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On q-orthogonal polynomials, dual to little and big q-Jacobi polynomials

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Abstract

We derive discrete orthogonality relations for polynomials, dual to little and big q-Jacobi polynomials. This derivation essentially requires use of bases, consisting of eigenvectors of certain self-adjoint operators, which are representable by a Jacobi matrix. Recurrence relations for these polynomials are also given.

Keywords: Duality; Little q-Jacobi polynomials; Big q-Jacobi polynomials; Discrete orthogonality relations; Jacobi matrix

1. Introduction

Properties of q-orthogonal polynomials are closely related to operators, which can be represented by a Jacobi matrix [1–3]. In the case under consideration we diagonalize certain self-adjoint bounded operators with the aid of big or little q-Jacobi polynomials. An explicit form of all eigenvectors of these operators is found. Since their spectra are simple, eigenvectors of each such operator form an orthogonal basis in a Hilbert space. One can normalize this basis. As a result, for each operator (one of them is related to little q-Jacobi polynomials and another to big q-Jacobi polynomials) two orthonormal bases in the Hilbert space emerge: the initial basis and the basis of eigenvectors of this operator. They

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are interrelated by a unitary matrix U, whose entries u_{mn} are explicitly expressed in terms of little or big q-Jacobi polynomials. Since the matrix U is unitary (and in fact real in our case), there are two orthogonality relations for its elements:

$$\sum_n u_{mn}u_{m'n} = \delta_{mm'}, \qquad \sum_m u_{mn}u_{mn'} = \delta_{nn'}. \tag{1}$$

The first relation expresses the orthogonality relation for little or big q-Jacobi polynomials. In order to interpret the second relation, we consider little and big q-Jacobi polynomials $P_n(q^{-m})$ as functions of n. In this way one obtains two sets of orthogonal functions (one for little and another for big q-Jacobi polynomials), which can be expressed in terms of q-orthogonal polynomials. These two sets of q-orthogonal polynomials differ from the initial ones and they can be considered as dual sets of polynomials with respect to little and big q-Jacobi polynomials. The second formula in (1) thus naturally leads to the orthogonality relations for these q-orthogonal polynomials on non-uniform lattices.

In fact, this idea extends the notion of the duality of polynomials, orthogonal on a finite set, to the case of polynomials, orthogonal on an infinite set of points. We have already employed this idea in [4] to show that q-Meixner polynomials are dual to big q-Laguerre polynomials. It is worth noting at this point that there are known theorems on dual orthogonality properties of q-polynomials, whose weight functions are supported on a discrete set of points (see, for example, [5] and [6]). However, they are formulated in terms of orthogonal functions (see (9) and (23)–(24) below for their explicit forms in the case of little and big q-Jacobi polynomials, respectively) as dual objects with respect to given orthogonal polynomials. Therefore, one still needs to make one step further in order to single out an appropriate family of dual polynomials from these functions. So, our main motif in this paper is to show explicitly how to accomplish that for little and big q-Jacobi polynomials.

The orthogonality measure for polynomials, dual to little q-Jacobi polynomials, is extremal (for the values of parameters, for which the corresponding moment problem is indeterminate), that is, these polynomials form a complete set in the space L^2 with respect to their orthogonality measure. The orthogonality measure for polynomials, dual to big q-Jacobi polynomials, is not extremal: these polynomials do not form a complete set in the corresponding space L^2. We have found the complementary set of orthogonal functions in this space L^2. These functions are expressed in terms of the same polynomials but with different values of parameters.

Throughout the sequel we always assume that q is a fixed positive number such that $q < 1$. We use (without additional explanation) notations of the theory of special functions (see, for example, [7] and [8]).

2. The operator I_1

Let $\ell^2(\mathbb{N})$ be the Hilbert space with the orthonormal basis $|n\rangle$, $n = 0, 1, 2, \ldots$. We define on $\ell^2(\mathbb{N})$ the symmetric operator I_1, acting on the basis $|n\rangle$, $n = 0, 1, 2, \ldots,$ by the formula

$$I_1|n\rangle = -a_n|n+1\rangle - a_{n-1}|n-1\rangle + b_n|n\rangle, \tag{2}$$

where

$$a_n = a^{1/2} q^{n+1/2} \frac{\sqrt{(1-q^{n+1})(1-aq^{n+1})(1-bq^{n+1})(1-abq^{n+1})}}{(1-abq^{2n+2})\sqrt{(1-abq^{2n+1})(1-abq^{2n+3})}},$$
$$b_n = \frac{q^n}{1-abq^{2n+1}} \left(\frac{(1-aq^{n+1})(1-abq^{n+1})}{1-abq^{2n+2}} + a \frac{(1-q^n)(1-bq^n)}{1-abq^{2n}} \right).$$

In order to assure that expressions for a_n and b_n are well defined, we suppose that $0 < a < q^{-1}$ and $b < q^{-1}$.

Since $a_n \to 0$ and $b_n \to 0$ when $n \to \infty$, the operator I_1 is bounded. Therefore, we assume that it is defined on the whole space $\ell^2(\mathbb{N})$. For this reason, I_1 is a self-adjoint operator. Let us show that I_1 is a trace class operator (we remind that a bounded self-adjoint operator is a trace class operator if a sum of its matrix elements in an orthonormal basis is finite; a spectrum of such an operator is discrete, with a single accumulation point at 0). For the coefficients a_n and b_n from (2), we have $a_{n+1}/a_n \to q$ and $b_{n+1}/b_n \to q$ when $n \to \infty$. Therefore, for the sum of all matrix elements of the operator I_1 in our basis we have $\sum_n (2a_n + b_n) < \infty$. This means that I_1 is a trace class operator. Thus, a spectrum of I_1 is discrete. Moreover, a spectrum of I_1 is simple, since I_1 is representable by a Jacobi matrix with $a_n \neq 0$ (see [2, Chapter VII]).

Let us find eigenvectors ξ_λ of the operator I_1, $I_1 \xi_\lambda = \lambda \xi_\lambda$. We set $\xi_\lambda = \sum_{n=0}^{\infty} \beta_n(\lambda)|n\rangle$. Acting by the operator I_1 upon both sides of this relation and then collecting all factors, which multiply $|n\rangle$ with fixed n, one derives that $\beta_{n+1}(\lambda)a_n + \beta_{n-1}(\lambda)a_{n-1} - \beta_n(\lambda)b_n = -\lambda\beta_n(\lambda)$. Making the substitution

$$\beta_n(\lambda) = \left(\frac{(abq, aq; q)_n (1-abq^{2n+1})}{(bq, q; q)_n (1-abq)(aq)^n} \right)^{1/2} \beta_n'(\lambda),$$

one reduces this relation to the following one:

$$A_n \beta_{n+1}'(\lambda) + C_n \beta_{n-1}'(\lambda) - (A_n + C_n)\beta_n'(\lambda) = -\lambda \beta_n'(\lambda)$$

with

$$A_n = \frac{q^n(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \qquad C_n = \frac{aq^n(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

This is the recurrence relation for the little q-Jacobi polynomials

$$p_n(\lambda; a, b|q) := {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, q\lambda) \tag{3}$$

(see, for example, formula (7.3.1) in [7]). Therefore, $\beta_n'(\lambda) = p_n(\lambda; a, b|q)$ and

$$\beta_n(\lambda) = \left(\frac{(abq, aq; q)_n (1-abq^{2n+1})}{(bq, q; q)_n (1-abq)(aq)^n} \right)^{1/2} p_n(\lambda; a, b|q). \tag{4}$$

For the eigenvectors ξ_λ of the operator I_1 we thus have the expression

$$\xi_\lambda = \sum_{n=0}^{\infty} \left(\frac{(abq, aq; q)_n (1-abq^{2n+1})}{(bq, q; q)_n (1-abq)(aq)^n} \right)^{1/2} p_n(\lambda; a, b|q)|n\rangle. \tag{5}$$

Since the spectrum of the operator I_1 is discrete, only a discrete set of these vectors belongs to the Hilbert space $\ell^2(\mathbb{N})$. This discrete set of vectors determines a spectrum of I_1.

Recall that the self-adjoint operator I_1 is represented by a Jacobi matrix in the basis $|n\rangle$, $n = 0, 1, 2, \ldots$. According to the theory of operators of such type (see [2, Chapter VII]), eigenvectors ξ_λ of I_1 are expanded into series in the basis vectors $|n\rangle$, $n = 0, 1, 2, \ldots$, with coefficients, which are polynomials in λ. These polynomials are orthogonal with respect to some positive measure $d\mu(\lambda)$ (moreover, for self-adjoint operators this measure is unique). The set (a subset of $\mathbb{R}$), on which the polynomials are orthogonal, coincides with the spectrum of the operator under consideration and the spectrum is simple.

The orthogonality relation for the little q-Jacobi polynomials $p_m(q^n) \equiv p_m(q^n; a, b|q)$ is of the form

$$\sum_{n=0}^{\infty} \frac{(bq;q)_n\,(aq)^n}{(q;q)_n} p_m(q^n)p_{m'}(q^n) = \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(1-abq)(aq)^m(bq,q;q)_m}{(1-abq^{2m+1})(abq,aq;q)_m} \delta_{mm'}. \tag{6}$$

Taking into account this orthogonality relation, we arrive at the following statement: *The spectrum of the operator I_1 coincides with the set of points q^n, $n = 0, 1, 2, \ldots,$ and this spectrum is simple.*

According to this statement the eigenvectors $\xi_{q^n}(x)$, $n = 0, 1, 2, \ldots,$ form a basis in the Hilbert space $\ell^2(\mathbb{N})$. Since these vectors belong to pairwise different eigenvalues, they are orthogonal to each other. But they are not normalized. They can be normalized by multiplying each of these vectors by corresponding constants. Let $\hat{\xi}_{q^n}(x) = c_n\xi_{q^n}(x)$ be normalized basis vectors. In order to find constants c_n, note that according to (5) one has

$$\hat{\xi}_{q^n}(x) = \sum_m c_n\beta_m(q^n)|m\rangle, \tag{7}$$

where $\beta_m(q^n)$ are given by (4). It follows from (7) that $\langle\hat{\xi}_{q^n}, \hat{\xi}_{q^n}\rangle = \sum_m c_n^2\beta_m(q^n)^2$. Taking into account the expression (4) for $\beta_m(q^n)$ and orthogonality relation (6), we find that

$$c_n = \left(\frac{(aq;q)_\infty}{(abq;q)_\infty}\frac{(bq;q)_n\,(aq)^n}{(q;q)_n}\right)^{1/2}.$$

The equality (7) connects two orthonormal bases in the space $\ell^2(\mathbb{N})$. This means that the matrix (a_{mn}), $m, n = 0, 1, 2, \ldots,$ with entries

$$a_{mn} = c_n\beta_m(q^n) = \left(\frac{(aq;q)_\infty}{(abq;q)_\infty}\frac{(bq;q)_n}{(q;q)_n}\frac{(abq,aq;q)_m\,(1-abq^{2m+1})}{(aq)^{m-n}(bq,q;q)_m\,(1-abq)}\right)^{1/2} p_m(q^n; a, b|q)$$

is unitary and real, that is,

$$\sum_n a_{mn}a_{m'n} = \delta_{mm'}, \qquad \sum_m a_{mn}a_{mn'} = \delta_{nn'}. \tag{8}$$

The first relation in (8) is equivalent to the orthogonality relation (6).

3. Dual little q-Jacobi polynomials

Now we consider the second identity in (8), which gives the orthogonality relation for the matrix elements a_{mn}, considered as functions of m. Up to multiplicative factors these functions coincide with the functions

$$F_n(x;a,b|q) = {}_2\phi_1(x, abq/x; aq; q, q^{n+1}), \tag{9}$$

considered on the set $x \in \{q^{-m} \mid m = 0,1,2,\dots\}$. Consequently,

$$a_{mn} = \left(\frac{(aq;q)_\infty}{(abq;q)_\infty} \frac{(bq;q)_n}{(q;q)_n} \frac{(abq,aq;q)_m(1-abq^{2m+1})}{(aq)^{m-n}(bq,q;q)_m} \right)^{1/2} F_n(q^{-m};a,b|q)$$

and the second identity in (8) gives the orthogonality relation for the functions (9),

$$\sum_{m=0}^{\infty} \frac{(1-abq^{2m+1})(abq,aq;q)_m}{(1-abq)(aq)^m(bq,q;q)_m} F_n(q^{-m})F_{n'}(q^{-m}) = \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(q;q)_n(aq)^{-n}}{(bq;q)_n} \delta_{nn'}. \tag{10}$$

The functions $F_n(x;a,b|q)$ can be represented in another form. Indeed, one can use the relation (III.8) of Appendix III in [7] in order to obtain that

$$F_n(q^{-m};a,b|q) = \frac{(-a)^m(bq;q)_m}{(aq;q)_m} q^{m(m+1)/2} \times {}_3\phi_1(q^{-m}, abq^{m+1}, q^{-n}; bq; q, q^n/a). \tag{11}$$

The basic hypergeometric function ${}_3\phi_1$ in (11) is a polynomial of degree n in the variable $\mu(m) := q^{-m} + abq^{m+1}$, which represents a q-quadratic lattice; we denote it as

$$d_n\big(\mu(m);a,b|q\big) := {}_3\phi_1(q^{-m}, abq^{m+1}, q^{-n}; bq; q, q^n/a). \tag{12}$$

Then formula (10) yields the orthogonality relation

$$\sum_{m=0}^{\infty} \frac{(1-abq^{2m+1})(abq,bq;q)_m}{(1-abq)(aq,q;q)_m a^{-m} q^{-m^2}} d_n\big(\mu(m)\big) d_{n'}\big(\mu(m)\big) = \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(q;q)_n(aq)^{-n}}{(bq;q)_n} \delta_{nn'} \tag{13}$$

for the polynomials (12). We call the polynomials $d_n(\mu(m);a,b|q)$ *dual little q-Jacobi polynomials*. Note that these polynomials can be expressed in terms of the Al-Salam–Chihara polynomials $Q_n(x;a,b|q)$ (see formula (3.8.1) in [9]) with the parameter $q > 1$. An explicit relation between them is

$$d_n\big(\mu(x);\beta/\alpha, 1/\alpha\beta q|q\big) = q^{n(n-1)/2}(-\beta)^{-n}(1/\alpha\beta;q)_n^{-1} Q_n\big(\alpha\mu(x)/2;\alpha,\beta|q^{-1}\big).$$

Ch. Berg and M.E.H. Ismail studied this type of Al-Salam–Chihara polynomials in [10] and derived continuous complex orthogonality measures for them. But [10] does not contain any discussion of the duality of this family of polynomials with respect to little q-Jacobi polynomials.

A recurrence relation for the polynomials $d_n(\mu(m); a, b|q)$ is derived from q-difference formula for little q-Jacobi polynomials, given by formula (3.12.5) in [9]. It has the form

$$(q^{-m} + abq^{m+1})d_n(\mu(m)) = -A_n d_{n+1}(\mu(m)) + q^{-n}(1+a)d_n(\mu(m)) \\ - C_n d_{n-1}(\mu(m)),$$

where $A_n = aq^{-n}(1 - bq^{n+1})$ and $C_n = q^{-n}(1 - q^n)$. Comparing this relation with the recurrence relation (3.69) in [11], we see that the polynomials (12) are multiple to the polynomials (3.67) in [11]. Moreover, if one takes into account this multiplicative factor, the orthogonality relation (13) for polynomials (12) turns into relation (3.82) for the polynomials (3.67) in [11], although the derivation of the orthogonality relation in [11] is more complicated than our derivation of (13). The authors of [11] do not give an explicit form of their polynomials in the form similar to (12). Note that connection of Al-Salam–Chihara polynomials in base q^{-1} with polynomials (3.67) in [11] and their connection with little q-Jacobi polynomials is considered in [12, Section 3.1]. The author of [12] states that the dual orthogonality relation for the Al-Salam–Chihara polynomials in base q^{-1} is the orthogonality relation for the little q-Jacobi polynomials. However, this assertion can be accepted only for those values of the parameters a and b for the Al-Salam–Chihara polynomials in base q^{-1}, for which the corresponding moment problem is determined. We move from the little q-Jacobi polynomials to their duals. For this reason, our orthogonality relation (13) is proved for all values of a and b, for which the orthogonality relation (6) is true. Besides, the paper [12] does not contain the expression (12) for dual q-Jacobi polynomials.

Let $\mathfrak{l}^2$ be the Hilbert space of functions on the set $m = 0, 1, 2, \dots$ with the scalar product

$$\langle f_1, f_2 \rangle = \sum_{m=0}^{\infty} \frac{(1 - abq^{2m+1})(abq, bq; q)_m}{(1 - abq)(aq, q; q)_m} a^m q^{m^2} f_1(m)\overline{f_2(m)}.$$

The polynomials (12) are in one-to-one correspondence with the columns of the unitary matrix (a_{mn}) and the orthogonality relation (13) is equivalent to the orthogonality of these columns. Due to (8) the columns of the matrix (a_{mn}) form an orthonormal basis in the Hilbert space of sequences $\mathbf{a} = \{a_n \mid n = 0, 1, 2, \dots\}$ with the scalar product $\langle \mathbf{a}, \mathbf{a}' \rangle = \sum_n a_n a'_n$. Therefore, the set of polynomials $d_n(\mu(m); a, b|q)$, $n = 0, 1, 2, \dots,$ form an orthogonal basis in the Hilbert space $\mathfrak{l}^2$. This means that *the point measure in* (13) *is extremal for those sets of the dual little q-Jacobi polynomials, for which the corresponding moment problem is indeterminate* (for the values of the parameters, for which the moment problem is indeterminate, see [11]).

Remark. For those values of parameters of dual little q-Jacobi polynomials, for which the associated moment problem is determinate, it is possible to invert our reasoning. Namely, one can take a self-adjoint operator J_1 (representable by a Jacobi matrix), diagonalization of which leads to dual little q-Jacobi polynomials. This operator is unbounded and has a discrete spectrum. Nevertheless, we still have two orthogonal bases in the Hilbert space: the initial one and the basis of eigenfunctions of J_1. If one normalizes the latter basis, then these two bases are interconnected by a unitary matrix. Orthogonalities by rows and by columns of this matrix are equivalent to orthogonality relations for little q-Jacobi polynomials and their duals. However, it is not possible to carry out this reasoning for those values of parameters, for which the associated moment problem is indeterminate, since in this case the operator J_1 is not self-adjoint. That is why we find it more convenient to start with orthogonal polynomials, which are related to a bounded self-adjoint operator (with a discrete spectrum), for in this case all values of parameters of these orthogonal polynomials correspond to determinate moment problem.

Of course, the above procedure can be also inverted when we deal with indeterminate moment problem, but one has to find first self-adjoint extensions of the corresponding symmetric operator and this is always a very complicated problem.

4. The operator I_2

Let a, b and c be real numbers such that $0 < a < q^{-1}$, $0 < b < q^{-1}$ and $c < 0$. We consider the bounded self-adjoint operator I_2 on the Hilbert space $\ell^2(\mathbb{N})$, which in the basis $\{|n\rangle\}$ has a form

$$I_2|n\rangle = a_n|n+1\rangle + a_{n-1}|n-1\rangle - b_n|n\rangle, \tag{14}$$

where

$$a_{n-1} = \frac{\sqrt{(1-q^n)(1-aq^n)(1-bq^n)(1-abq^n)(1-cq^n)(1-abc^{-1}q^n)}}{(-acq^{n+1})^{-1/2}(1-abq^{2n})\sqrt{(1-abq^{2n-1})(1-abq^{2n+1})}},$$

$$b_n = \frac{(1-aq^{n+1})(1-abq^{n+1})(1-cq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})} - acq^{n+1}\frac{(1-q^n)(1-bq^n)(1-abq^n/c)}{(1-abq^{2n})(1-abq^{2n+1})} - 1.$$

Actually, I_2 is a trace class operator. To show this we note that for the coefficients a_n and b_n from (14) one obtains that $a_{n+1}/a_n \to q^{1/2}$ and $b_{n+1}/b_n \to q$ when $n \to \infty$. Therefore, $\sum_n (2a_n + b_n) < \infty$ and this means that I_2 is a trace class operator. Thus, the spectrum of I_2 is simple (since it is representable by a Jacobi matrix with $a_n \neq 0$) and discrete.

In order to find eigenvectors ψ_λ of the operator I_2, $I_2\psi_\lambda = \lambda\psi_\lambda$, we set $\psi_\lambda = \sum_{n=0}^{\infty} \beta_n(\lambda)|n\rangle$. Acting by the operator I_2 on both sides of this relation and then collecting factors, which multiply $|n\rangle$ with fixed n, we arrive at the relation $a_n\beta_{n+1}(\lambda) + a_{n-1}\beta_{n-1}(\lambda) - b_n\beta_n(\lambda) = \lambda\beta_n(\lambda)$. Making the substitution

$$\beta_n(\lambda) = \left(\frac{(abq, aq, cq; q)_n (1-abq^{2n+1})}{(abq/c, bq, q; q)_n (1-abq)(-ac)^n}\right)^{1/2} q^{-n(n+3)/4}\beta'_n(\lambda),$$

we reduce this relation to the following one:

$$A_n\beta'_{n+1}(\lambda) + C_n\beta'_{n-1}(\lambda) - (A_n + C_n - 1)p'_n(\lambda) = \lambda\beta'_n(\lambda)$$

with

$$A_n = \frac{(1-aq^{n+1})(1-cq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},$$

$$C_n = \frac{-ac(1-q^n)(1-bq^n)(1-abc^{-1}q^n)}{q^{-n-1}(1-abq^{2n})(1-abq^{2n+1})}.$$

This is the recurrence relation for the big q-Jacobi polynomials

$$P_n(\lambda; a, b, c; q) := {}_3\phi_2(q^{-n}, abq^{n+1}, \lambda; aq, cq; q, q), \tag{15}$$

introduced by G.E. Andrews and R. Askey [13]. Therefore, $\beta_n'(\lambda) = P_n(\lambda; a, b, c; q)$ and

$$\beta_n(\lambda) = \left(\frac{(abq, aq, cq; q)_n (1 - abq^{2n+1})}{(abq/c, bq, q; q)_n (1 - abq)(-ac)^n} \right)^{1/2} \times q^{-n(n+3)/4} P_n(\lambda; a, b, c; q). \tag{16}$$

So, for the eigenvectors $\psi_\lambda(x)$ we have the expansion

$$\psi_\lambda = \sum_{n=0}^{\infty} \left(\frac{(abq, aq, cq; q)_n (1 - abq^{2n+1})}{(abq/c, bq, q; q)_n (1 - abq)(-ac)^n} \right)^{1/2} \times q^{-n(n+3)/4} P_n(\lambda; a, b, c; q) |n\rangle. \tag{17}$$

Since the spectrum of the operator I_2 is discrete, only a discrete set of these functions belongs to the Hilbert space $\ell^2(\mathbb{N})$ and this discrete set determines a spectrum of I_2.

In order to find a spectrum of I_2 explicitly, we act as in the previous case. Namely, we take into account that the orthogonality relation for the polynomials (15) is of the form

$$\begin{aligned} & r(a, b, c) \sum_{n=0}^{\infty} \frac{(aq, abq/c; q)_n q^n}{(aq/c, q; q)_n} P_m(aq^{n+1}) P_{m'}(aq^{n+1}) \\ & \quad + r(b, a, ab/c) \sum_{n=0}^{\infty} \frac{(bq, cq; q)_n q^n}{(cq/a, q; q)_n} P_m(cq^{n+1}) P_{m'}(cq^{n+1}) \\ & = \frac{(1 - abq)(bq, abq/c, q; q)_m}{(1 - abq^{2m+1})(aq, abq, cq; q)_m} (-ac)^m q^{m(m+3)/2} \delta_{mm'}, \end{aligned} \tag{18}$$

where $r(a, b, c) = (bq, cq; q)_\infty / (abq^2, c/a; q)_\infty$. *Thus, the spectrum of the operator I_2 is simple and consists of two sets of points aq^{n+1} and cq^{n+1}, $n = 0, 1, 2, \ldots$.*

The above assertion means that the vectors ψ_{aq^n} and ψ_{cq^n}, $n = 1, 2, \ldots,$ are linearly independent elements of the Hilbert space $\ell^2(\mathbb{N})$ and constitute a basis of $\ell^2(\mathbb{N})$. Moreover, these vectors are orthogonal (but not normalized). In order to normalized these vector, we have to multiply them by corresponding constants. Let the vectors $\hat{\psi}_{aq^n} = c_n \psi_{aq^n}$ and $\hat{\psi}_{cq^n} = c_n' \psi_{cq^n}$, $n = 1, 2, \ldots,$ be normalized. Then due to the orthogonality relation for big q-Jacobi polynomials we derive (as in the previous case) that

$$c_n = \left(\frac{(bq, cq; q)_\infty}{(abq^2, c/a; q)_\infty} \frac{(abq/c, aq; q)_n q^n}{(aq/c, q; q)_n} \right)^{1/2},$$

$$c_n' = \left(\frac{(aq, abq/c; q)_\infty}{(abq^2, a/c; q)_\infty} \frac{(bq, cq; q)_n q^n}{(cq/a, q; q)_n} \right)^{1/2}.$$

In the expansions

$$\hat{\psi}_{aq^n} = \sum_m c_n \beta_m(aq^n)|m\rangle = \sum_m a_{mn}|m\rangle,$$
$$\hat{\psi}_{cq^n} = \sum_m c'_n \beta_m(cq^n)|m\rangle = \sum_m a'_{mn}|m\rangle, \tag{19}$$

where $\beta_m(\lambda)$ is given by (16), the matrix $M := (a_{mn}\ a'_{mn})$ with entries

$$a_{mn} = c_n \beta_m(aq^n), \qquad a'_{mn} = c'_n \beta_m(cq^n) \tag{20}$$

is unitary and real. (Note that the matrix M is formed by adding the columns of the matrix (a'_{mn}) to the columns of the matrix (a_{mn}).) The orthogonality of the matrix $M \equiv (a_{mn}\ a'_{mn})$ means that

$$\sum_m a_{mn} a_{mn'} = \delta_{nn'}, \quad \sum_m a'_{mn} a'_{mn'} = \delta_{nn'}, \quad \sum_m a_{mn} a'_{mn'} = 0, \tag{21}$$
$$\sum_n (a_{mn} a_{m'n} + a'_{mn} a'_{m'n}) = \delta_{mm'}. \tag{22}$$

The relation (22) is equivalent to the orthogonality relation (18).

5. Dual big q-Jacobi polynomials

Now we consider the relations (21). They give the orthogonality relations for the set of matrix elements a_{mn} and a'_{mn}, viewed as functions of m. Up to multiplicative factors, they coincide with the functions

$$F_n(x; a, b, c; q) := {}_3\phi_2(x, abq/x, aq^{n+1}; aq, cq; q, q), \quad n = 0, 1, 2, \ldots, \tag{23}$$
$$F'_n(x; a, b, c; q) := {}_3\phi_2(x, abq/x, cq^{n+1}; aq, cq; q, q) \equiv F_n(x; c, ab/c, a),$$
$$n = 0, 1, 2, \ldots, \tag{24}$$

considered on the corresponding sets of points. Namely, we have

$$a_{mn} \equiv a_{mn}(a, b, c) = AF_n(q^{-m}; a, b, c; q),$$
$$a'_{mn} = A'F'_n(q^{-m}; a, b, c; q) \equiv a_{mn}(c, ab/c, a),$$

where A and A' are expressions, which multiply $P_m(aq^{n+1}; a, b, c; q)$ and $P_m(cq^{n+1}; a, b, c; q)$ in formulas (20) for a_{mn} and a'_{mn}, respectively. The relations (21) lead to the orthogonality relations for the functions (23) and (24),

$$\frac{(bq, cq; q)_\infty}{(abq^2, c/a; q)_\infty} \sum_{m=0}^{\infty} \rho(m) F_n(q^{-m})\, F_{n'}(q^{-m}) = \frac{(aq/c, q; q)_n}{(aq, abq/c; q)_n q^n} \delta_{nn'}, \tag{25}$$
$$\frac{(aq, abq/c; q)_\infty}{(abq^2, a/c; q)_\infty} \sum_{m=0}^{\infty} \rho(m) F'_n(q^{-m})\, F'_{n'}(q^{-m}) = \frac{(cq/a, q; q)_n}{(bq, cq; q)_n q^n} \delta_{nn'}, \tag{26}$$
$$\sum_{m=0}^{\infty} \rho(m) F_n(q^{-m})\, F'_{n'}(q^{-m}) = 0, \tag{27}$$

where

$$\rho(m) := \frac{(1-abq^{2m+1})(aq, abq, cq; q)_m}{(1-abq)(bq, abq/c, q; q)_m (-ac)^m} q^{-m(m+3)/2}.$$

There is another form for the functions $F_n(q^{-m}; a, b, c; q)$. Indeed, one can use the relation (III.12) of Appendix III in [7] to obtain that

$$F_n(q^{-m}; a, b, c; q) = \frac{(abq/c; q)_m}{(cq; q)_m} (-c)^m q^{m(m+1)/2} \times {}_3\phi_2\left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ aq, abq/c \end{matrix} \middle| q, aq^{n+1}/c \right).$$

The basic hypergeometric function ${}_3\phi_2$ in this formula is a polynomial in $\mu(m) := q^{-m} + abq^{m+1}$. So if we introduce the notation

$$D_n\big(\mu(m); a, b, c|q\big) := {}_3\phi_2\left(\begin{matrix} q^{-m}, abq^{m+1}, q^{-n} \\ aq, abq/c \end{matrix} \middle| q, aq^{n+1}/c \right), \tag{28}$$

then

$$F_n(q^{-m}; a, b, c; q) = \frac{(abq/c; q)_m}{(cq; q)_m} (-c)^m q^{m(m+1)/2} D_n\big(\mu(m); a, b, c|q\big).$$

Formula (25) directly leads to the orthogonality relation for the polynomials (28):

$$\sum_{m=0}^{\infty} \frac{(1-abq^{2m+1})(aq, abq, abq/c; q)_m}{(1-abq)(bq, cq, q; q)_m} (-c/a)^m q^{m(m-1)/2} D_n\big(\mu(m)\big) D_{n'}\big(\mu(m)\big) = \frac{(abq^2, c/a; q)_\infty}{(bq, cq; q)_\infty} \frac{(aq/c, q; q)_n}{(aq, abq/c; q)_n q^n} \delta_{nn'}. \tag{29}$$

We call the polynomials $D_n(\mu(m); a, b, c|q)$ *dual big q-Jacobi polynomials*. It is natural to ask whether they can be identified with some known and thoroughly studied set of polynomials. The answer is: they can be obtained from the q-Racah polynomials $R_n(\mu(x); a, b, c, d|q)$ of Askey and Wilson [14] by setting $a = q^{-N-1}$ and sending $N \to \infty$, that is,

$$D_n\big(\mu(x); a, b, c|q\big) = \lim_{N\to\infty} R_n\big(\mu(x); q^{-N-1}, a/c, a, b|q\big).$$

Observe that the orthogonality relation (29) can be also derived from formula (4.16) in [15]. But the derivation of this formula (4.16) is rather complicated.

It is easy to show that the polynomials (28) can be expressed in terms of continuous q^{-1}-Hahn polynomials. A more complicated case than our is considered in [16], where the authors study duality connection between big q-Jacobi functions and continuous q^{-1}-Hahn polynomials (a paper [17] also deals with the duality on the non-polynomial level). It is shown in [16] that the orthogonality relation, dual to that for big q-Jacobi functions, is continuous.

It is worth noting here that in the limit as $c \to 0$ the dual big q-Jacobi polynomials $D_n(\mu(x); a, b, c|q)$ coincide with the dual little q-Jacobi polynomials $d_n(\mu(x); b, a|q)$, defined in Section 3. The dual little q-Jacobi polynomials $d_n(\mu(x); a, b|q)$ reduce, in turn, to the Al-Salam–Carlitz II polynomials $V_n^{(a)}(s;q)$ on the q-linear lattice $s=q^{-x}$ (see [9, p. 114]) in the case when the parameter b vanishes, that is, $d_n(\mu(x);a,0|q)=(-a)^{-n}q^{n(n-1)/2}\,V_n^{(a)}(q^{-x};q)$. This means that we have a complete chain of reductions

$$\begin{aligned}R_n\big(\mu(x);a,b,c,d|q\big) &\underset{a\to\infty}{\longrightarrow} D_n\big(\mu(x);b,c,d|q\big) \underset{d\to 0}{\longrightarrow} d_n\big(\mu(x);c,b|q\big)\\ &\underset{b=0}{\longrightarrow} V_n^{(c)}(q^{-x};q)\end{aligned}$$

from the four-parameter family of q-Racah polynomials, which occupy the upper level in the Askey-scheme of basic hypergeometric polynomials (see [9, p. 62]), down to the one-parameter set of Al-Salam–Carlitz II polynomials from the second level in the same scheme.

The recurrence relations for $D_n(\mu(m);a,b,c|q)$ are obtained from the q-difference equation for big q-Jacobi polynomials, given by formula (3.5.5) in [9]. They are of the form

$$\begin{aligned}(q^{-m}-1)(1-abq^{m+1})D_n\big(\mu(m)\big) &= A_n D_{n+1}\big(\mu(m)\big)-(A_n+C_n)D_n\big(\mu(m)\big)\\ &\quad + C_n D_{n-1}\big(\mu(m)\big),\end{aligned}$$

where $A_n=q^{-2n-1}(1-aq^{n+1})(c/a-bq^{n+1})$ and $C_n=q^{-2n}(1-q^n)(c/a-q^n)$.

The relation (27) leads to an interesting equality

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{(-1)^m(1-abq^{2m+1})(abq;q)_m}{(1-abq)(q;q)_m\,q^{-m(m-1)/2}}D_n\big(\mu(m);a,b,c|q\big)\\ &\quad\times D_{n'}\big(\mu(m);b,a,abq/c|q\big)=0.\end{aligned}$$

Observe also that from the expression (28) for the dual big q-Jacobi polynomials it follows that they possess the symmetry property $D_n(\mu(m);a,b,c|q)=D_n(\mu(m);ab/c,c,b|q)$.

The set of functions (23) and (24) form an orthogonal basis in the Hilbert space $\mathfrak{l}^2$ of functions, defined on the set of points $m=0,1,2,\ldots,$ with the scalar product $\langle f_1,f_2\rangle=\sum_{m=0}^{\infty}\rho(m)\,f_1(m)\,\overline{f_2(m)}$, where $\rho(m)$ is the same as in formulas (25)–(27). Consequently, *the dual big q-Jacobi polynomials $D_n(\mu(m);a,b,c|q)$ correspond to indeterminate moment problem and the orthogonality measure for them, given by formula* (29)*, is not extremal*.

Note that in [16] a set of functions, dual to big q-Jacobi functions, is found, which contains as a subset the family of continuous q^{-1}-Hahn polynomials. Of course, as in our case, the whole dual set does not exhausted by continuous q^{-1}-Hahn polynomials (which means that these polynomials correspond to indeterminate moment problem).

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