Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 53 (2007) 1098–1106
An International Journal
computers & mathematics
with applications
www.elsevier.com/locate/camwa

Mean square numerical solution of random differential equations: Facts and possibilities☆

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Received 25 April 2006; accepted 5 May 2006

Abstract

This paper deals with the construction of numerical solutions of random initial value differential problems. The random Euler method is presented and the conditions for the mean square convergence are established. Numerical examples show that random Euler method gives good results even if the sufficient convergence conditions are not satisfied.
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Keywords: Random differential equation; Mean square; Numerical solution

1. Introduction

Differential equations are powerful tools for representing reality up to certain point. Parameters and functions appearing in the mathematical models are subject to uncertainty because of error measurements or complexity, or for technical reasons [1–4]. These facts make it more realistic dealing with random models where parameters are random variables and data functions are stochastic processes. This paper deals with random differential initial value problems of the form

$$\begin{aligned} &\dot{X}(t) = f(X(t), t), \quad t \in T = [t_0, t_1], \\ &X(t_0) = X_0, \end{aligned} \tag{1.1}$$

where X_0 is a random variable, and the unknown $X(t)$ as well as the right-hand side $f(X, t)$ are stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, P)$. We are interested in second-order random variables (2-r.v.'s) Y, having a density function f_Y,

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\mathrm{d}y < \infty,$$

☆ This work was partially supported by the Spanish M.C.Y.T. and FEDER Grant DPI2004-08383-C03-03 and the Generalitat Valenciana.
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doi:10.1016/j.camwa.2006.05.030

where E denotes the expectation operator, and it allows the introduction of the Banach space L_2 of all the 2-r.v.'s endowed with the norm

$$\|Y\| = \sqrt{E[Y^2]},$$

[5, Chap 4]. A stochastic process $p(t)$ defined on the same probability space $(\Omega, \mathcal{F}, P)$ is called a second-order stochastic process (2-s.p.) if for each t, $p(t)$ is a 2-r.v. Hence the meaning of $\dot{X}(t)$ in (1.1) is the mean square limit in L_2 of the expression

$$\frac{X(t+\Delta t) - X(t)}{\Delta t} \quad \text{as } \Delta t \to 0. \tag{1.2}$$

Furthermore, we are interested not only in the existence of the above mean square limit of (1.2), but that for each $\omega \in \Omega$, the deterministic derivative of the realization $X(t)(\omega)$ coincides with $f(X(t), t)(\omega)$, i.e., the solution process of the problem (1.1) has differential deterministic realizations. Thus, if the uncertainty of the model (1.1) were to be known information, our solution would coincide with the deterministic solution. This fact makes that our approach differ from the stochastic differential approach based on Ito calculus [6]. Even in the deterministic case, constructing exact solutions of differential equations is a difficult task apart from in a few particular cases. This motivates the main goal of this paper that is to introduce the random Euler method for constructing reliable numerical solutions of the problem (1.1) in the sense that approximations be mean square (m.s.) convergent to the exact solution process of the problem. We are also interested in computing the expectation and the variance of the constructed approximating process.

This paper is organized as follows. In Section 2, some sufficient conditions are given in order to guarantee the existence and uniqueness of a process of solution of problem (1.1), having differentiable realizations. Section 3 deals with the introduction of the random Euler method as well as its convergence properties. Section 4 includes numerical examples of the random Euler method under conditions of Section 3, where the m.s. convergence of the random Euler method is granted. Finally in Section 4 we show with illustrative examples that outside of the framework of Section 3, the random Euler method offers possibilities for obtaining good numerical results.

For the sake of clarity in the presentation we conclude this section by recalling that if $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_M$ are r.v.'s and a_i, b_j are real numbers for $1 \le i \le N$, $1 \le j \le M$, then [7]

$$\operatorname{Cov}\left[\sum_{i=1}^{N} a_i X_i, \sum_{j=1}^{M} b_j Y_j\right] = \sum_{i=1}^{N}\sum_{j=1}^{M} a_i b_j \operatorname{Cov}[X_i, Y_j]. \tag{1.3}$$

2. Random differential equations

It is well know that under the m.s. Lipschitz conditions of $f(X, t)$ together with its m.s. continuity, the problem (1.1) admits a unique local solution s.p. [5, p. 119]. There are several reasons why an alternative approach to the m.s. Lipschitz condition is suitable. One is that such a hypothesis is not often satisfied [5, p. 119]; and the second is that we seek solution processes that are not only m.s. differentiable but also with differentiable realizations.

The following result shows under which conditions a 2-s.p. $p(t)$, $t \in T$, which for a fixed event $\omega \in \Omega$ satisfies the property that the realization $p(t)(\omega)$ is differentiable in the deterministic sense, is m.s. differentiable.

Theorem 2.1. *Let $p(t) = p(t, \beta)$ be a 2-s.p. defined on $(\Omega, \mathcal{F}, P)$ which depends on the 2-r.v. β. Assume that for each $\omega \in \Omega$, the realization $p(t, \beta)(\omega)$ is a twice-differentiable deterministic function with respect to the variable t, and assume that its second derivative satisfies the property*

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}y^2} p(y, \beta)(\omega)\right| \le M < +\infty, \quad \forall (y, \omega) \in [t-\delta, t+\delta] \times \Omega,\ \delta > 0. \tag{2.1}$$

Then the process $p(t, \beta)$ is m.s. differentiable and $\dot{p}(t)$ is defined for each $\omega \in \Omega$, by

$$\dot{p}(t)(\omega) = \frac{\mathrm{d}}{\mathrm{d}t} p(t, \beta)(\omega) = \lim_{\Delta t \to 0} \frac{p(t+\Delta t, \beta)(\omega) - p(t, \beta)(\omega)}{\Delta t}. \tag{2.2}$$

Proof. Let $\omega \in \Omega$ be fixed, and consider Taylor's expansion about t of the deterministic differentiable function $p(t, \beta)(\omega)$,

$$p(t + \Delta t, \beta)(\omega) = p(t, \beta)(\omega) + \frac{\mathrm{d}}{\mathrm{d}t} p(t, \beta)(\omega)\Delta t + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} p(t_\omega, \beta)(\omega)(\Delta t)^2, \tag{2.3}$$

for some t_ω between t and $t + \Delta t$. Let $\dot{p}(t)$ be the s.p. defined by (2.2). As for each Δt, $p(t + \Delta t, \beta)$ is a 2-r.v. that is a function of the r.v. β, then by [8, p. 93] it follows that $\dot{p}(t)$ is also a function of β, and by (2.3) one gets

$$E\left[\left(\frac{p(t + \Delta t) - p(t)}{\Delta t} - \dot{p}(t)\right)^2\right] = \left(\int_\Omega \frac{\mathrm{d}^2}{\mathrm{d}t^2} p(t_\omega, \beta)(\omega)^2 \mathrm{d}P(\omega)\right)\frac{(\Delta t)^2}{4}. \tag{2.4}$$

Under hypothesis (2.1), taking limits in (2.4) as $\Delta t \to 0$, one gets

$$\lim_{\Delta t \to 0} E\left[\left(\frac{p(t + \Delta t) - p(t)}{\Delta t} - \dot{p}(t)\right)^2\right] = 0. \tag{2.5}$$

Thus the result is established.

Let γ and X_0 be 2-r.v. defined on the probability space $(\Omega, \mathcal{F}, P)$ and let $f(X, t)$ be such that:

(H_1) $f : S \times T \to L_2$, $S \subset L_2$ is a function of the 2-r.v. γ, continuous in both variables (X, t).

(H_2) For each $\omega \in \Omega$, the realization $f(X, t)(\omega)$ is differentiable and its derivative with respect to the variable t satisfies

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} f(X, t)(\omega)\right| \leq M < \infty, \quad \forall (X, t, \omega) \in S \times T \times \Omega.$$

(H_3) There is a constant $K > 0$ such that

$$|f(X, t)(\omega) - f(Y, t)(\omega)| \leq K\,|X(\omega) - Y(\omega)|, \qquad \forall (X, t, \omega) \in S \times T \times \Omega, \quad \forall (Y, t, \omega) \in S \times T \times \Omega.$$

Theorem 2.2. *Let $f(x, t)$ be a 2-s.p. defined on $(\Omega, \mathcal{F}, P)$ satisfying conditions* (H_1), (H_2) *and* (H_3), *and let X_0 be a given 2-r.v. Then there exists one 2-s.p. $X(t)$ solution of the problem (1.1) with differentiable realizations.*

Proof. Let $\omega \in \Omega$ be a fixed event, and let us consider the deterministic differential equation

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} x_\omega(t) = f(x_\omega(t), t), \\ &x_\omega(t_0) = x_0(\omega). \end{aligned}\right\} \tag{2.6}$$

By the continuity of f and property (H_3) [9, p. 15], there exists a unique solution $x_\omega(t)$ of the problem (2.6) such that

$$x_\omega(t) = x_0(\omega) + \int_{t_0}^{t} f(x_\omega(s), s)\mathrm{d}s. \tag{2.7}$$

Let us introduce the s.p. defined for each $\omega \in \Omega$ by

$$X(t, \omega) = x_\omega(t), \tag{2.8}$$

and note that by (H_1), $X(t, \cdot)$ is a function of the 2-r.v. γ. By hypothesis (H_2), the process $X(t, \omega)$ admits twice-differentiable realizations with a bounded second derivative. Furthermore, by Theorem 2.1, $X(t, \omega)$ is m.s. differentiable and

$$\dot{X}(t, \omega) = \frac{\mathrm{d}}{\mathrm{d}t} x_\omega(t) = f(x_\omega(t), t) = f(X(t, \omega), t),$$

and

$$X(t_0, \omega) = x_0(\omega).$$

Thus $X(t,\omega)$ is an m.s. solution of the problem (1.1). In order to show the uniqueness of this solution, note that

$$\|f(X,t)-f(Y,t)\|^2 = E\left[(f(X,t)-f(Y,t))^2\right] = \int_\Omega (f(X,t)-f(Y,t))^2(\omega)\mathrm{d}P(\omega)$$
$$\leq \int_\Omega (K\,|X(\omega)-Y(\omega)|)^2\mathrm{d}P(\omega) = K^2\,\|X-Y\|^2\,.$$

Hence, by Theorem 5.12 of [5, p. 118] there exists only one 2-s.p. solution for the problem (1.1) satisfying

$$X(t) = X_0 + \int_{t_0}^{t} f(X(s),s)\mathrm{d}s. \tag{2.9}$$

Example 2.3. Let X_0 be a 2-r.v. defined on $(\Omega,\mathcal{F},P)$ and $b(t)$ be a 2-s.p. m.s. integrable defined on $(\Omega,\mathcal{F},P)$ such that $b(t)$ is independent of X_0 for each t lying in the interval $T=[t_0,t_1]$. Let $a(t)$ be a deterministic continuous function defined on T and consider the random initial value problem.

$$\dot{X}(t) = a(t)X(t)+b(t); \quad X(t_0)=X_0,\ t\in T. \tag{2.10}$$

By Theorem 2.2, problem (2.10) admits a unique solution process and, as suggested by the solution of the deterministic case, we will show that this solution is given by

$$X(t) = \exp\left(\int_{t_0}^{t} a(s)\mathrm{d}s\right)\left\{X_0+\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}a(r)\mathrm{d}r\right)b(s)\mathrm{d}s\right\}. \tag{2.11}$$

In fact, by a corollary of [5, p. 103] the process $X(t)$ given by (2.11) is well defined and by [5, p. 96], $X(t)$ is m.s. differentiable,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(\int_{t_0}^{t}a(s)\mathrm{d}s\right)\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}a(r)\mathrm{d}r\right)b(s)\mathrm{d}s\right]\\ &\quad= a(t)\exp\left(\int_{t_0}^{t}a(s)\mathrm{d}s\right)\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}a(r)\mathrm{d}r\right)b(s)\mathrm{d}s+\exp\left(\int_{t_0}^{t}a(s)\mathrm{d}s\right)\exp\left(-\int_{t_0}^{t}a(r)\mathrm{d}r\right)b(t)\\ &\quad= a(t)\exp\left(\int_{t_0}^{t}a(s)\mathrm{d}s\right)\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}a(r)\mathrm{d}r\right)b(s)\mathrm{d}s+b(t),\end{aligned} \tag{2.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(\int_{t_0}^{t}a(s)\mathrm{d}s\right)X_0\right) = a(t)\exp\left(\int_{t_0}^{t}a(s)\mathrm{d}s\right)X_0. \tag{2.13}$$

By (2.12) and (2.13) one gets that $X(t)$ defined by (2.11) is the solution of (2.10). By Theorem 4.5.2 of [5, p. 104], the expectation of $X(t)$ takes the form

$$E\left[X(t)\right] = \exp\left(\int_{t_0}^{t}a(s)\mathrm{d}s\right)\left\{E[X_0]+\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}a(r)\mathrm{d}r\right)E[b(s)]\mathrm{d}s\right\}. \tag{2.14}$$

According to the Theorem 4.3.1 of [5, p. 88], the variance of $X(t)$ is given by

$$\mathrm{Var}\left(X(t)\right) = \exp\left(2\int_{t_0}^{t}a(s)\mathrm{d}s\right)\left\{\mathrm{Var}[X_0]+\mathrm{Var}\left[\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}a(r)\mathrm{d}r\right)b(s)\mathrm{d}s\right]\right\}, \tag{2.15}$$

where

$$\begin{aligned}&\mathrm{Var}\left[\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}a(r)\mathrm{d}r\right)b(s)\mathrm{d}s\right]\\ &\quad=\int_{t_0}^{t}\int_{t_0}^{t}\exp\left(-\int_{t_0}^{s}a(u)\mathrm{d}u\right)\exp\left(-\int_{t_0}^{r}a(u)\mathrm{d}u\right)\mathrm{cov}[b(r),b(s)]\mathrm{d}r\mathrm{d}s,\end{aligned}$$

and

$$\operatorname{cov}[b(r), b(s)] = E[b(r)b(s)] - E[b(r)]E[b(s)]. \tag{2.16}$$

3. On the mean square random Euler numerical method

As is natural, the random Euler numerical scheme associated with the random initial value problem (1.1) has the expression

$$\left.\begin{array}{l} X_n = X_{n-1} + hf(X_{n-1}, t_{n-1}), \\ X(t_0) = X_0. \end{array}\right\} \quad n \geq 1, \tag{3.1}$$

where X_n, $f(X_{n-1}, t_{n-1})$ are 2-r.v. and $h = t_n - t_{n-1}$. The aim of this section is to show that under the hypothesis of Theorem 2.2, the sequence $\{X_n\}$ is m.s. convergent to the theoretical value $X(t)$, in the fixed station sense, i.e., $h \to 0$ and $t_n = t$; see [10, p. 26].

Assuming the hypothesis of Theorem 2.2, let $X(t)$ be the solution of the random initial value problem (1.1). From condition (H_2) of Theorem 2.2 and expression (2.6) of the proof of Theorem 2.2, for each $\omega \in \Omega$ one gets

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2} x_\omega(t)\right| = \left|\frac{\mathrm{d}}{\mathrm{d}t} f(x_\omega(t), t)\right| < M, \quad \forall (t, \omega) \in T \times \Omega. \tag{3.2}$$

Taking into account Taylor's expansion of x_ω about $t_n = t$, it follows that

$$x_\omega(t_{n+1}) = x_\omega(t) + hf(x_\omega(t), t) + \frac{h^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}t^2} x_\omega\left(\xi(\omega)\right), \tag{3.3}$$

where $t_n = t < \xi(\omega) < t_{n+1}$. We wish to show that the error random variable

$$e_n = X_n - X(t), \tag{3.4}$$

is m.s. convergent to zero in the fixed station sense. In order to show that, let us take the realizations of e_n,

$$e_n(\omega) = X_n(\omega) - X(t, \omega) = X_n(\omega) - x_\omega(t), \quad n \geq 0, \tag{3.5}$$

and the realization of the Euler scheme,

$$X_{n+1}(\omega) = X_n(\omega) + hf(X_n(\omega), t), \quad X_0(\omega) = x_\omega(t_0). \tag{3.6}$$

From (3.3)–(3.6) it follows that

$$e_{n+1}(\omega) = e_n(\omega) + h[f(X_n(\omega), t) - f(x_\omega(t), t)] - \frac{h^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} x_\omega(\xi(\omega)). \tag{3.7}$$

From condition (H_3) of Theorem 2.2 one gets

$$|e_{n+1}(\omega)| \leq (1 + hK)|e_n(\omega)| + \frac{h^2 M}{2}. \tag{3.8}$$

By Lemma 1.2 of [9, p. 18] it follows that

$$|e_n(\omega)| \leq \mathrm{e}^{nhK}|e_0(\omega)| + \frac{(\mathrm{e}^{nhK} - 1)}{Kh}\left(\frac{h^2 M}{2}\right). \tag{3.9}$$

As $e_0(\omega) = 0$ for all $\omega \in \Omega$, and $nh = t - t_0$, by (3.9) one gets

$$|e_n(\omega)| \leq \frac{hM}{2K}\left(\mathrm{e}^{(t-t_0)K} - 1\right), \quad (t, \omega) \in T \times \Omega. \tag{3.10}$$

By (3.4) and (3.10) it follows that

$$\begin{aligned}\|e_n\|^2 &= \|X_n - X(t)\|^2 = E[(X_n - X(t))^2] \\ &= \int_\Omega (X_n - X(t))^2(\omega)\mathrm{d}P(\omega) = \int_\Omega (e_n)^2(\omega)\mathrm{d}P(\omega) \\ &\leq \left[\left(\frac{hM}{2K}\right)\left(\mathrm{e}^{(t-t_0)K} - 1\right)\right]^2.\end{aligned}$$

Thus $\{e_n\}$ is m.s. convergent to zero as $h \to 0$, $nh = t - t_0$ and the following result has been established:

Theorem 3.1. *If $f(x,t)$ satisfies the hypothesis of Theorem 2.2, then the random Euler numerical method is m.s. convergent.*

Remark 3.2. From a practical point of view the m.s. convergence of the random Euler numerical method is relevant because it guarantees that $E[X_n]$ and $\mathrm{Var}[X_n]$ both converge to the exact values $E[X(t)]$ and $\mathrm{Var}[X(t)]$, respectively, of the theoretical solution process $X(t)$, as $h \to 0$, $nh = t - t_0$; see Theorem 4.2.1 of [5, p. 76] and Theorem 4.3.1 of [5, p. 88].

Remark 3.3. Results of Sections 2 and 3 including the convergence of the random Euler method are extended without additional effort, when function $f(X,t)$ appearing in the right-hand side of Eq. (1.1) depends on a finite number $\gamma_1, \gamma_2, \ldots, \gamma_s$ of 2-r.v.'s defined on the same probability space $(\Omega, \mathcal{F}, P)$. For the sake of simplicity in the notation we assumed that $s = 1$.

4. Numerical results

In this section numerical results obtained using the Euler method for problem (2.10) are included. Note that the random Euler scheme for problem (2.10) takes the form

$$\left.\begin{aligned} X_n &= X_{n-1} + h\left[a(t_{n-1})X_{n-1} + b(t_{n-1})\right], \\ &= (1 + ha(t_{n-1}))X_{n-1} + hb(t_{n-1}) \end{aligned}\right\} \quad n \geq 1. \tag{4.1}$$

It is easy to show using the induction principle, see [11, p. 58–59], that X_n can be written as

$$X_n = \left(\prod_{i=0}^{n-1}(1 + ha(t_i))\right)X(t_0) + h\sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1}(1 + ha(t_j))\right)b(t_i), \quad n \geq 1. \tag{4.2}$$

Let us assume that $a(t)$ is a determinist continuous function and that $b(t)$ is a 2-s.p. independent of X_0 for each $t \in T$. From (4.2), (2.16), (1.3) and (2.16) it follows that

$$E[X_n] = \left(\prod_{i=0}^{n-1}(1 + ha(t_i))\right)E[X(t_0)] + h\sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1}(1 + ha(t_j))\right)E[b(t_i)], \tag{4.3}$$

$$\begin{aligned}\mathrm{Var}[X_n] &= \left(\prod_{i=0}^{n-1}(1 + ha(t_i))\right)^2 \mathrm{Var}[X_0] \\ &\quad + h^2\sum_{i=0}^{n-1}\sum_{k=0}^{n-1}\left(\prod_{j=i+1}^{n-1}(1 + ha(t_j))\right)\left(\prod_{l=k+1}^{n-1}(1 + ha(t_l))\right)\mathrm{cov}[b(t_i), b(t_k)].\end{aligned} \tag{4.4}$$

Example 4.1. Consider the random initial value problem (2.10) in $T = [0,1]$ where $a(t) = t + 2$ and $b(t) = \cos(\pi t + \rho)$ is a Rice noise [5, p. 47], ρ is a 2-r.v. uniformly distributed in $[0, 2\pi]$ and $X_0 = \eta$ is an exponential 2-r.v. with parameter $\lambda = 11/2$, truncated on the interval $[0, 40]$ and independent of ρ.

In this case we have $E[X_0] = 0.18$ and $\mathrm{Var}[X_0] = 0.03$. Figs. 1 and 2 show that the numerical values of $E[X_n]$ and $\mathrm{Var}[X_n]$ are closer to the theoretical values $E[X(t)]$ and $\mathrm{Var}[X(t)]$ as well as the parameter h decreasing. This fact agrees with Theorem 3.1 that guarantees the convergence of the random Euler method.

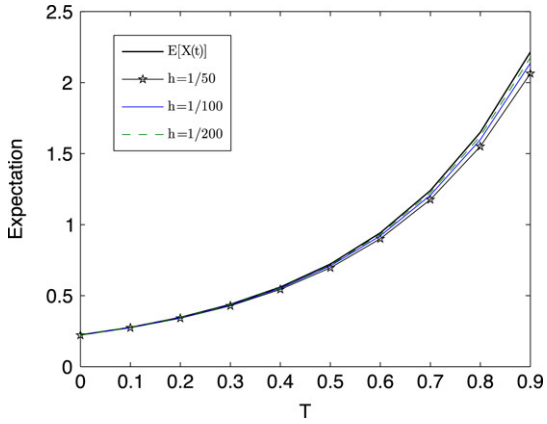

Fig. 1. Expectations of $X(t)$ and X_n for Example 4.1.

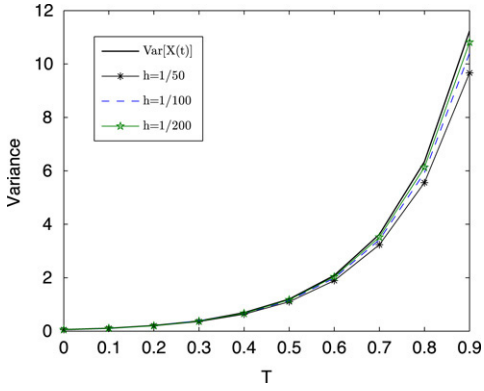

Fig. 2. Variances of $X(t)$ and X_n for Example 4.1.

5. Possibilities of the random Euler method

As has been stated in Theorem 3.1 of Section 3, the random Euler method converges under specific conditions. However, from a practical point of view, in situations where the exact solution is not available, the discrete 2-s.p. constructed using the random Euler method can be useful if the approximations become closer as h decreases. This fact is illustrated in the following examples.

Example 5.1. Consider the random initial value problem (2.10) where $a(t) = 1/4$, $b(t)$ is the Brownian motion process and X_0 is an exponential 2-r.v. with parameter $\lambda = 1/4$, truncated on the interval $[0, 100]$ and independent of $b(t)$ for each $t \in [0, 1]$.

In this case the exact solution process is given by (2.11) although the conditions of Theorem 3.1 are not satisfied while the convergence of the random Euler method is not guaranteed. Tables 1 and 2 show the comparisons between the values of the expectation $E[X(t)]$ and $E[X_n]$ as well as those of the variances $\mathrm{Var}[X(t)]$ and $\mathrm{Var}[X_n]$, using expressions (2.14), (2.15), (4.3) and (4.4).

Example 5.2. Consider the random initial value problem (2.10) where $a(t)$ is a uniform r.v. on $[0, 1]$, $b(t)$ is the Brownian motion and X_0 is a Gaussian 2-r.v. $N(1/2, 1/12)$ and X_0, $a(t)$ and $b(t)$ are pairwise independent for each $t \in [0, 1]$.

Tables 3 and 4 provide the values of $E[X_n]$ and $\mathrm{Var}[X_n]$ for different values of h and they show that as h decreases the differences of the numerical values for several values of h become negligible as $h \to 0$.

Table 1
Numerical results for the expectation for Example 5.1

t	$E[X_n], h = \frac{1}{50}$	$E[X_n], h = \frac{1}{100}$	$E[X_n], h = \frac{1}{200}$	$E[X_n], h = \frac{1}{400}$	$E[X(t)]$
0.1	4.1010	4.1011	4.1012	4.1012	4.1012
0.2	4.2045	4.2048	4.2049	4.2050	4.2050
0.3	4.3107	4.3111	4.3113	4.3114	4.3115
0.4	4.4195	4.4201	4.4204	4.4205	4.4206
0.5	4.5311	4.5318	4.5322	4.5324	4.5325
0.6	4.6459	4.6464	4.6469	4.6471	4.6473
0.7	4.7629	4.7639	4.7644	4.7647	4.7649
0.8	4.8831	4.8843	4.8850	4.8853	4.8856
0.9	5.0064	5.0078	5.0085	5.0089	5.0092
1	5.1329	5.1345	5.1353	5.1357	5.1361

Table 2
Numerical results for the variance for Example 5.1

t	$\mathrm{Var}[X_n], h = \frac{1}{50}$	$\mathrm{Var}[X_n], h = \frac{1}{100}$	$\mathrm{Var}[X_n], h = \frac{1}{200}$	$\mathrm{Var}[X_n], h = \frac{1}{400}$	$\mathrm{Var}[X(t)]$
0.1	16.8185	16.8196	16.8201	16.8204	16.8207
0.2	17.6807	17.6831	17.6843	17.4869	17.6855
0.3	18.5909	18.5949	18.5969	18.5878	18.5989
0.4	19.5539	19.5596	19.5625	19.5640	19.5655
0.5	20.5744	20.5823	20.5862	20.5882	20.5902
0.6	21.6578	21.6681	21.6673	21.6758	21.6784
0.7	22.8095	22.8226	22.8291	22.8324	22.8357
0.8	24.0355	24.0517	24.0598	24.0639	24.068
0.9	25.3419	25.3617	25.3717	25.3767	25.3817
1	26.7356	26.7594	26.7714	26.7774	26.7834

Table 3
Numerical results for the expectation for Example 5.2

t	$E[X_n], h = \frac{1}{50}$	$E[X_n], h = \frac{1}{100}$	$E[X_n], h = \frac{1}{200}$	$E[X_n], h = \frac{1}{400}$
0.1	0.5256	0.5257	0.5258	0.5258
0.2	0.5531	0.5533	0.5534	0.5534
0.3	0.5824	0.5827	0.5829	0.5830
0.4	0.6138	0.6143	0.6145	0.6146
0.5	0.6475	0.6481	0.6484	0.6485
0.6	0.6835	0.6843	0.6847	0.6849
0.7	0.7221	0.7231	0.7236	0.7238
0.8	0.7635	0.7647	0.7653	0.7656
0.9	0.8079	0.8094	0.8101	0.8105
1	0.8555	0.8573	0.8573	0.8586

Table 4
Numerical results for the variance for Example 5.2

t	$\mathrm{Var}[X_n], h = \frac{1}{50}$	$\mathrm{Var}[X_n], h = \frac{1}{100}$	$\mathrm{Var}[X_n], h = \frac{1}{200}$	$\mathrm{Var}[X_n], h = \frac{1}{400}$
0.1	0.0926	0.0927	0.0927	0.0928
0.2	0.1057	0.1060	0.1062	0.1062
0.3	0.1254	0.1261	0.1264	0.1266
0.4	0.1549	0.1562	0.1569	0.1573
0.5	0.1982	0.2004	0.2016	0.2022
0.6	0.2597	0.2632	0.2650	0.2652
0.7	0.3447	0.3500	0.3527	0.3540
0.8	0.4594	0.4670	0.4709	0.4728
0.9	0.6111	0.6217	0.6270	0.6298
1	0.8082	0.8227	0.8300	0.8337

References

[1] C.R. MacCluer, Industrial Mathematics, Prentice Hall, New Jersey, 2000.
[2] J. Chilés, P. Delfiner, Geostatistics. Modelling Spatial Uncertainty, John Wiley, London, 1999.
[3] R. Barron, A.G. Ayala, El método de yuxtaposición de dominios en la solución numérica de ecuaciones diferenciales estocásticas, in: E. Oñate, et al. (Eds.), Proc. Métodos numéricos en Ingeniería y Ciencias Aplicadas, CIMNE, Barcelona, 2002, pp. 267–276 (in Spanish).
[4] T.T. Soong, M. Grigoriu, Random Vibration of Mechanical and Structural Systems, Prentice Hall, México, 1993.
[5] T.T. Soong, Random Differential Equations in Science and Engineering, Academic Press, New York, 1973.
[6] E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1992.
[7] J.C. Cortés, P. Sevilla-Peris, L. Jódar, Analytic–numerical approximating processes of diffusion equation with data uncertainty, Computers and Mathematics with Applications 49 (2005) 1255–1266.
[8] G.R. Grimmett, D.R. Stirzaker, Probability and Random Processes, Clarendon Press, Oxford, 2000.
[9] P. Henrici, Discrete Variable Methods in Ordinary Differential Equations, John Wiley & Sons, New York, 1962.
[10] J.D. Lambert, Numerical Methods for Ordinary Differential Systems, Wiley, Chichester, 1991.
[11] R.P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 1992.