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Some notes on fixed points of quasi-contraction maps

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ABSTRACT

In this paper, we shall give some results about fixed points of quasi-contraction maps on cone metric spaces. These results generalize some recent results.

1. Introduction

Let (E, τ) be a topological vector space and P a subset of E. Then, P is called a cone whenever

(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int\, P$, where $int\, P$ denotes the interior of P. If E is a normed space, then the cone P is called normal (with respect to this norm) whenever there is a number $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq M\|y\|$. The least positive number satisfying this norm inequality is called the normal constant of P [1]. Of course, there are non-normal cones [2]. Some authors use the notion of normality in their works, but most of the fundamental results in normal cone metric spaces hold in non-normal case.

Definition 1.1 (*[1]*). Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Let (X, d) be a metric space. A self-map $T : X \to X$ is called a quasi-contraction whenever there exists $\lambda \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

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for all $x, y \in X$ [3]. As we know, there are some works about fixed points of quasi-contractions. Recently, Ilić and Rakočević [1] generalized this notion to cone metric spaces. Suppose that (X, d) is a cone metric space. A self-map $T : X \to X$ such that for some $\lambda \in (0, 1)$ and for every $x, y \in X$ there exists

$$u \in C(T, x, y) = \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

such that

$$d(Tx, Ty) \leq \lambda u,$$

is called a quasi-contraction [4]. They proved the following result [4, Theorem2.1].

Theorem 1.1. *Let (X, d) be a complete cone metric space and P a normal cone. Suppose that $T : X \to X$ is a quasi-contraction. Then, T has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

Then, Kadelburg, Radenović and Rakočević generalized this result by omitting the assumption of normality and for $\lambda \in (0, \frac{1}{2})$ [5, Theorem 2.2]. We shall prove this recent result for any $\lambda \in (0, 1)$.

As we know, Rhoades has defined the property (P) on metric spaces in his works [6–8]. Recently, Kadelburg, Radenović and Rakočević [5] generalized notion of the property (P) to cone metric spaces. Denote as usual, by $F(T)$ the set of fixed points of the mapping $T : X \to X$. We say that the map T has the property (P) if $F(T) = F(T^n)$ for all $n \geq 1$, that is it has no periodic points. They proved the following results (respectively, [5, Theorems 2.3 and 3.2]).

Theorem 1.2. *Let (X, d) be a complete cone metric space and $T : X \to X$ a quasi-contraction such that there exists a point $x \in X$ having a bounded orbit with respect to T. Then, T has a unique fixed point.*

We shall show that by using the assumptions, usually there exists $x \in X$ having a bounded orbit.

Theorem 1.3. *Let (X, d) be a cone metric space and $T : X \to X$ a quasi-contraction with $\lambda \in (0, \frac{1}{2})$. Then, T has the property* (P).

We shall prove this result for all $\lambda \in (0, 1)$.

Let (X, d) be a cone metric space (with values of the metric in a Banach space E), $T : X \to X$ a mapping, $x \in X$ and

$$O(x; \infty) = \{x, Tx, T^2x, \ldots, T^n x, \ldots\}.$$

In fact, $O(x; \infty)$ is called the orbit of x with respect to T. A vector u is a bound for a set $A \subseteq X$ if $d(x, y) \leq u$ for all $x, y \in A$. A space (X, d) is called T-orbitally complete whenever every sequence $\{T^{n_i}x\}_{i\geq 1}$, $x \in X$, which is a Cauchy sequence, has a limit point in X [9]. Also, a map $f : X \to \mathbf{R}$ is called lower semicontinuous at $x \in X$ if for any sequence $\{x_n\}$ in X such that $x_n \to x$, we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$ [9]. It is obvious that each complete cone metric space is a T-orbitally complete space, but Pathak and Shahzad showed that the converse does not hold [9, Example 3.1]. Also, they proved the following result [9, Theorem 3.5].

Theorem 1.4. *Let (X, d) be a cone metric space, P a regular cone with normal constant K and $T : X \to X$ a self-map. Suppose that (X, d) is T-orbitally complete and there exists $x \in X$ and $\lambda \in [0, 1)$ such that*

$$d(Ty, T^2y) \leq \varphi(y) d(y, Ty)$$

for all $y \in O(x; \infty)$, where $\varphi : O(x; \infty) \to [0, 1)$ is a function such that

$$\sup_{y\in O(x;\infty)} \varphi(y) \leq \lambda < 1.$$

Then, the following statements hold.

(i) *$\lim_{n\to\infty} T^n x = \bar{x}$ exists,*
(ii) *$T(\bar{x}) = \bar{x}$ if the mapping on $X \ni z \mapsto \|d(z, Tz)\|$ is lower semicontinuous at $\bar{x}$,*
(iii) *$\bar{x}$ is a unique fixed point of T if $\bigcap_{n=0}^{\infty} T^n(X)$ is a singleton set, where $T^n(X) = T(T^{n-1}(X))$ for all $n \geq 1$ and $T^0(X) := X$.*

We shall generalize this result.

2. Main results

Now, we are ready to state and prove our main results. First, we give the following result which improves Theorems 1.1 and 1.2.

Theorem 2.1. *Let (X, d) be a complete cone metric space and $T : X \to X$ a quasi-contraction and $x \in X$. Then, the following statements hold.*

(i) *The orbit $O(x; \infty)$ is bounded,*
(ii) *$\lim_{n\to\infty} T^n x = \bar{x}$ exists and $T(\bar{x}) = \bar{x}$,*
(iii) *$\bar{x}$ is a unique fixed point of T.*

Proof. (i) Put $\delta_1(x) = d(x, Tx)$. Since $d(x, T^2x) \leq d(x, Tx) + d(Tx, T^2x)$ and T is quasi-contraction, we have two cases:

$$d(Tx, T^2x) \leq \lambda d(x, Tx) \quad \text{or} \quad d(Tx, T^2x) \leq \lambda d(x, T^2x).$$

If $d(Tx, T^2x) \leq \lambda d(x, Tx)$, then $d(x, T^2x) \leq (1+\lambda)d(x, Tx)$. In this case, put $\delta_2(x) = (1+\lambda)d(x, Tx)$. If $d(Tx, T^2x) \leq \lambda d(x, T^2x)$, then $d(x, T^2x) \leq \frac{1}{1-\lambda} d(x, Tx)$. In this case, put $\delta_2(x) = \frac{1}{1-\lambda} d(x, Tx)$. Note that, in both cases we have $\delta_1(x) \leq \delta_2(x)$ and $d(x, T^2x) \leq \delta_2(x)$. Suppose that $\delta_{n-1}(x)$ has been chosen. Now by induction, we show that for each $n \geq 2$ there exists $1 \leq m \leq n$ such that

$$d(T^n x, T^{n-1} x) \leq \lambda^{n-1} d(x, T^m x). \tag{$*$}$$

If $n = 2$, then $d(Tx, T^2x) \leq \lambda u$, where $u \in \{d(x, Tx), d(Tx, T^2x), d(x, T^2x)\}$. Thus, $(*)$ holds for $n = 2$. Suppose that $(*)$ holds for each $k < n$. We show that $(*)$ holds for $k = n$. In this case we have $d(T^n x, T^{n-1} x) \leq \lambda u$, where

$$u \in \{d(T^{n-1}x, T^{n-2}x), d(T^n x, T^{n-1}x), d(T^n x, T^{n-2}x)\}.$$

It is trivial that $(*)$ holds if $u = d(T^{n-1}x, T^{n-2}x)$ or $u = d(T^n x, T^{n-1}x)$. Now suppose that $u = d(T^n x, T^{n-2}x)$. In this case we have $d(T^n x, T^{n-2}x) \leq \lambda u_1$, where

$$u_1 \in \{d(T^{n-1}x, T^{n-3}x), d(T^{n-1}x, T^{n-2}x), d(T^n x, T^{n-3}x), d(T^{n-2}x, T^{n-3}x), d(T^n x, T^{n-1}x)\}.$$

Again, it is trivial that $(*)$ holds if $u_1 = d(T^n x, T^{n-1}x)$ or $u_1 = d(T^{n-2}x, T^{n-3}x)$. If $u_1 = d(T^{n-1}x, T^{n-2}x)$, then $d(T^n x, T^{n-1}x) \leq \lambda^2 d(T^{n-1}x, T^{n-2}x)$. By assumption of the induction, there exists $1 \leq m \leq n-1$ such that $d(T^{n-1}x, T^{n-2}x) \leq \lambda^{n-2} d(x, T^m x)$. Hence, $d(T^n x, T^{n-1}x) \leq \lambda^n d(x, T^m x) \leq \lambda^{n-1} d(x, T^m x)$. If $u_1 = d(T^{n-1}x, T^{n-3}x)$, then $d(T^n x, T^{n-1}x) \leq \lambda^2 d(T^{n-3}x, T^{n-1}x)$. If $u_1 = d(T^n x, T^{n-3}x)$, then $d(T^n x, T^{n-1}x) \leq \lambda^2 d(T^n x, T^{n-3}x)$. Therefore, by continuing this process, we see that $(*)$ holds for each $n \geq 2$. Now, we have

$$d(x, T^n x) \leq d(x, T^{n-1}x) + d(T^{n-1}x, T^n x) \leq d(x, T^{n-1}x) + \lambda^{n-1} d(x, T^m x)$$

for some $1 \leq m \leq n$. If $n > m$, then

$$d(x, T^n x) \leq d(x, T^{n-1}x) + \lambda^{n-1} d(x, T^m x) \leq (1 + \lambda^{n-1})\delta_{n-1}(x).$$

In this case, put $\delta_n(x) = (1 + \lambda^{n-1})\delta_{n-1}(x)$. If $n = m$, then

$$d(x, T^n x) \leq \frac{1}{1 - \lambda^{n-1}} d(x, T^{n-1}x) \leq \left(\frac{1}{1 - \lambda^{n-1}}\right) \delta_{n-1}(x).$$

In this case, put $\delta_n(x) = \frac{1}{1-\lambda^{n-1}} \delta_{n-1}(x)$. Note that, in both cases we have

$$\delta_{n-1}(x) \leq \delta_n(x) \quad \text{and} \quad d(x, T^n x) \leq \delta_n(x).$$

Thus, $\{\delta_n(x)\}_{n\geq 1}$ is an increasing sequence and $\delta_n(x)$ is a bound for the set

$$\{x, Tx, T^2x, \ldots, T^n x\}$$

for all $n \geq 1$. Note that, for each $m \geq 2$ there exists a subset I of $\{1, 2, \ldots, m-1\}$ such that

$$\delta_m(x) = \frac{\prod_{k\in I}(1 + \lambda^k)}{\prod_{k\in\{1,2,\ldots,m-1\}\setminus I}(1 - \lambda^k)} \delta_1(x).$$

Put $u_m = \prod_{i=1}^m (1 + \lambda^i)$ for all $m \geq 1$. Since $\ln u_m = \sum_{i=1}^m \ln(1 + \lambda^i)$ and the series $\sum_{i=1}^\infty \ln(1 + \lambda^i)$ converges, the sequence $\{u_m\}_{m\geq 1}$ is bounded and increasing. Thus, $\prod_{i=1}^\infty (1 + \lambda^i)$ converges. Similarly, one can prove that $\prod_{i=1}^\infty \frac{1}{1-\lambda^i}$ converges. Now, note that

$$\delta_m(x) \leq \frac{\prod_{i=1}^{\infty}(1 + \lambda^i)}{\prod_{i=1}^{\infty}(1 - \lambda^i)} \delta_1(x)$$

for all $m \geq 1$. If we put

$$\delta(x) = \frac{\prod_{i=1}^{\infty}(1 + \lambda^i)}{\prod_{i=1}^{\infty}(1 - \lambda^i)} \delta_1(x),$$

then $\delta(x)$ is a bound for the orbit $O(x; \infty)$.

(ii) For each $m > n \geq 1$ we have

$$d(T^n x, T^m x) \leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^m x) \leq \lambda^n d(x, T^s x) + \lambda^{n+1} d(x, T^p x)$$

for some $s, p \leq m$. Thus, $d(T^n x, T^m x) \leq (\lambda^n + \lambda^{n+1})\delta(x)$ for all $m > n \geq 1$. Hence, $\{T^n x\}_{n\geq 1}$ is a Cauchy sequence. Since (X, d) is a complete cone metric space, there exists $\bar{x} \in X$ such that $\lim_{n\to\infty} T^n x = \bar{x}$. For each $n \geq 1$ there exists

$$u_n \in \{d(\bar{x}, T^n x), d(\bar{x}, T\bar{x}), d(T^n x, T^{n+1} x), d(\bar{x}, T^{n+1} x), d(T^n x, T\bar{x})\}$$

such that

$$d(T\bar{x}, T^{n+1} x) \leq \lambda u_n.$$

Let $0 \ll c$ be given. Choose a natural number N such that $d(T^n x, \bar{x}) \ll c$ for all $n \geq N$. Then, we have $d(\bar{x}, T\bar{x}) \ll \frac{2\lambda+1}{(1-\lambda)} c$ for all $n \geq N$. Since c was arbitrary, we obtain $T\bar{x} = \bar{x}$.

(iii) It is obvious. □

Now, we give the next result which improves Theorem 1.3.

Theorem 2.2. *Let (X, d) be a cone metric space and $T : X \to X$ a quasi-contraction. Then, T has the property* (P).

Proof. It is clear that $F(T) \subseteq F(T^n)$ for all $n \geq 1$. Now, let $n \geq 1$ be given and $u \in F(T^n)$. Then, $T^n u = u$ and so $T^{kn} u = u$ for all $k \geq 1$. Hence, for every $k \geq 1$ we have

$$d(u, Tu) = d(T^{kn} u, T^{kn+1} u) \leq \lambda^{kn} d(u, T^s u) \leq \lambda^{kn} \delta(u),$$

where $s \leq kn + 1$ and $\delta(u)$ is defined as in Theorem 2.1. If $k \to \infty$, then we obtain $d(u, Tu) = 0$. Therefore, $Tu = u$ and so T has the property (P). □

Finally, we give the following result which generalizes Theorem 1.4.

Theorem 2.3. *Let (X, d) be a cone metric space and $T : X \to X$ a self-map. Suppose that (X, d) is T-orbitally complete and there exists $x \in X$ and $\lambda_1, \lambda_2 \in [0, 1)$ such that $\lambda_1 + 2\lambda_2 < 1$ and*

$$d(Ty, T^2 y) \leq \lambda_1 d(y, Ty) + \lambda_2 d(y, T^2 y)$$

for all $y \in O(x; \infty)$. Then, the following statements hold.

(i) *$\lim_{n\to\infty} T^n x = \bar{x}$ exists,*
(ii) *$T(\bar{x}) = \bar{x}$ if the mapping on $X \ni z \mapsto \|d(z, Tz)\|$ is lower semicontinuous at $\bar{x}$ and P is a normal cone,*
(iii) *$\bar{x}$ is a unique fixed point of T if $\bigcap_{n=0}^{\infty} T^n(X)$ is a singleton set.*

Proof. (i) First, note that

$$d(Ty, T^2 y) \leq \lambda_1 d(y, Ty) + \lambda_2 (d(y, Ty) + d(Ty, T^2 y)).$$

Hence, $d(Ty, T^2 y) \leq \frac{\lambda_1+\lambda_2}{1-\lambda_2} d(y, Ty)$. Since $\lambda_1 + 2\lambda_2 < 1$, $\mu = \frac{\lambda_1+\lambda_2}{1-\lambda_2} < 1$. Thus,

$$d(T^n x, T^{n+1} x) \leq \mu^n d(x, Tx)$$

for all $n \geq 1$. Hence, for each $m > n \geq 1$ we have

$$d(T^n x, T^m x) \leq (\mu^n + \mu^{n+1} + \cdots + \mu^m) d(x, Tx).$$

Therefore, $\{T^n x\}_{n\geq 1}$ is a Cauchy sequence. Since (X, d) is T-orbitally complete, there exists $\bar{x} \in X$ such that $\lim_{n\to\infty} T^n x = \bar{x}$.

(ii) Since the mapping $z \mapsto \|d(z, Tz)\|$ is lower semicontinuous at $\bar{x}$, we have

$$\|d(\bar{x}, T\bar{x})\| \leq \liminf_{n\to\infty} \|d(T^n x, T^{n+1} x)\| = 0.$$

Thus, $T(\bar{x}) = \bar{x}$.

(iii) It is obvious. □

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