame semiregular branch enlargement of (wild) canonical type $(2, 4, 7)$, and in particular neither tilted nor tubular (see [17, Sect. 4]). The Auslander–Reiten quiver Γ_Λ of Λ is of the form

$$\mathscr{P}_0 \vee \mathscr{T}_0 \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathscr{T}_q \right) \vee \mathscr{T}_\infty \vee \mathscr{Q}_\infty,$$

where $\mathbb{Q}^+$ is the set of all positive rational numbers and

- $\mathscr{P}_0$ is a preprojective component of Euclidean type $\tilde{\mathbb{D}}_9$ containing the projective modules corresponding to the vertices $1, 2, \ldots, 9, 10$;

• $\mathcal{T}_0$ is a $\mathbb{P}_1(K)$-family of semiregular tubes formed by one coray tube with 7 corays and 3 injective modules (corresponding to the vertices $8, 9, 10$), one ray tube with 4 rays and two projective modules (corresponding to the vertices $11, 12$), one stable tube of rank 2, and the remaining tubes being stable tubes of rank 1;

• For each $q \in \mathbb{Q}^+$, $\mathcal{T}_q$ is a $\mathbb{P}_1(K)$-family of stable tubes of tubular type $(2, 4, 4)$;

• $\mathcal{T}_\infty$ is a $\mathbb{P}_1(K)$-family of coray tubes formed by a coray tube with 4 corays and containing the injective module given by the vertex 1, a coray tube with 4 corays and containing the injective module given by the vertex 2, one stable tube of rank 2, and the remaining tubes being stable tubes of rank 1;

• $\mathcal{Q}_\infty$ is the preinjective component of type $\tilde{\mathbb{E}}_6$ containing the injective modules given by the vertices 3, 4, 5, 6, 7, 11, and 12 (forming a tame hereditary algebra of type $\tilde{\mathbb{E}}_6$).

2. KNOWN FACTS ON QUASI-TILTED ALGEBRAS

In this section we shall collect some known facts on quasi-tilted algebras, applied in our proofs.

Let Λ be an algebra. A path in $\operatorname{mod} A$ is a sequence of non-zero non-isomorphisms

$$X = X_0 \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_t} X_t = Y,$$

where $t \geq 1$ and all X_i belong to $\operatorname{ind} A$. In this case we write $X \preccurlyeq Y$ and say that X is a predecessor of Y and that Y is a successor of X in $\operatorname{mod} A$. A module M in $\operatorname{mod} \Lambda$ is said to be directing [12] provided there do not exist indecomposable direct summands M_1, M_2 of M and an indecomposable non-projective Λ-module W such that $M_1 \preccurlyeq \tau_\Lambda W$ and $W \preccurlyeq M_2$. For a Λ-module M, we denote by $\operatorname{pd}_\Lambda M$ (respectively, $\operatorname{id}_\Lambda M$) the projective dimension (respectively, the injective dimension) of M. Following [10] we denote by $\mathcal{L}_\Lambda$ the full subcategory of $\operatorname{ind} \Lambda$ consisting of all modules such that $\operatorname{pd}_\Lambda Y \leq 1$ for any predecessor Y of X. Dually, we denote by $\mathcal{R}_\Lambda$ the full subcategory of $\operatorname{ind} \Lambda$ consisting of all modules X such that $\operatorname{id}_\Lambda Y \leq 1$ for any successor Y of X. The one-point extension algebra $\Lambda[M]$ of Λ by a Λ-module M is by definition the algebra

$$\Lambda[M] = \begin{bmatrix} K & M \\ 0 & \Lambda \end{bmatrix}.$$

If $\Gamma = \Lambda[M]$ then the category $\operatorname{mod}\Gamma$ is equivalent to the category of triples (V, X, φ) where V is a finite dimensional K-vector space, X is a Λ-module, and φ: $V \to \operatorname{Hom}_\Lambda(M, X)$ is a K-linear map (see [23, (2.5)].

Assume now that Λ is a quasi-tilted algebra and $\mathscr{L} = \mathscr{L}_\Lambda$, $\mathscr{R} = \mathscr{R}_\Lambda$. Then the following facts hold:

(2.1) There is a trisection

$$\operatorname{ind}\Lambda = (\mathscr{L}\setminus\mathscr{R}) \vee (\mathscr{L}\cap\mathscr{R}) \vee (\mathscr{R}\setminus\mathscr{L})$$

such that

$$\operatorname{Hom}_A(\mathscr{L}\cap\mathscr{R}, \mathscr{L}\setminus\mathscr{R}) = 0, \qquad \operatorname{Hom}_A(\mathscr{R}\setminus\mathscr{L}, \mathscr{L}\cap\mathscr{R}) = 0, \qquad \text{and}$$

$$\operatorname{Hom}_A(\mathscr{R}\setminus\mathscr{L}, \mathscr{L}\setminus\mathscr{R}) = 0.$$

Moreover, $\mathscr{L}$ (respectively, $\mathscr{R}$) contains all indecomposable projective (respectively, injective) Λ-modules [10, II, (1.13) and (1.14)].

(2.2) Γ_Λ admits a preprojective (respectively, preinjective) component [7].

(2.3) If Λ is not tilted then every component of Γ is semiregular, that is, does not contain simultaneously a projective module and an injective module [8, Corollary E].

(2.4) If a component $\mathscr{C}$ of Γ_Λ contains an oriented cycle then $\mathscr{C}$ is a ray or coray tube [8, Theorem A].

(2.5) Assume that Λ is not tilted and $\mathscr{C}$ a component of Γ_Λ. If $\mathscr{C}$ contains a projective module, then $\mathscr{C}$ is contained in $\mathscr{L}\setminus\mathscr{R}$ [8, Theorem C].

(2.6) Λ is triangular [10, III, (1.1)].

(2.7) If Λ is representation-finite, then Λ is tilted, and hence Γ_Λ is directed [10, II, (3.6)].

(2.8) If $\mathscr{R}$ contains a projective module then Λ is tilted [10, II, (3.4)].

(2.9) Every full subcategory of Λ is quasi-tilted [10, II, (1.15)].

(2.10) If T is a tilting Λ-module in $\operatorname{add}(\mathscr{L})$ then $\operatorname{End}_\Lambda(T)$ is quasi-tilted [10, II, (2.4)].

(2.11) If $\Lambda[M]$ is quasi-tilted then each indecomposable direct summand of M is contained in $\mathscr{L}$ [10, III, (2.4)].

(2.12) Let M_1 and M_2 be non-zero Λ-modules such that $\Lambda[M_1 \oplus M_2]$ is quasi-tilted. Then each indecomposable direct summand of M_1 is contained in $\mathscr{R}$ or M_2 is projective [7, (2.1)].

(2.13) Assume Λ is tame concealed and M is an indecomposable regular Λ-module. Then $\Lambda[M]$ is quasi-tilted if and only if M is simple regular [10, III, (3.9)].

3. NEW FACTS ON QUASI-TILTED ALGEBRAS

We shall prove here several new facts on quasi-tilted algebras playing an important role in our investigations.

3.1. LEMMA. *Let C be a tame concealed algebra, E a simple regular C-module, and Λ the triangular matrix algebra*

$$C[E][E] = \begin{bmatrix} K & 0 & E \\ 0 & K & E \\ 0 & 0 & C \end{bmatrix}.$$

Then Λ is not quasi-tilted.

Proof. Since E is simple regular, it is a source of a unique sectional path

$$E = E_0 \to E_1 \to E_2 \to \cdots$$

of a standard stable tube in Γ_C. Then, applying [23, p. 88], we infer that Γ_Λ contains a mesh-complete full translation subquiver of the form

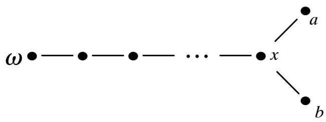
$$\begin{array}{ccccccc}
P_\Lambda(x) & \longrightarrow & & L_1 & \longrightarrow & L_2 & \longrightarrow \cdots \\
\uparrow & & \nearrow & \uparrow & & \uparrow & \\
& P_\Lambda(y) & & & & & \\
& \nearrow & & & & & \\
E_0 & \longrightarrow & & E_1 & \longrightarrow & E_2 & \longrightarrow \cdots
\end{array}$$

with $\tau_\Lambda E_{i+2} \not\simeq E_i$ for all $i \geq 0$, where $P_\Lambda(x)$ and $P_\Lambda(y)$ are the indecomposable projective Λ-modules given by the extension vertices x and y of $C[E][E]$, respectively. From the shape of the Auslander−Reiten components of tilted algebras (see [13, 14, 18, 23]) we conclude that Λ is not tilted. Consider now the almost split sequence

$$0 \to E \to Y \xrightarrow{g} X \to 0$$

in Γ_C, and take a unique indecomposable Λ-module Z whose restriction to C is Y, and Z has one dimensional vector spaces at the vertices x and y. Let $P(Z) \xrightarrow{\pi} Z$ be the projective cover of Z in $\operatorname{mod} \Lambda$ and $\Omega(Z) = \operatorname{Ker} \pi$. Then $\Omega(Z) \simeq E \oplus \Omega(X)$. Since E is not projective, we get $\operatorname{pd}_\Lambda Z = 2$. But then $\operatorname{Hom}_\Lambda(I, \tau_\Lambda Z) \neq 0$ for an indecomposable injective Λ-module I (see [23, p. 74]). Further, g induces a non-zero map $Z \to X$, again by [23, p. 88], and $E = \tau_C^{-r} X$ for some $r \geq 0$. Therefore, we conclude that I is a predecessor of $P_A(x)$ in $\operatorname{mod} \Lambda$, and so $\mathscr{R}_\Lambda$ contains a projective module. Hence, by (2.8), Λ is not quasi-tilted.

3.2. LEMMA. *Let C be a tame concealed algebra, E a simple regular C-module, ω the extension vertex of $C[E]$, and Λ the algebra obtained from $C[E]$ by rooting a hereditary quiver Δ*

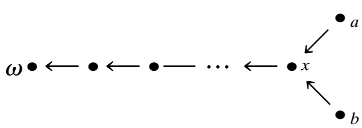


at ω, where $\bullet$—$\bullet$ means $\bullet \longrightarrow \bullet$ or $\bullet \longleftarrow \bullet$, and possibly $\omega = x$. Then Λ is not quasi-tilted.

Proof. We first claim that Λ is not tilted. Observe that one of the vertices of the above quiver Δ is a source. Then, applying [23, p. 88], we infer that Γ_Λ has a full subquiver of the form

$$\begin{array}{ccccccccc} & U & & V & & & & & \\ & \uparrow & \nearrow & & & & & & \\ P_\Lambda(x) = & W_0 & \longrightarrow & W_1 & \longrightarrow & W_2 \longrightarrow & W_3 & \longrightarrow & \cdots \end{array}$$

with $\tau_\Lambda W_{i+2} \not\simeq W_i$ for all $i \geq 0$. This implies that Γ_A is not the Auslander–Reiten quiver of a tilted algebra, again by [13, 14, 18, 23]. Suppose now that (for an orientation of Δ) Λ is quasi-tilted. Since Λ is not tilted, we conclude by (2.5) that every component of Γ_Λ containing a projective module consists entirely of modules from $\mathscr{L}_\Lambda$. In particular, the APR-tilting modules [1] induced by the simple projective Λ-modules are in the additive category $\operatorname{add}(\mathscr{L}_\Lambda)$ of $\mathscr{L}_\Lambda$. Applying now an appropriate sequence of APR-tilts and (2.10) we get a quasi-tilted algebra Λ' obtained from $C[E]$ by rooting at ω the quiver



Then the full subcategory Λ'' of Λ' given by the objects of C, a, and b is the double one-point extension $C[E][E]$ of C by E. Hence, by (3.1), Λ' is not quasi-tilted. This contradicts (2.9).

3.3. PROPOSITION. *Let B be a tubular extension of a tame concealed algebra C, $\mathscr{P}$ the preprojective component of Γ_C (and hence of Γ_B), $\mathscr{T} = (\mathscr{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ the $\mathbb{P}_1(K)$-family of ray tubes in Γ_B obtained from the unique family $\mathscr{T}' = (\mathscr{T}_\lambda')_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes of Γ_C by ray insertions, and $\mathscr{Q}$ the* *family of the remaining components of* Γ_B. *Assume that* M *is a* B*-module satisfying the following conditions*:

(i) M *has no indecomposable direct summand from* $\mathscr{P}$.

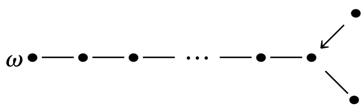
(ii) M *has at least one indecomposable direct summand from* $\mathscr{T}$.

(iii) *The one-point extension* $B[M]$ *is quasi-tilted.*

(iv) $\chi_{B[M]}$ *is weakly non-negative.*

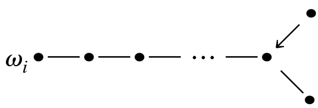
Then M *is indecomposable and has exactly one direct successor in* Γ_B (*hence is a ray module in the sense of* [23]).

Proof. We have two cases to consider. Assume first that M is indecomposable. Then, by our assumption, M lies in a ray tube $\mathscr{T}_\lambda$. We know (see [23, (4.7)]) that the restriction F of M to C is indecomposable or zero. If F is indecomposable then $C[F]$ is a full subcategory of $B[M]$, and hence F is simple regular, by (2.9) and (2.13). Therefore, we may assume that $F \neq M$. Suppose now that M has two direct successors in Γ_B. Since $\mathscr{T}_\lambda$ is a ray tube we then conclude that $B[M]$ contains a full subcategory D obtained from a one-point extension $\Lambda = C[N]$ of C by a simple regular C-module N, say with the extension vertex ω, by rooting a hereditary quiver of the form



at ω, where $\bullet\!-\!\!-\!\bullet$ means $\bullet \longrightarrow \bullet$ or $\bullet \longleftarrow \bullet$. It follows from (3.2) that D is not quasi-tilted. Therefore, applying (2.9) we conclude that $B[M]$ is not quasi-tilted which contradicts our assumption (iii).

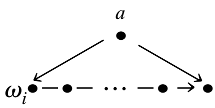
Assume now that M is decomposable. Then, by our assumption (ii), we have $M = X \oplus Y$ where $Y \neq 0$ and X is an indecomposable direct summand of M lying in a tube $\mathscr{T}_\mu$ of $\mathscr{T}$. If $\mathscr{T}_\mu$ contains a projective module, then there exist indecomposable modules Z and P in $\mathscr{T}_\mu$ such that P is projective, $\operatorname{Hom}_B(\tau_B^- Z, P) \neq 0$, and X is predecessor of Z in $\mathscr{T}_\mu$. Hence, $\operatorname{id}_B Z = 2$, and so $X \notin \mathscr{R}_B$. Then, by (2.12), Y is projective. If $\mathscr{T}_\mu$ is a stable tube, then it is a stable tube of Γ_C, and Y is projective by [10, III, (2.9)(a)]. Since all indecomposable projective B-modules lie in $\mathscr{P} \vee \mathscr{T}$, it follows from our assumption (i) that Y is a direct sum of indecomposable projective modules from $\mathscr{T}$. Replacing now X by an indecomposable (projective) direct summand U of Y, we conclude as above that $M = U \oplus V$ with V projective. Therefore, M is projective. We shall show now that this leads to a contradiction. Observe first that B, as a tubular extension of C, is obtained from a multiple one-point extension $C[E_1][E_2]\cdots[E_t]$ of C by pairwise non-isomorphic simple regular C-modules $E_1, \ldots, E_t$, say with the extension vertices $\omega_1, \ldots, \omega_t$, by rooting (extension) branches $K_1, \ldots, K_t$ at $\omega_1, \ldots, \omega_t$, respectively (see [23, (4.7)]). Moreover, M is a direct sum of indecomposable projective B-modules given by some vertices of the branches $K_1, \ldots, K_t$. Further, if the restriction M' of M to C is non-zero then M' is simple regular (2.13), and hence isomorphic to one of the modules $E_1, \ldots, E_t$, because M is projective. From Lemma 3.2 we know that $B[M]$ does not contain a full subcategory which is obtained from the one-point extension $C[E_i]$ by rooting at the extension vertex ω_i a hereditary quiver of the form



Further, $B[M]$ does not contain a full subcategory L given by the quiver

$$\omega_i \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \cdots \longleftarrow \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \longleftarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \, \omega_i$$

bounded only by $\alpha\beta = 0$. Indeed, L is representation-finite and Γ_L has an oriented cycle, and hence by (2.9) and (2.7), L cannot be a full subcategory of $B[M]$. Moreover, since M is decomposable projective, the restriction M' of M to C is zero or simple regular, and $\chi_{B[M]}$ is weakly non-negative, a simple analysis of the supports of indecomposable direct summands of M shows that $B[M]$ contains a full subcategory R obtained from $C[E_1]\ldots[E_t]$ by rooting a hereditary quiver



at ω_i or a hereditary quiver

$$\omega_i \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \cdots \longrightarrow \bullet \longleftarrow \underset{a}{\bullet} \longrightarrow \bullet \longleftarrow \cdots \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \, \omega_j$$

at the vertices ω_i and ω_j. Observe also that R is not tilted. Indeed, the radical of the indecomposable projective R-module $P_R(a)$ at a has at least two indecomposable direct summands being sources of infinite sectional paths in Γ_R, which for tilted algebras is not possible. In particular, the APR-tilting modules given by simple projective R-modules are in $\mathrm{add}(\mathscr{L}_R)$, by (2.3). Applying now an appropriate sequence of APR-tilts and taking a full subcategory we get, by (2.9) and (2.10), a quasi-tilted algebra of the form $C[E]$, where E has a direct summand $E_i \oplus E_i$ or $E_i \oplus E_j$. It is a contradiction with (2.13). Hence, the case when M is decomposable does not hold. This finishes the proof.

3.4. COROLLARY. *Let B be a tubular extension of a tame concealed algebra C, $\mathscr{T}$ the $\mathbb{P}_1(K)$-family of ray tubes in Γ_B, obtained from the $\mathbb{P}_1(K)$-family of stable tubes in Γ_C by ray insertions, and M an indecomposable B-module in $\mathscr{T}$. Assume that Λ is an algebra, N is a non-zero Λ-module, and the one-point extension $(B \times \Lambda)[M \oplus N]$ is quasi-tilted. Then N is uniserial and each submodule of N is projective.*

Proof. It follows from the above proof and (2.9) that N is an indecomposable projective Λ-module whose support is a convex subcategory of Λ given by a linear hereditary quiver

$$a_1 \to a_2 \to \cdots \to a_r$$

and the support of any projective modules $P_\Lambda(a_i)$, $1 \le i \le r$, consists of the objects $a_i, \dots, a_r$. In particular, N is uniserial and each submodule of N is uniserial and projective.

3.5. PROPOSITION. *Let B be a tame semiregular branch enlargement of a tame concealed algebra C and M be a non-zero B-module having an indecomposable preprojective direct summand. Assume $\Lambda = B[M]$ is quasi-tilted and χ_Λ is weakly non-negative. Then M is preprojective.*

Proof. Let $\mathscr{T}$ be the $\mathbb{P}_1(K)$-family of semiregular tubes in Γ_B obtained from the unique $\mathbb{P}_1(K)$-family of stable tubes in Γ_C by the corresponding ray and coray insertions. It is known that the tubes in $\mathscr{T}$ are standard and pairwise orthogonal. Denote by B^- (respectively, B^+) the maximal tubular coextension (respectively, extension) of C inside B. Then the preprojective component $\mathscr{P}$ of Γ_{B^-} is the unique preprojective component of Γ_B. Let $M = M_1 \oplus M_2$ where M_1 is a direct sum of modules from $\mathscr{P}$ and M_2 has no direct summands from $\mathscr{P}$. It follows from our assumption that $M_1 \neq 0$. Since χ_Λ is weakly non-negative we infer, by [20, (2.5); 22, (3.3); 23, (4.9)], that B^- is a representation-infinite tilted algebra of Euclidean type having a complete slice in the preprojective component $\mathscr{P}$, the family $\mathscr{T}$ contains at least one injective module, and the restriction of M_1 to C is zero. Let $\mathscr{T}'$ be the family of all tubes in $\mathscr{T}$ containing injective modules and $\mathscr{T}''$ the family of all remaining tubes of $\mathscr{T}$. We know that $\Gamma_B = \mathscr{P} \vee \mathscr{T} \vee \mathscr{Q}$ where $\mathscr{Q}$ is either a preinjective component (if B^+ is tilted of Euclidean type) or of the form $(\bigvee_{q \in \mathbb{Q}^+} \mathscr{T}_q) \vee \mathscr{T}_\infty \vee \mathscr{Q}_\infty$ (if B^+ is tubular), see [23, (4.9) and (5.2)]. Observe that each tube in $\mathscr{T}'$ is a coray tube containing an injective module, and hence a module of projective dimension 2. Therefore, $\mathscr{T}'$ is entirely contained in $\mathscr{R}_B \setminus \mathscr{L}_B$. Further, it follows from the shape of Γ_B that any module in $\mathscr{Q}$ is a successor in ind B of a module from $\mathscr{T}'$, and so $\mathscr{Q}$ is also contained $\mathscr{R}_B \setminus \mathscr{L}_B$. Suppose now that $M_2 \neq 0$. Then invoking (2.11) and our assumption that $\Lambda = B[M]$ is quasi-tilted we conclude that

M_2 is a direct sum of modules from $\mathscr{T}''$. We know that $\mathrm{Hom}_B(\mathscr{T}'', \mathscr{P} \vee \mathscr{T}') = 0$ and $\mathrm{Hom}_B(\mathscr{T}', \mathscr{T}'') = 0$. Observe also that the one-point extension $B^+[M_2]$ is a convex subcategory of $\Lambda = B[M]$, and so is quasi-tilted. Applying (3.3) we then infer that M_2 is a ray module of a tube $\mathscr{T}_\lambda''$ of $\mathscr{T}''$. Further, there is a tube $\mathscr{T}_\mu'$ in $\mathscr{T}'$ such that $\mathrm{Hom}_B(M_1, \mathscr{T}_\mu') \neq 0$. Finally, observe that $\mathrm{Hom}_B(M_1, \mathscr{T}'') = 0$ because the restriction of M_1 to C is zero. Take now indecomposable non-directing modules X in $\mathscr{T}_\mu'$ and $Y \in \mathscr{T}_\lambda''$ with $\mathrm{Hom}_B(M_1, X) \neq 0$, $\mathrm{Hom}_B(M_2, Y) \neq 0$, and $M_2 \not\simeq Y$. Let f: $M_1 \to X$ and g: $M_2 \to Y$ be non-zero maps in mod B, and consider the Λ-module Z given by the triple $(K, X \oplus Y, \varphi)$ where the K-linear map φ: $K \to \mathrm{Hom}_B(M, X \oplus Y) = \mathrm{Hom}_B(M_1, X) \oplus \mathrm{Hom}_B(M_2, Y)$ assigns to the identity of K the pair (f, g). Since X and Y are pairwise orthogonal and indecomposable, Z is indecomposable. Observe that $\mathrm{pd}_B U = \mathrm{pd}_\Lambda U$ for any B-module U. Hence, since $X \in \mathscr{R}_B \setminus \mathscr{L}_B$ and is a predecessor of Z in mod Λ, we get $Z \notin \mathscr{L}_\Lambda$. We shall prove that also $Z \notin \mathscr{R}_\Lambda$. This will lead to a contradiction because Λ is quasi-tilted (see (2.1)). Let h: $Y \to N$ be an irreducible map in mod B where N is a module in $\mathscr{T}_\lambda''$ which does not lie on the ray starting at M_2. Since $\mathrm{Hom}_B(M, h) = 0$, h induces a non-zero map $Z = (K, X \oplus Y, \varphi) \to (0, N, 0) = N$ in ind Λ. Observe that N is a non-directing module of $\mathscr{T}_\lambda''$. We have two cases to consider. Assume first that $\mathscr{T}_\lambda''$ is a stable tube. Then $\tau_\Lambda M_2 = \tau_B M_2$, $\mathrm{Hom}_\Lambda(\tau_\Lambda^-(\tau_\Lambda M_2), \Lambda) = \mathrm{Hom}_\Lambda(M_2, \Lambda) \neq 0$, and so $\mathrm{id}_\Lambda \tau_\Lambda M_2 = 2$. Clearly, $\tau_\Lambda M_2 = \tau_B M_2$ is a successor of N in $\mathscr{T}_\lambda''$, and hence $\tau_\Lambda M_2$ is a successor of Z in mod Λ. This implies $Z \notin \mathscr{R}_\Lambda$. Assume now that $\mathscr{T}_\lambda''$ contains a projective module. Since $\mathscr{T}_\lambda''$ is a ray tube, there exists an indecomposable non-projective non-directing B-module V which is a direct predecessor of a projective module in $\mathscr{T}_\lambda''$. Then again $\tau_\Lambda V = \tau_B V$, $\mathrm{Hom}_\Lambda(\tau_\Lambda^-(\tau_\Lambda V), \Lambda) = \mathrm{Hom}_\Lambda(V, \Lambda) \neq 0$, and so $\mathrm{id}_\Lambda \tau_\Lambda V = 2$. On the other hand, since N and $\tau_\Lambda V = \tau_B V$ are non-directing modules in $\mathscr{T}_\lambda''$, we conclude that $\tau_B V$ is a successor of N in $\mathscr{T}_\lambda''$. Then $\tau_\Lambda V$ is a successor of Z in mod Λ and hence $Z \notin \mathscr{R}_\Lambda$. Therefore, we proved that $M_2 = 0$, and $M = M_1$ is preprojective.

4. PROOF OF THEOREM A

Let A be a quasi-tilted algebra. Then, by (2.6), A is triangular. Since gl.dim $A \leq 2$ we then get $q_A = \chi_A$. In particular (i) $\Rightarrow$ (iv). The implication (ii) $\Rightarrow$ (i) is obvious. Further, (iii) $\Rightarrow$ (ii) follows from [23, (4.9), (5.2); 22, (2.4); 26, (3.6)]. The implications (iii) $\Rightarrow$ (vi) and (iii) $\Rightarrow$ (vii) follow from [23, (4.9), (5.2); 13]. Moreover, if (vi) (respectively, (vii)) holds then $\mathrm{rad}^\infty(X, X) = 0$ for any $X \in$ ind A, and hence A is tame (see [29, (2.8)]). We shall prove now that (iv) is equivalent to (v). Observe first that, for $X \in$ ind A, we have $\dim_K \mathrm{End}_A(X) - \dim_K \mathrm{Ext}_A^1(X, X) = \chi_A(\underline{\dim} X) =$

$q_A(\underline{\dim} X)$ because $\operatorname{pd}_A X \le 1$ or $\operatorname{id}_A X \le 1$. Hence (iv) implies (v). The implication (v) $\Rightarrow$ (iv) follows by reducing the general case to the indecomposable one via generic decomposition. Namely, if $x \in K_0(A)$ has non-negative coordinates, then there is a decomposition $x = x_1 + \cdots + x_s$ and $X_1, \dots, X_s \in \operatorname{ind} A$ with $x_i = \underline{\dim} X_i$, $1 \le i \le s$, such that

$$q_A(x) \ge \sum_{i=1}^{s} \left(\dim_K \operatorname{End}_A(X_i) - \dim_K \operatorname{Ext}_A^1(X_i, X_i)\right)$$

(see [21, (1.3), (3.4) and (5.1)], and so (v) implies (iv). Therefore, it remains to show that (iv) implies (iii). We divide the proof of this implication into several steps.

Assume that $\chi_A = q_A$ is weakly non-negative. We may assume that A is not tilted. Hence, by (2.3), every component of Γ_A is semiregular. From (2.2) we know also that Γ_A admits a preprojective component. Let $\mathscr{P}_0$ be a preprojective component of Γ_A and D the support algebra of $\mathscr{P}_0$. Since $\mathscr{P}_0$ is semiregular, hence without injective modules, and consists entirely of directing modules, we infer (see [31, dual of (2.6)]) that there exists a hereditary algebra H of infinite type and a tilting H-module T without preprojective direct summands such that $D = \operatorname{End}_H(T)$. Moreover, D is a convex subcategory of A, by a modified argument from [5, (3.2)]. Hence, q_D is weakly non-negative and D is of Euclidean type. Therefore, there exists a tame concealed convex subcategory C_0 of D such that D is a tubular coextension of C_0 [23]. Since A is triangular, it can be obtained from D by a sequence of one-point extensions and coextensions. We know that $\mathscr{P}_0$ contains all indecomposable projective D-modules, and hence for each $Y \in \operatorname{ind} D$ there exists $X \in \mathscr{P}_0$ such that $\operatorname{Hom}_D(X, Y) \ne 0$. Invoking now [23, p. 88] and the fact that $\mathscr{P}_0$ is a complete component in Γ_A, we deduce that A does not contain a full subcategory which is a one-point coextension $[N]D$ of D by a non-zero D-module N. We know also that the Γ_D consist of $\mathscr{P}_0$, a $\mathbb{P}_1(K)$-family Γ of coray tubes, obtained from the unique $\mathbb{P}_1(K)$-family of stable tubes in Γ_{C_0} by coray insertions, and a preinjective component Ω (consisting of C_0-modules). Denote by Γ' the family of all tubes in Γ containing injective modules, and by Γ'' the family of all remaining (stable) tubes in Γ. Observe that, if Γ' is not empty, then it consists of modules from $\mathscr{R}_D \setminus \mathscr{L}_D$. Assume now that there is a one-point extension $D[M]$ of D inside A by a non-zero D-module M. Since $\mathscr{P}_0$ is a complete component of Γ_A, M has no indecomposable direct summand from $\mathscr{P}_0$. Then applying (2.9), (2.11), and (3.3) we infer that either M belongs to $\operatorname{add}(\Omega)$ or M is an indecomposable simple regular module from Γ''. In particular, all components from Γ' are full components of Γ_A. Let B_0 be a maximal tubular extension of C_0 which is a convex subcategory of A. By the above remarks, the indecomposable projective B_0-modules given by the vertices of $Q_{B_0} \setminus Q_{C_0}$ lie in the ray tubes obtained from stable tubes of Γ'' by ray insertions. Moreover, by our assumption, χ_{B_0} is weakly non-negative, and then B_0 is either tubular or a tilted algebra of Euclidean type, by [22, (3.3)]. Let Λ_0 be the full subcategory of A given by the objects of B_0 and D. Clearly, Λ_0 is a tame semiregular branch extension of C_0 and a convex subcategory of A. Denote by $\mathscr{T}_0$ the $\mathbb{P}_1(K)$-family of semiregular tubes in Γ_Λ, obtained from the family of stable tubes in Γ_{C_0} by the corresponding coray and ray insertions, and by $\mathscr{Q}_0$ the family of all remaining components different from $\mathscr{P}$. From the maximality of B_0, the above remarks, (3.3), and (3.4), we conclude that if $\Lambda_0[R]$ is a one-point extension of Λ_0 inside A then R is a direct sum of modules from $\mathscr{Q}_0$. Moreover, there is no one-point coextension $[R']\Lambda_0$ of Λ_0, inside A, by a non-zero Λ_0-module R'. In particular, all components of Γ_{Λ_0} from $\mathscr{P}_0 \vee \mathscr{T}_0$ are full components of Γ_A. Assume now that $\mathscr{T}_0$ contains an injective module, that is, Γ' is not empty. Since B_0 is tubular or tilted of Euclidean type, we conclude that any module from $\mathscr{Q}_0$ is a successor of a module from Γ', and so belongs to $\mathscr{R}_{\Lambda_0} \setminus \mathscr{L}_{\Lambda_0}$. Hence, by (2.11), there is no one-point extension of Λ_0 inside A, and consequently $A = \Lambda_0$. Thus we may assume that $D = C_0$, and so $\Lambda_0 = B_0$.

Consider now the case when B_0 is tubular. Then, by [23, (5.2)], Γ_{B_0} is of the form

$$\Gamma_{B_0} = \mathscr{P}_0 \vee \mathscr{T}_0 \vee \left(\bigvee_{q \in \mathbb{Q}^+} \mathscr{T}_q \right) \vee \mathscr{T}_\infty \vee \mathscr{Q}_\infty,$$

where $\mathbb{Q}^+$ is the set of all positive rational numbers, $\mathscr{P}_0$ is the preprojective component of Γ_{C_0}, $\mathscr{T}_0$ is the considered above family of ray tubes, $\mathscr{T}_\infty$ is a $\mathbb{P}_1(K)$-family of coray tubes containing at least one injective module, $\mathscr{Q}_\infty$ is a preinjective component, and, for each $q \in \mathbb{Q}^+, \mathscr{T}_q$ is a $\mathbb{P}_1(K)$-family of stable tubes. Let $\mathscr{T}_\infty'$ be the family of all tubes in $\mathscr{T}_\infty$ containing injective modules, and $\mathscr{T}_\infty''$ the family of all remaining tubes of $\mathscr{T}_\infty$. It follows also from [23, (5.2)] that there is a tame concealed convex subcategory C_∞ of B_0 such that B_0 is a tubular coextension of C_∞, Q_∞ is the preinjective component of Γ_{C_∞}, and $\mathscr{T}_\infty$ is obtained from the unique $\mathbb{P}_1(K)$-family of stable tubes in Γ_{C_∞} by coray insertions. It is shown in [30, (2.5)] that if N is an indecomposable B_0-module and $\chi_{B_0[N]}$ is weakly non-negative, then N lies in $\mathscr{T}_\infty \vee \mathscr{Q}_\infty$. Furthermore, $\mathscr{T}_\infty'$ is not empty and consists of modules from $\mathscr{R}_B \setminus \mathscr{L}_B$. Therefore, for any one-point extension $B[X]$ of B inside A, X is a direct sum of modules from $\mathscr{T}_\infty''$. Let B_∞ be a maximal tubular extension of C_∞ which is a convex subcategory of A. Then the indecomposable projective B_∞-modules given by the vertices of $Q_{B_\infty} \setminus Q_{C_\infty}$ lie in ray tubes obtained from stable tubes of $\mathscr{T}_\infty''$ by ray insertions. Again, since B_∞

is a convex subcategory of A and χ_A is weakly non-negative, we conclude that B_∞ is either tubular or a tilted algebra of Euclidean type. Denote by B the full subcategory of A given by the objects of B_0 and B_∞. Clearly, B is a tame semiregular branch enlargement of the tame concealed algebra $C = C_\infty$ and $\Gamma_B = \mathscr{P} \vee \mathscr{T} \vee \mathscr{Q}$ where $\mathscr{P} = \mathscr{P}_0 \vee \mathscr{T}_0 \vee (\bigvee_{q \in \mathbb{Q}^+} \mathscr{T}_q)$, $\mathscr{T}$ is a $\mathbb{P}_1(K)$-family of semiregular tubes separating $\mathscr{P}$ from Q, and Q is either a preinjective component (if B_∞ is tilted) or of the form $(\bigvee_{q \in \mathbb{Q}^+} \mathscr{T}_q^*) \vee \mathscr{T}_\infty^* \vee \mathscr{Q}_\infty^*$ (if B_∞ is tubular), where Q_∞^* is a preinjective component, $\mathscr{T}_\infty^*$ is a $\mathbb{P}_1(K)$-family of coray tubes, and, for each $q \in \mathbb{Q}^+$, $\mathscr{T}_q^*$ is a $\mathbb{P}_1(K)$-family of stable tubes. In particular, the component quiver Σ_B of B is directed. Further, since the family $\mathscr{T}'$ of tubes in $\mathscr{T}$ containing injective modules (equal $\mathscr{T}_\infty'$) is not empty and any module in $\mathscr{Q}$ is a successor of a module from $\mathscr{T}'$, A does not contain a full subcategory which is a one-point extension of B by a non-zero B-module. Clearly, A does not contain a full subcategory which is a one-point coextension of B by a non-zero B-module, because $\mathscr{P} \vee \mathscr{T}$ is a full translation subquiver of Γ_B. Hence, $B = A$.

Let now $\mathscr{P}_1, \dots, \mathscr{P}_r$ be the family of all preprojective components of Γ_A. From the above discussion, we may assume that, for each $1 \le i \le r$, the following hold:

(1) The support algebra C_i of $\mathscr{P}_i$ is a tame concealed algebra.

(2) The maximal tubular extension B_i of C_i inside A is a tilted algebra of Euclidean type.

(3) The unique $\mathbb{P}_1(K)$-family $\mathscr{T}_i$ of ray tubes in Γ_{B_i} is a $\mathbb{P}_1(K)$-family of tubes in Γ_A.

(4) A does not contain a full subcategory which is a one-point coextension of B_i by a non-zero B_i-module.

(5) For any one-point extension $B_i[R_i]$ of B_i inside A, the B_i-module R_i is preinjective.

(6) The preinjective component $\mathscr{Q}_i$ of Γ_{B_i} contains a full translation subquiver $\mathscr{E}_i$ which is a full translation subquiver of Γ_A and is closed under predecessors.

(7) $\Gamma_i = \mathscr{P}_i \vee \mathscr{T}_i \vee \mathscr{E}_i$ is closed under predecessors in ind A.

Denote by B the direct product of $B_1, \dots, B_r$. If $A = B$ then $A = B_1$ and there is nothing to show. By symmetry we may assume that the dual statements (1′)–(7′) related with the family of all preinjective components of Γ_A also hold. Hence we may assume that $A \neq B$ and is a one-point extension $A = \Lambda[M]$ where Λ is a convex subcategory of A containing B, and M is a Λ-module. Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_t$ with $\Lambda_1, \dots, \Lambda_t$ connected, and $M = M_1 \oplus \cdots \oplus M_t$ with $M_j \in \operatorname{mod} \Lambda_j$ for any $1 \le j \le t$. Since A is connected, the modules $M_1, \ldots, M_t$ are non-zero. Clearly, B is a convex subcategory of Λ and the translation quivers Γ_i are full translation subquivers of Γ_A which are closed under predecessors in ind Λ. Each of Λ_j is a proper convex connected subcategory of A, and hence Λ_j is quasi-tilted with χ_{Λ_j} weakly non-negative. Therefore, we may assume that each Λ_j is either tame tilted or a tame semiregular branch enlargement of a tame concealed algebra. We shall first prove that, for each $1 \leq j \leq t$, M_j is a direct sum of indecomposable modules lying in one connected component of Γ_{Λ_j} consisting of directing modules. Fix $1 \leq j \leq t$. Observe that if Λ_j contains an algebra B_i then Λ_j is tilted, because Γ_i is a full translation subquiver of Γ_{Λ_j}. Conversely, each algebra B_i is contained in some Λ_j. Suppose now that Λ_j is not tilted. Then Λ_j is a tame semiregular branch enlargement of a tame concealed algebra C'_j, and the unique preprojective component, say $\mathscr{P}'_j$, of Γ_{Λ_j} is different from $\mathscr{P}_1, \ldots, \mathscr{P}_r$. Since the components $\mathscr{P}_1, \ldots, \mathscr{P}_r$ exhaust all preprojective components of Γ_A, invoking [23, p. 88], we conclude that M_j has at least one indecomposable direct summand lying in $\mathscr{P}'_j$. Applying (3.5) we then conclude that M_j is a direct sum of (preprojective) modules from $\mathscr{P}'_j$. Assume now that Λ_j is tilted, say of the form $\Lambda_j = \operatorname{End}_{H_j}(T_j)$ for a hereditary algebra H_j and a tilting H_j-module T_j. Then T_j induces a torsion theory $(\mathscr{Y}(T_j), \mathscr{X}(T_j))$ on mod Λ_j, where $\mathscr{Y}(T_j) = \{Z \in \operatorname{mod} \Lambda_j,\ \operatorname{Tor}_1^{\Lambda_j}(X, T_j) = 0\}$ and $\mathscr{X}(T_j) = \{Z \in \operatorname{mod} \Lambda_j,\ Z \otimes_{\Lambda_j} T_j = 0\}$. Denote by $\mathscr{C}_j$ the connecting component of Γ_{Λ_j} determined by T_j. It is well known (see [13]) that the support algebra of any preprojective component of Γ_{Λ_j} different from $\mathscr{C}_j$ is tame concealed, and so after the extension of Λ to $A = \Lambda[M]$ it remains (because q_A is weakly non-negative) a preprojective component of Γ_A. Therefore, the preprojective components of Γ_{Λ_j} different from $\mathscr{C}_j$ (if they exist) are given by the preprojective components $\mathscr{P}_i$ of all algebras B_i which are contained in Λ_j. Clearly, we have at least one of such components $\mathscr{P}_i$ if $\mathscr{C}_j$ is not preprojective. Assume $\mathscr{C}_j$ is not preprojective. We claim that then $\mathscr{C}_j$ contains at least one injective module. Suppose it is not the case. Then $\mathscr{C}_j$ is a semiregular component and hence T_j has no preprojective direct summands (see [24]). Moreover, the slice of $\mathscr{C}_j$ contains as a convex subquiver the Euclidean slice of the preinjective component of one of the algebras B_i. Consequently, H_j is wild and hence χ_{Λ_j} is not weakly non-negative (see [13, (6.2)]), a contradiction. Assume $\mathscr{C}_j$ contains an injective module. Observe that if $\mathscr{D}$ is a component of Γ_{Λ_j} which is contained entirely in $\mathscr{X}(T_j)$ then any module of $\mathscr{D}$ is a successor of a module Z in $\mathscr{C}_j$ with $\operatorname{Hom}_{\Lambda_j}(D(\Lambda_j), \tau_{\Lambda_j} Z) \neq 0$, and so a successor of a module Z of projective dimension 2. Hence $\mathscr{D}$ is contained in $\mathscr{R}_{\Lambda_j} \setminus \mathscr{L}_{\Lambda_j}$.

Applying now (2.11) and the property (7) of the algebras B_i, $1 \leq i \leq r$, we conclude that M_j is a direct sum of modules from $\mathscr{C}_j$. Finally, consider the case when $\mathscr{C}_j$ is a preprojective component without injective modules. Then Λ_j is a tilted algebra of Euclidean type having a complete slice in $\mathscr{C}_j$. If $\mathscr{C}_j$ is different from any of the components $\mathscr{P}_1, \dots, \mathscr{P}_r$ then M_j has at least one indecomposable direct summand from $\mathscr{C}_j$, and invoking again (3.5), we conclude that M_j is a direct sum of modules from $\mathscr{C}_j$. If $\mathscr{C}_j$ is one of the components $\mathscr{P}_i$, $1 \leq i \leq r$, then it follows from our assumption (7) that M_j is a direct sum of modules from the preinjective component of Γ_{B_i}. Therefore, we have proved that, for each $1 \leq j \leq t$, M_j is a direct sum of indecomposable modules lying in one component of Γ_{Λ_j} formed by directing modules. Applying arguments as in the final part of the proof of Theorem 2.3 in [7] we infer that in fact M is a directing Λ-module. Let P be the indecomposable projective A-module whose radical is M, and $\mathscr{C}$ be the connected component of Γ_A containing P. We know from [12, 32] that P is an indecomposable directing A-module. Moreover, it follows from the above discussion and [23, p. 88] that the full translation subquiver of $\mathscr{C}$ formed by all proper predecessors of P in Γ_A consists of all predecessors of indecomposable direct summands of M in Γ_Λ, and all these modules are directing. Since $M_1, \dots, M_t$ are non-zero, we also deduce that $\mathscr{E}_1, \dots, \mathscr{E}_t$ are full translation subquivers of $\mathscr{C}$ which are closed under predecessors. Invoking now (2.3) and (2.4) we infer that $\mathscr{C}$ is a component without injective modules and oriented cycles. Finally, observe that we may repeat the above considerations for any indecomposable projective A-module P' corresponding to a source of Q_A which is not in Q_B. Consequently, the component $\mathscr{C}$ contains the indecomposable projective A-modules corresponding to all sources of Q_A which do not belong to Q_B, and the predecessors of these modules form a translation subquiver of $\mathscr{C}$ consisting of directing modules. Hence we conclude that $\mathscr{C}$ is directing with finitely many τ_A-orbits and for any left stable module U in $\mathscr{C}$ there are $1 \leq i \leq r$ and a positive integer m such that $\tau_A^m U$ belongs to the translation quiver $\mathscr{E}_i$. Since we assumed that A is not tilted, applying (2.3), we obtain that $\mathscr{C}$ is a generalized standard [28] semiregular component without oriented cycles and injective modules. Moreover, since $\mathscr{E}_1, \dots, \mathscr{E}_r$ are full translation subquivers of $\mathscr{C}$ which are closed under predecessors, we deduce as above that $\mathscr{C}$ is the connecting component of a tilted algebra $F = \operatorname{End}_H(T)$ given by a wild hereditary algebra and a tilting H-module without preprojective direct summands. But F is a convex subcategory of A with χ_F not weakly non-negative, a contradiction with our assumption. Therefore, A is either tame tilted or a tame semiregular branch enlargement of a tame concealed algebra. This finishes the proof.

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