Quantile hedging for equity-linked life insurance contracts with stochastic interest rate

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Abstract

This paper studies the problem of pricing equity-linked life insurance contracts, and also focuses on the valuation of insurance contracts with stochastic guarantee. The contracts under consideration are based on two risky assets which satisfy a two-factor jump-diffusion model: one asset is responsible for future gains, and the other one is a stochastic guarantee. As most life insurance products are long-term contracts, it is more practical to consider the problem in a stochastic interest rate environment. In our setting, the stochastic interest rate behaviour is also described by a jump-diffusion model. In addition, quantile hedging technique is developed and exploited to price such finance/insurance contracts with initial capital constraints. Explicit formulas for both the price of the contracts and the survival probability are obtained. Our results are illustrated by numerical example based on financial indexes Russell 2000 and S&P 500.

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Keywords: quantile hedging, jump-diffusion, stochastic interest rate, equity-linked life insurance

1. Introduction

Equity-linked life insurance contracts have been studied since the middle of the 1970s. This type of contracts links the benefit payable at the maturity time with the market value of some reference portfolio, such as stocks, foreign currencies etc. Thus, the benefit of such contracts is uncertain while it is fixed for the traditional contracts. Compared with traditional ones, these innovative products can bring the insurance companies as well as the clients more benefit and improve the insurance companies’ competitiveness in the modern financial system.

In North America and the UK, equity-linked life insurance contracts are typically provided with guarantee. Therefore, the topic of pricing equity-linked life insurance contracts with guarantee has attracted most scholars’ attention. Brennan and Schwartz (1976), Boyle and Schwartz (1977) are the first papers appeared in this area. The authors decomposed the benefit of the contracts into a guaranteed

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Open access under CC BY-NC-ND license. doi:10.1016/j.sepro.2011.11.044
amount and a call (put) option on the reference portfolio, then they used Black-Scholes model to evaluate the contracts. Moreover, Moeller (1998, 2001) applied the mean-variance hedging method to calculate the price of the contracts. The guarantee of the contracts in all those papers is deterministic or fixed. Ekern and Persson (1996) priced the contracts with different guarantees, fixed and stochastic using fair pricing valuation. Kirch and Melnikov (2005), Melnikov and Romanyuk (2008) also applied efficient hedging method to price the equity-linked life insurance contracts with stochastic guarantee.

Quantile hedging technique, as an imperfect hedging technique, was developed in several publications by Foellmer and Leukert (1999), and we exploit further the most important paper on this topic. It can successfully hedge the option with maximal probability in the class of self-financing strategies with restricted initial capital. This technique has been proposed by Melnikov (2004) as pricing and hedging methodology for equity-linked life insurance contracts in the Black-Scholes framework. Later it was extended by Melnikov and Skornyakova (2005) to a two factor jump-diffusion model with constant interest rate, where the second risky asset could be considered as a stochastic guarantee for the contracts.

Up till now, many research papers in the area work with a constant interest rate $r$. However, as insurance products are usually long-term contract, they could be more sensitive to the changes in the interest rates. Therefore, it is more practical to consider a stochastic interest rate in the financial market. Gao, et al. (2010) considered the problem of pricing equity-linked life insurance contracts by means of quantile hedging and stochastic interest rate. They studied this topic in the framework of the Black-Scholes market model driven by two independent Wiener processes and a stochastic interest rate via HJM model (See [13]). The guarantee of the contracts in their study depends on a constant rate of return $g$ and time $t$.

It is well-known that discontinuous models for both the stochastic interest rate and the value of risky assets are more realistic. Extending the paper of Gao et al. (2010), we consider two risky assets $S^1$ and $S^2$ satisfying a two-factor jump-diffusion model, where the asset $S^2$ is less risky than $S^1$, and it can be seen as a stochastic guarantee of the equity-linked life insurance contract. We study the problem in the framework of Melnikov and Skornyakova (2005). But in contrast with that paper, we use a generalised HJM jump-diffusion model for the term structure of interest rate $r(t)$, which is similar to the framework of Shirakawa (1991), and Chiarella & Sklibosios (2003). Assuming independence of financial and insurance (mortality) risks, we apply quantile hedging to price equity-linked life insurance contracts with initial capital constraints.

The paper is structured as follows. In Section 2, we review jump-diffusion models and introduce the HJM term structure framework. Then we describe finance/insurance contracts under consideration. In Section 3, we briefly describe quantile hedging technique and present our main pricing results. Section 4 illustrates our results with a numerical example. In Section 5 some future work is discussed. Appendix A, B and C contain technical details of proofs.

2. The financial and insurance setting

2.1. Financial model

Let $(\Omega, F, (F_t)_{t \geq 0}, P)$ be a filtered probability space, where the filtration $(F_t)_{t \geq 0}$ satisfies the usual conditions and represents a flow of available information. It is supposed that all processes are adapted to this filtration. Considering a financial market with two risky assets $S^1$ and $S^2$, we use the same two factor jump-diffusion model as in Melnikov and Skornyakova (2005):

$$dS^i_t = S^i_{t-} \left( \mu_t dt + \sigma_t dW^i_t - \nu_t d\Pi_t \right), \quad i = 1, 2$$  \hspace{1cm} (2.1)
where $\mu_i \in R, \sigma_i > 0, \nu_i < 1$, and $W$, $\Pi$ are a standard Wiener and Poisson process with intensity $\lambda$ correspondently. We assume that the processes $W$ and $\Pi$ generate $(F)$ and $S^1$ is more riskier than $S^2$, or $\sigma_1 > \sigma_2$. Besides, all trades are assumed to take place in a frictionless market (no transaction costs or taxes, and short-sale allowed).

2.2. HJM framework

As in Chiarella and Sklibosios (2003), we consider a default-free bond market where arbitrary maturity bonds are traded continuously within a time horizon $[0,T]$ and denote $f(t,T)$ the instantaneous forward interest rate at time $t$ for instantaneous borrowing at time $T(\geq t)$. Let $P(t,T)$ be the price at time $t$ of a default-free discount zero-coupon bond with maturity $T$, i.e.

$$P(t,T) = \exp\left(-\int_t^T f(t,s) \, ds\right), \quad (2.2)$$

so that $P(T,T) = 1$.

The spot interest rate at time $t$, $(r(t))_{t \leq T}$, is given by the instantaneous forward rate, i.e.

$$r(t) = f(t,t) \quad (2.3)$$

Let $B(t)$ be the accumulated money account (a money market account starting with a dollar investment at time 0) and

$$B(t) = \exp\left(\int_0^t r(s) \, ds\right). \quad (2.4)$$

The stochastic differential equation for the instantaneous forward rate $f(t,T)$ driven by both Wiener and Poisson processes $W$ and $\Pi$ is given by

$$df(t,T) = \alpha(t,T) \, dt + \sigma(t,T) \, dW_t + \beta [d\Pi_t - \lambda \, dt] \quad (2.5)$$

where $\alpha : [0,T] \rightarrow R_+$ is the drift function, $\sigma : [0,T] \rightarrow R_+$ is the volatility function, $\lambda$ is the constant intensity for the Poisson process $\Pi_T$, $\beta$ is the constant jump size.

From (2.5), the forward rate $f(t,T)$ can be expressed as

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T) \, ds + \int_0^t \sigma(s,T) \, dW_s + \int_0^t \beta [d\Pi_s - \lambda \, ds], \quad (2.6)$$

where $f(0,T)$ is the given initial forward rate curve.

Putting $T = t$ in (2.6), we arrive at the stochastic integral equation for the instantaneous spot rate

$$r(t) = f(0,t) + \int_0^t \alpha(s,t) \, ds + \int_0^t \sigma(s,t) \, dW_s + \int_0^t \beta [d\Pi_s - \lambda \, ds] \quad (2.7)$$
From Itô’s lemma, the dynamics for the bond price $P(t,T)$ and the accumulated factor $B(t)$ are expressed as

\[ dP(t,T) = P(t,T) \left[ \left( \frac{r(t)}{2} - \alpha^*(t,T) \right) + \beta(T-t) \lambda + \left( \sigma^*(t,T) \right)^2 \right] dt \]
\[ -\sigma^*(t,T) dW_t + \left( \exp(-\beta(T-t)) - 1 \right) d\Pi_t \]  
(2.8)

and

\[ dB(t) = B(t) r(t) dt . \]  
(2.9)

where $\alpha^*(t,T) = \int_t^T \alpha(t,s) ds$, $\sigma^*(t,T) = \int_t^T \sigma(t,s) ds$.

2.3. Risk neutral dynamics

For a security market under consideration, one can determine conditions under which a unique equivalent martingale measure $P^*$ does exist. According to Melnikov et al. (2002), we can do this if

\[ \left( \mu_1 - r(t) \right) \sigma_2 - \left( \mu_2 - r(t) \right) \sigma_1 > 0, \quad \sigma_3 v_1 - \sigma_1 v_2 \neq 0 \]  
(2.10)

By Girsanov’s theorem, under this probability measure $P^*$, $W_t^* = W_t - \int_0^t \phi_s ds$ is a standard Wiener process and $\Pi_t$ is a Poisson process associated with intensity $\lambda_t^*$, and processes $W_t^*$ and $\Pi_t$ are two independent processes. The process $\phi_t$ can be interpreted as the market price of diffusion risk generated by Wiener process, while $\lambda_t^*$ represents the market price of jump risk generated by the Poisson process.

The risk neutral measure $P^*$ has a local density (See Shirakawa (1991), Melnikov and Skornyakova (2005)),

\[ Z_t = \frac{dP^*}{dP} \mid_t = \exp \left[ \int_0^t \phi_s dW_s - \frac{1}{2} \int_0^t \phi_s^2 ds + \int_0^t (\lambda_s^* - \ln \lambda_s^*) ds + \left( \ln\lambda^*_s - \ln\lambda_s^* \right) \Pi_t \right] \]  
(2.11)

The pair $(\phi_t, \lambda_t^*)$ satisfies the following equations:

\[ \begin{cases} 
\mu_1 - r(t) + \phi_s \sigma_1 - v_1 \lambda_t^* = 0 \\
\mu_2 - r(t) + \phi_s \sigma_2 - v_2 \lambda_t^* = 0 
\end{cases} \]  
(2.12)

Solving these equations, we get

\[ \phi_t = \frac{\left( \mu_1 - r(t) \right) v_2 - \left( \mu_2 - r(t) \right) v_1}{\sigma_2 v_1 - \sigma_1 v_2} \]  
(2.13)

\[ \lambda_t^* = \frac{\left( \mu_1 - r(t) \right) \sigma_2 - \left( \mu_2 - r(t) \right) \sigma_1}{\sigma_2 v_1 - \sigma_1 v_2} \]  
(2.14)
Under the risk neutral measure $P^*$, the dynamics of the forward interest rate $f(t, T)$ and spot interest rate $r(t)$ are in the form

$$
\begin{align*}
&f(t, T) = f(0, t) + \int_0^t \sigma(s, T) \sigma^*(s, T) ds + \int_0^t \sigma(s, T) dW^*_s \\
&\quad + \int_0^t \beta \lambda^*_s \left[ 1 - e^{-\beta(T-s)} \right] ds + \int_0^t \beta \left[ d\Pi_s - \lambda^*_s ds \right] \\
&= f(0, t) + \int_0^t \sigma(s, T) \sigma^*(s, T) ds + \int_0^t \sigma(s, T) dW^*_s \\
&\quad + \int_0^t \beta \lambda^*_s \left[ 1 - e^{-\beta(T-s)} \right] ds + \int_0^t \beta \left[ d\Pi_s - \lambda^*_s ds \right] \\
&= f(0, t) + \int_0^t \sigma(s, T) \sigma^*(s, T) ds + \int_0^t \sigma(s, T) dW^*_s \\
&\quad + \int_0^t \beta \lambda^*_s \left[ 1 - e^{-\beta(T-s)} \right] ds + \int_0^t \beta \left[ d\Pi_s - \lambda^*_s ds \right]
\end{align*}
$$

(2.15)

$$
\begin{align*}
&r(t) = f(0, t) + \int_0^t \sigma(s, T) \sigma^*(s, T) ds + \int_0^t \sigma(s, T) dW^*_s \\
&\quad + \int_0^t \beta \lambda^*_s \left[ 1 - e^{-\beta(t-s)} \right] ds + \int_0^t \beta \left[ d\Pi_s - \lambda^*_s ds \right] \\
&= f(0, t) + \int_0^t \sigma(s, T) \sigma^*(s, T) ds + \int_0^t \sigma(s, T) dW^*_s \\
&\quad + \int_0^t \beta \lambda^*_s \left[ 1 - e^{-\beta(t-s)} \right] ds + \int_0^t \beta \left[ d\Pi_s - \lambda^*_s ds \right]
\end{align*}
$$

(2.16)

In this circumstances, the asset price $S^i_t$, $i = 1, 2$ admits the following exponential form

$$
S^i_t = S^i_0 \exp \left\{ \sigma W^*_t + \left[ \mu_i - \frac{1}{2} \left( \sigma_i \right)^2 \right] t + \int_0^t \sigma_i \phi_i ds + \Pi_t \ln(1 - v_t) \right\}
$$

(2.17)

which is the solution of the stochastic differential equation (2.1).

Follow the approach of Amin and Jarrow (1992), we can write out the explicit representations of $B(t)$ and $S^i_t$ in terms of the parameters of the system:

$$
\begin{align*}
B(t) &= \frac{1}{P(0, t)} \exp \left\{ \frac{1}{2} \int_0^t \left( \sigma^*(s, T) \right)^2 ds + \int_0^t \left[ e^{-\beta(T-s)} - 1 \right] \lambda^*_s ds \right\} \\
&\quad + \int_0^t \sigma^*(s, T) dW^*_s - \int_0^t \beta(T-s)d\Pi_s \\
S^i_t &= S^i_0 B(t) \exp \left\{ \sigma W^*_t + \Pi_t \ln(1 - v_t) + \int_0^t \left( v_t \lambda^*_s - \frac{1}{2} \sigma^2_i \right) ds \right\}
\end{align*}
$$

(2.18)

(2.19)

2.4. Insurance settings

In this section, we work with an equity-linked life insurance contract, which is also called “pure endowment contract with a flexible guarantee”. The contract links the amount of benefit to both the financial assets $S^1_t$, $S^2_t$ and insurer’s life. The risky asset $S^1_t$ is responsible for the maximal size of future profits, while the asset $S^2_t$ provides a stochastic guarantee to the insured. The insurer does not receive any economic compensation for accepting mortality risk. Assuming financial and mortality risks as independent, we can hedge them separately (see Bacinello and Persson (2002)).

Let $T_x$ be a nonnegative random variable, defined on another probability space $(\tilde{\Omega}, \tilde{F}, \tilde{P})$. This random variable represents the remaining life time of an $x$-year old policyholder. Denote $p_x = \tilde{P}(T_x > t)$ the survival probability of this policyholder. It follows from the financial and mortality risk assumptions that $T_x$ is independent of all processes reflecting financial quantities.

We use $C(t)$ to denote a benefit payable at time $t$, which depends on the market value of $S^i_t$ and on the guaranteed value $S^i_t$, i.e.

$$
C(t) = \max \left( S^1_t, S^2_t \right).
$$

(2.20)
Taking the expected value of \( C(T) \cdot I\{T_x > T \} \) with respect to \( P^* \times \tilde{P} \), we can find the following initial price of the contract \( H(0) \):

\[
H(0) = E^* \left\{ \tilde{E} \left[ C(T) B^{-1}(T) I\{T_x > T \} \right] \right\} \\
= E^* \left[ C(T) B^{-1}(T) \right] \tilde{E} \left[ I\{T_x > T \} \right] \\
= E^* \left[ C(T) B^{-1}(T) \right] \cdot \tau P_s
\]

(2.21)

where \( I\{ \} \) is the indicator function. We call \( H(0) \) as the Brennan-Schwartz price.

3. Quantile hedging and valuation of equity-linked life insurance contract

3.1. Quantile hedging technique

The Quantile hedging technique is utilized when we cannot provide perfect hedge for the claim \( C(t) \), especially because of the initial budget constraint \( H(0) < C(0) \). Therefore, we would like to maximize the probability of successful hedging. Let \( A^* \) be the maximal set of successful hedging which is in the form \( A^* = \{ (dP / dP^*) \geq a \cdot C(T) B^{-1}(T) \} \), where \( a \) is a constant (see details in Follmer and Leukert (1999), Melnikov et al. (2002)). There exists a unique corresponding strategy \( \pi^* \) (quantile hedge) which becomes a perfect hedge for the modified claim \( C(T) I\{A^* \} \).

Taking into account (2.22), we find that

\[
H(0) = E^* \left[ C(T) B^{-1}(T) I\{A^* \} \right] = E^* \left[ C(T) B^{-1}(T) \right] \cdot \tau P_s
\]

and hence the survival probability \( \tau P_s \) has the following expression

\[
\tau P_s = \frac{E^* \left[ C(T) B^{-1}(T) I\{A^* \} \right]}{E^* \left[ C(T) B^{-1}(T) \right]}
\]

3.2. Application to equity-linked life insurance

In this section, we extend the quantile hedging approach to price the equity-linked life insurance contract with flexible guarantee and maturity time \( T \). We also get the expression of the survival probability.

**Theorem 1:** Consider a financial market model (2.1), and an equity-linked life insurance contract with payoff \( C(t) = \max \left( S_1^t, S_2^t \right) \). Then the Brennan-Schwartz price of the contract is

\[
H(0) = \sum_{n=0}^{\infty} P_{n,T}^* \left[ e^{\int_0^T \rho dt} \left(1-v_1\right)^n S_0^1 \Psi_1 \left( \Gamma_1, \Gamma_2, \rho, \delta_1, \delta_2 \right) \right. \\
+ \left. e^{\int_0^T \rho dt} \left(1-v_2\right)^n S_0^2 \Psi_2 \left( \Gamma_1, \Gamma_2, \tilde{\rho}, \tilde{\delta}_1, \tilde{\delta}_2 \right) \right]
\]

(3.1)
with \( P_{n,T}^* = e^{-\int_0^T \lambda^*_t dt} \left( \int_0^T \lambda^*_t dt \right)^n \) are the probabilities of a non-homogeneous Poisson distribution with intensity \( \lambda^*_t \); \( S_0^1, S_0^2 \) are the initial assets prices; \( \Psi_i (\cdot, \cdot), i=1,2 \), is the bivariate normal distribution function; \( \rho \) and \( \tilde{\rho} \) are correlation coefficients,

\[
\Gamma_1 = -\ln \left( \frac{a_n \cdot S_0^1 \left( \lambda^*_t \right)^n}{\lambda^n} \right) - \frac{1}{2} \delta^2_1 - \int_0^T \left( \lambda - \lambda^*_1 \right) ds, \\
\delta^2_1 = \int_0^T (\phi_1 + \sigma_1)^2 ds,
\]

\[
\Gamma_2 = \ln \left( \frac{S_0^1}{S_0^2} \right) (1-v_2) \left( \lambda^*_2 \right)^n - \frac{1}{2} \delta^2_2 - \int_0^T \left( \lambda - \lambda^*_2 \right) ds, \\
\delta^2_2 = \left( \sigma_2 - \sigma_1 \right)^2 T,
\]

\[
\Gamma_1 = -\ln \left( \frac{a_n \cdot S_0^2 \left( \lambda^*_t \right)^n}{\lambda^n} \right) - \frac{1}{2} \delta^2_1 - \int_0^T \left( \lambda - \lambda^*_1 \right) ds, \\
\delta^2_1 = \int_0^T (\phi_1 + \sigma_1)^2 ds,
\]

\[
\Gamma_2 = \ln \left( \frac{S_0^2}{S_0^1} \right) (1-v_1) \left( \lambda^*_2 \right)^n - \frac{1}{2} \delta^2_2 - \int_0^T \left( \lambda - \lambda^*_2 \right) ds, \\
\delta^2_2 = \left( \sigma_2 - \sigma_1 \right)^2 T,
\]

\[
\lambda^1_t = \lambda^*_1 (1-v_1), \quad \lambda^2_t = \lambda^*_2 (1-v_2).
\]

**Proof:** See Appendix A.

**Remark 3.1:** To calculate the price of the equity-linked life insurance contract \( H(0) \), we can also use the “multi-asset theorem” from Melnikov and Romanyuk (2008) (see Appendix B), and find that this approach leads to the same result as that in Theorem 1.

**Remark 3.2:** The payoff of the equity-linked life insurance contract with flexible guarantee \( C(t) \) can be decomposed into the payoff of a European exchange option plus a pure equity-linked life insurance contract:

\[
C(t) = \max \left( S^1_t, S^2_t \right) = \left( S^1_t - S^2_t \right)^+ + S^2_t
eq
\]

It gives a possibility to reduce the valuation of initial contract to the embedded exchange option \( \left( S^1_t - S^2_t \right)^+ \), building up the maximal successful heding set \( A^* \) for it (see Melnikov and Skornyakova (2005)).

**Theorem 2:** Consider a financial market model (2.1), and an equity-linked life insurance contract with payoff \( C(t) = \max \left( S^1_t, S^2_t \right) \). The survival probability of an insured is as following:
\[
\tau P_x = \sum_{n=0}^\infty p^n \left[ p \Psi_n (\Gamma_1, \Gamma_2, \rho, \delta, \tilde{\delta}) + q \Psi_n (\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\rho}, \tilde{\delta}, \tilde{\delta}) \right]
\]
\[
= \sum_{n=0}^\infty p^n \left[ p \Phi \left( \frac{d_1}{(\sigma_1 - \sigma_2)\sqrt{T}} \right) + q \Phi \left( \frac{d_2}{(\rho_1 - \rho_2)\sqrt{T}} \right) \right]
\]
\[(3.4)\]

where the notations \( p^n \) and \( \Psi_i (\cdot, \cdot) \), \( i = 1, 2, \Gamma_1, \Gamma_2, \rho, \delta, \tilde{\rho}, \tilde{\delta} \), \( \tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\delta}, \tilde{\delta} \), \( \tilde{\rho} \) are the same as in Theorem 1. \( \Phi \) denotes the cumulative distribution function of the standard normal distribution, and other notations \( d_1, d_2, p, q \) are given by:

\[
d_1 = \ln \frac{S^0_0 (1-v_1)}{S^0_0 (1-v_2)} - \int_0^T \lambda^2_s (v_2 - v_1) ds + \frac{1}{2} \sigma_2^2 , \quad p = e^{-\int_0^T \lambda^2_s ds} (1-v_1)^n S^0_0 ,
\]

\[
d_2 = \ln \frac{S^0_0 (1-v_2)}{S^0_0 (1-v_1)} - \int_0^T \lambda^2_s (v_1 - v_2) ds + \frac{1}{2} \sigma_2^2 , \quad q = e^{-\int_0^T \lambda^2_s ds} (1-v_2)^n S^0_0 .
\]

**Proof:** See Appendix C.

**Remark 3.3:** Using quantile hedging technique for risk management of equity-linked life insurance contracts, we can fix the financial risk level \( \varepsilon, \varepsilon > 0 \), and find the probability of the successful hedging set as \( 1 - \varepsilon \). Applying the formula (3.1), we obtain the price of the contract, and using the formula (3.4) we determine the survival probability \( p_x \). After that, based on available life tables (see [21]), we can find the age of the corresponding clients.

### 4. Numerical example

In this section, we give a numerical example to illustrate how to use quantile hedging technique to price an equity-linked life insurance contract. First, we estimate the parameters in both interest rate model and two-factor jump-diffusion model. Then, we specify the structure of maximal set of successful hedging. Furthermore, survival probabilities and ages of the clients are calculated for different financial risk level.

#### 4.1. Specification of parameters

We consider a simplified stochastic interest rate model without its jump component: one factor Vasicek-Hull-White model. As in Gao et al. (2010), we set

\[
\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s) ds
\]

where \( \alpha(t,T) \) is the mean rate of return. We also assume the volatility structure \( \sigma(t,T) \) satisfies

\[
\sigma(t,T) = \beta \exp(-\alpha(T-t))
\]
where $\alpha > 0, \beta > 0$. This expression leads to one factor Vasicek-Hull-White model so that the dynamics of the instantaneous spot rate $r(t)$ is

$$dr(t) = \alpha (m(t) - r(t))dt + \beta dW_t$$

where $m(t) = f_0 + \eta + \eta t + \frac{\beta^2}{2\alpha} (1 - e^{-2\alpha t})$ by setting $f(0,t) = f_0 + \eta t$.

We assume $f_0 = 0.01$, $\eta = 0$ and use maximum likelihood estimate method to specify the constant parameters $\alpha$ and $\beta$. In this paper, we work with one month deposit rate data from September, 1987 till September, 2010.

For the two-factor jump-diffusion model, we apply the approach in Mancini (2004) to estimate the parameters. There is one Poisson process in our model which determines jumps in the prices for two assets. However, in Mancini’s paper there is one Poisson process which specifies jumps for only one asset. So we modify the estimator of number of jumps in Mancini’s approach slightly.

We consider financial index Russell 2000 (RUT-I) as risky asset $S^1$, and S&P 500 as risky asset $S^2$. As Russell 2000 measures the performance of small US companies, whereas S&P 500 is the index of the prices of 500 large-cap common stocks traded in US, it is supposed that RUT-I is more risky than S&P 500. Therefore, it is reasonable to consider S&P 500 as the flexible guarantee $S^2$. Using monthly observations of prices over 23 years from September 1987 to September 2010, we can estimate the parameters of the two-factor jump diffusion model as following:

$$\mu_1 = 0.2763, \sigma_1 = 0.19, \nu_1 = -0.27,$$

$$\mu_2 = 0.2898, \sigma_2 = 0.15, \nu_2 = -0.2, \lambda = 0.17$$

The initial indices of Russell 2000 and S&P 500 are 167.44 and 329.81. In order to make the initial values of two assets $S^1$, $S^2$ the same, we change the value of $S^1$ as $\frac{329.81}{167.44}$.

### 4.2. Structure of maximal hedging set

According to proof of Theorem 1, we can see that the maximal set of successful hedging $A^* = \left\{ a_n Z_T \middle| \frac{C(T)}{B(T)} \right\}$ admits two types of expression:

If $\Delta_1 > \Delta_2$, $A^* = \{ Y^* < \Delta_1 \}$, otherwise $A^* = \{ Y^* < \Delta_2 \}$, where variable $Y^*$ follows standard normal distribution

and

$$\Delta_1 = \frac{-\ln \left( a \cdot S_0 \left( \lambda_1^n \right)^n / \lambda_1 \right) - \frac{1}{2} \delta_1^2 - \int_0^T (\lambda - \lambda_1) ds}{\delta_1}, \quad \Delta_2 = \frac{-\ln \left( a \cdot S_0 \left( \lambda_2^n \right)^n / \lambda_2 \right) - \frac{1}{2} \delta_2^2 - \int_0^T (\lambda - \lambda_2) ds}{\delta_1}.$$

A sequence of constant $a_n$ can be found by fixing the probability of the set of successful hedging as $P(A^* | \pi_T = n) = 1 - \varepsilon = \Phi(\Delta_i), \ i = 1, 2$. Then we can use the log-normality of this conditional distribution to estimate constants $a_n$.

### 4.3. Numerical results

For the contracts with flexible guarantee, we choose the initial investment $S_0 = 1000$, and terms of the contracts $T = 1, 3, 5, 10, 15, 20$ years. Then we use the formula from Theorem 2 to calculate the Survival
probability $T_p_x$ for different levels of financial risk $\varepsilon = 0.01, 0.025, 0.05$. The results are displayed in Table 1.

Table 1: Survival probabilities with flexible guarantee

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\varepsilon=0.01$</th>
<th>$\varepsilon=0.025$</th>
<th>$\varepsilon=0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9885</td>
<td>0.9718</td>
<td>0.9447</td>
</tr>
<tr>
<td>3</td>
<td>0.9878</td>
<td>0.9705</td>
<td>0.9426</td>
</tr>
<tr>
<td>5</td>
<td>0.9874</td>
<td>0.9697</td>
<td>0.9413</td>
</tr>
<tr>
<td>10</td>
<td>0.9867</td>
<td>0.9684</td>
<td>0.9391</td>
</tr>
<tr>
<td>15</td>
<td>0.9859</td>
<td>0.9667</td>
<td>0.9364</td>
</tr>
<tr>
<td>20</td>
<td>0.9853</td>
<td>0.9656</td>
<td>0.9345</td>
</tr>
</tbody>
</table>

We determine the corresponding clients’ age using the 2005 United States life table (See [20]), the results are presented in Table 2.

Table 2: Age of insured with flexible guarantee

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\varepsilon=0.01$</th>
<th>$\varepsilon=0.025$</th>
<th>$\varepsilon=0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>62</td>
<td>73</td>
<td>79</td>
</tr>
<tr>
<td>3</td>
<td>49</td>
<td>59</td>
<td>68</td>
</tr>
<tr>
<td>5</td>
<td>41</td>
<td>52</td>
<td>61</td>
</tr>
<tr>
<td>10</td>
<td>31</td>
<td>41</td>
<td>50</td>
</tr>
<tr>
<td>15</td>
<td>22</td>
<td>34</td>
<td>42</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>28</td>
<td>36</td>
</tr>
</tbody>
</table>

In order to compare with the results in Melnikov and Skornyakova (2005), we also use the same life table in [21] to get the age of the clients, shown in Table 3.

Table 3: Age of insured with flexible guarantee

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\varepsilon=0.01$</th>
<th>$\varepsilon=0.025$</th>
<th>$\varepsilon=0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>58</td>
<td>69</td>
<td>78</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
<td>55</td>
<td>63</td>
</tr>
<tr>
<td>5</td>
<td>39</td>
<td>48</td>
<td>56</td>
</tr>
<tr>
<td>10</td>
<td>23</td>
<td>39</td>
<td>46</td>
</tr>
<tr>
<td>15</td>
<td>12</td>
<td>31</td>
<td>39</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>24</td>
<td>33</td>
</tr>
</tbody>
</table>

Based on the above results, we observe that as the insurance company’s financial risk level $\varepsilon$ increases, or the probability of successful hedging $1-\varepsilon$ decreases, the survival probability
In this paper, we generalized the results by Melnikov and Skornyakova (2005) and Gao, et al. (2010) on pricing equity-linked life insurance. We choose the two-factor jump-diffusion model and generalized HJM model in our study in order to better describe the real financial market. The presence of mortality risk usually causes budget constraint on a hedge and makes it impossible for insurance companies to exactly replicate the payoff of a contract. Thus, we apply the quantile hedging technique to price the contracts when the perfect hedging is impossible.

A natural extension of this work is to consider non-constant parameters in assets’ models, for example, a stochastic volatility $\sigma(t)$. Besides stochastic interest rate $r(t)$, stochastic volatility $\sigma(t)$ is another important factor in pricing long-term equity-linked life insurance contracts. It is possible to incorporate both factors into the jump-diffusion model. In this paper, we only consider a single premium contract. Further, we could work with a periodic premium contract. Such contract could be an equity-linked endowment insurance policy with asset value guarantee with periodic premiums, where the buyer is committed to pay regularly a predetermined premium to the insurance company.

Acknowledgment

This paper was supported by NSERC grant # 261855.

References


Appendix A. Proof of Theorem 1

Conditioning on each set \( \{\Pi_T = n\} \), \( n = 1, 2, \ldots \), We can decompose the initial price \( H(0) \) into two parts \( H_1(0), H_2(0) \).

\[
H(0) = E^* \left[ C(T) B^{-1}(T) I_{\{\mathcal{F}_T^+\}} \right] = E^* \left[ \max\left( S_T^1, S_T^2 \right) \right] \frac{B(T)}{B(T)} \left[ I_{\{1 \geq a_T Z_T \}} \max\left( S_T^1, S_T^2 \right) \frac{B(T)}{B(T)} \right]
\]

\[
= H_1(0) + H_2(0)
\]

Then, we can use similar approach to calculate \( H_1(0), H_2(0) \) separately. We calculate \( H_1(0) \) first.

\[
H_1(0) = E^* \left[ \frac{S_T^1}{B_T} I_{\{1 > a_T Z_T \}} \frac{S_T^2}{B(T)} \left[ I_{\{S_T^1 > S_T^2\}} \right] \right]
\]

where \( Z_T, B(T), S_T^i \) satisfy the risk neutral dynamics (2.11), (2.19), (2.20).

By the change of measure approach, we can define another measure \( Q_1 \) such that

\[
\frac{dQ_1}{dP}\bigg|_{T} = \exp \left[ \sigma_T W_T^* + \Pi_T \ln \left( 1 - v_1 \right) + \int_0^T v_1 d\lambda_T^* - \frac{1}{2} \sigma_T^2 \right] ds
\]
Under measure \( Q \), \( \tilde{W}^1_t = W^* - \sigma T \) is another Wiener process, and \( \Pi_t \) is the Poisson process with a new intensity \( \lambda^*_t = \lambda_t (1 - v_1) \).

So on each set \( \Pi_T = n \), \( n = 1, 2, \ldots, \) we arrive to

\[
H_1(0) = S_0^1 E^{Q_1}\left\{ I \left[ 1 > a_n \exp\left[ \int_0^T \phi_s d\tilde{W}^1_s + \int_0^T \phi_s \sigma T ds = \frac{1}{2} \left( \int_0^T \phi^2_s ds + \int_0^T \lambda_t - \lambda^*_t \right) ds + n \ln \frac{\lambda^*_t}{\lambda_t} \right] \right] \right. \\
-S_0^1 \cdot \exp\left[ \sigma_1 \tilde{W}^1_T - \frac{1}{2} \sigma^2_1 T + n \ln (1 - v_1) + \int_0^T \left( v_1 \lambda^*_s - \frac{1}{2} \sigma^2_1 \right) ds \right] \\
\cdot I\left[ \frac{S_0^1}{S_0^1} > \exp\left[ (\sigma_2 - \sigma_1) \tilde{W}^1_T + n \ln (1 - v_2) + \int_0^T \left( v_2 \lambda^*_s - \frac{1}{2} \sigma^2_2 \right) ds + \sigma_1 \sigma_2 T \right. \\
- \sigma^2_1 T - n \ln (1 - v_1) - \int_0^T \left( v_1 \lambda^*_s - \frac{1}{2} \sigma^2_1 \right) ds \right] \right) \right) \\
= S_0^1 E^{Q_1}\left\{ I \left[ -\ln \left( a_n \cdot S_0^1 \right) \geq \int_0^T \left( \phi_s + \sigma_1 \right) d\tilde{W}^1_s + n \left( \ln \lambda^*_T - \ln \lambda_T \right) \right. \\
+ \int_0^T \left[ \frac{1}{2} \left( \phi_s + \sigma_1 \right)^2 + \lambda_t - \lambda^*_t \right] ds \right] \right. \\
\cdot I\left[ \ln \left( \frac{S_0^1}{S_0^2} \right) > (\sigma_2 - \sigma_1) \tilde{W}^1_T + n \ln \frac{1 - v_2}{1 - v_1} + \int_0^T \left( \lambda^*_s (v_2 - v_1) - \frac{1}{2} (\sigma_2 - \sigma_1)^2 \right) ds \right] \right) \right) \right) \\
\]

Define \( y_1 = \int_0^T \left( \phi_s + \sigma_1 \right) d\tilde{W}^1_s \), \( y_2 = (\sigma_2 - \sigma_1) \tilde{W}^1_T \).

It is obvious that \( y_1, y_2 \) follow normal distribution under measure \( Q \),

\( y_1 \sim N \left( 0, \sigma^2_1 T \right) \), \( y_2 \sim N \left( 0, \sigma^2_2 T \right) \)

where \( \sigma^2_1 = \int_0^T \left( \phi_s + \sigma_1 \right)^2 ds \), \( \sigma^2_2 = (\sigma_2 - \sigma_1)^2 T \).

For any constants \( k_1, k_2 \neq 0 \), the linear combination of \( y_1, y_2 \) is

\[
k_1 y_1 + k_2 y_2 = \int_0^T k_1 \left( \phi_s + \sigma_1 \right) d\tilde{W}^1_s + k_2 \left( \sigma_2 - \sigma_1 \right) \tilde{W}^1_T \\
= \int_0^T \left[ k_1 \phi_s + \left( k_1 - k_2 \right) \sigma_1 + k_2 \sigma_2 \right] d\tilde{W}^1_s \]

Clearly, the above linear combination is still a normal random variable. So the random vector \( \left( y_1, y_2 \right) \) is normally distributed with mean equals \( \left( 0, 0 \right) \), and the correlation between \( y_1, y_2 \) is

\[
\rho = \int_0^T \left( \phi_s + \sigma_1 \right) \left( \sigma_2 - \sigma_1 \right) ds .
\]
We can obtain
\[
H_1(0) = S_0^i E^{Q_0} \left\{ \left[ y_1 \leq -\ln \frac{a_n \cdot S_0^i \left( \lambda^1_T \right)^n}{\lambda^n} - \frac{1}{2} \delta_1^2 - \int_0^T \left( \lambda - \lambda^1_x \right) ds \right] + \int \left[ y_2 < \ln \frac{S_0^i (1 - v_1)^n}{S_0^2 (1 - v_2)^n} - \int_0^T \lambda^1_x \left( v_2 - v_1 \right) ds + \frac{1}{2} \delta_2^2 \right] \right\}
\]
\[
= S_0^i \Omega_1 \left\{ y_1 \leq -\ln \frac{a_n \cdot S_0^i \left( \lambda^1_T \right)^n}{\lambda^n} - \frac{1}{2} \delta_1^2 - \int_0^T \left( \lambda - \lambda^1_x \right) ds \right\},
\]
\[
y_2 < \ln \frac{S_0^i (1 - v_1)^n}{S_0^2 (1 - v_2)^n} - \int_0^T \lambda^1_x \left( v_2 - v_1 \right) ds + \frac{1}{2} \delta_2^2 \right) \right\} = S_0^i \Psi \left( \Gamma_1, \Gamma_2; \rho, \delta_1, \delta_2 \right)
\]

where \( \Psi (\cdot, \cdot) \) is the bivariate normal distribution function with probability density function
\[
\varphi = \frac{1}{2\pi \delta_1 \delta_2 \sqrt{1 - \rho^2}} \exp \left[ -\frac{1}{2 \left( 1 - \rho^2 \right)} \left( \frac{y_1^2}{\delta_1^2} - \frac{2 \rho y_1 y_2}{\delta_1 \delta_2} + \frac{y_2^2}{\delta_2^2} \right) \right].
\]

We can also obtain the maximal set of successful hedging \( A_2^* \),
\[
A^* = \left\{ y_1 \leq -\ln \frac{a_n \cdot S_0^i \left( \lambda^1_T \right)^n}{\lambda^n} - \frac{1}{2} \delta_1^2 - \int_0^T \left( \lambda - \lambda^1_x \right) ds \right\}.
\]

We now turn to \( H_2(0) \). The calculation of \( H_2(0) \) can be treated in a similar way. Again, we can define another measure \( Q_2 \)
\[
\left. \frac{dQ_2}{dP} \right|_{\mathcal{F}_t} = \exp \left[ \sigma_2 \tilde{W}_t^2 + \Pi_t \ln (1 - v_2) + \int_0^T \left( v_2 \lambda^1_x - \frac{1}{2} \sigma_2^2 \right) ds \right].
\]

Under this new measure \( Q_2 \), \( \tilde{W}_t^2 = W_t^* - \sigma_2 T \) is another Wiener process, and \( \Pi_t \) is a Poisson process with new intensity \( \lambda^2_t = \lambda^1_t \left( 1 - v_2 \right) \).

On each set \( \left\{ \Pi_T = n \right\}, \ n = 0, 1, 2, \ldots \), we can obtain
\[ H_2(0) = S_0^2 E^{Q_2} \left\{ \int \left[ -\ln \left( a \cdot S_0^2 \right) \right] \geq \int_0^T \left( \phi_s + \sigma_s \right) d\hat{W}_s^2 + n \left( \ln \lambda_T^2 - \ln \lambda \right) \right. \]
\[ + \int_0^T \frac{1}{2} \left( \phi_s + \sigma_s \right)^2 \left( \lambda - \lambda_s^2 \right) ds \left. \right\} \cdot I \left[ \ln \left( \frac{S_0^2}{S_0^2} \right) \geq (\sigma_1 - \sigma_2) \hat{W}_T^2 + n \ln \frac{1-v_1}{1-v_2} + \int_0^T \lambda_s^* (v_1 - v_2) - \frac{1}{2} (\sigma_2 - \sigma_1)^2 ds \right] \}

Let us define new normal random variables
\[ \tilde{y}_1 = \int_0^T (\phi_s + \sigma_s) d\hat{W}_s^2, \quad \tilde{y}_2 = (\sigma_1 - \sigma_2) \hat{W}_T^2 \]
where \( \tilde{y}_1 \sim N(0, \delta_1^2) \), \( \delta_1^2 = \int_0^T (\phi_s + \sigma_s)^2 ds \); \( \tilde{y}_2 \sim N(0, \delta_2^2) \), \( \delta_2^2 = (\sigma_2 - \sigma_1)^2 T \).

The random vector \( \tilde{y}_1, \tilde{y}_2 \) is normally distributed with mean equals to \((0, 0)\). Correlation between \( \tilde{y}_1, \tilde{y}_2 \) is \( \rho = \int_0^T (\phi_s + \sigma_s)(\sigma_1 - \sigma_2) ds \), by checking the linear combination of \( \tilde{y}_1, \tilde{y}_2 \), which is still normal distributed.

Then, we arrive to
\[ H_2(0) = S_0^2 Q_2 \left\{ \tilde{y}_1 \leq -\ln \left( a \cdot S_0^2 \frac{\lambda_T^2}{\lambda} \right)^a - \frac{1}{2} \delta_1^2 - \int_0^T (\lambda - \lambda_s^2) \right. \]
\[ \left. \tilde{y}_2 < \ln \frac{S_0^2 (1-v_2)^n}{S_0^2 (1-v_1)^n} - \int_0^T \lambda_s^* (v_1 - v_2) ds + \frac{1}{2} \delta_2^2 \right\} = S_0^2 \Psi (\tilde{\Gamma}_1, \tilde{\Gamma}_2; \tilde{\rho}, \tilde{\delta}_1, \tilde{\delta}_2) \]

Also, the maximal set of successful hedging \( A^* \) is equal to
\[ A^* = \left\{ \tilde{y}_1 \leq -\ln \left( a \cdot S_0^2 \frac{\lambda_T^2}{\lambda} \right)^a - \frac{1}{2} \delta_1^2 - \int_0^T (\lambda - \lambda_s^2) ds \right\} \]

Finally, we combine the results (3.2) and (3.3), and obtain the expression of \( H(0) \)
\[ H(0) = H_1(0) + H_2(0) \]
\[ = \sum_{n=0}^{\infty} e^{-\int_0^T \lambda_t dt} \left( \int_0^T \lambda_t^2 dt \right)^n S_0^2 \Psi (\Gamma_1, \Gamma_2; \rho, \delta_1, \delta_2) \]
\[ + \sum_{n=0}^{\infty} e^{-\int_{0}^{T} \lambda^*_t \, dt} \left( \int_{0}^{T} \lambda^*_t \, dt \right)^n \frac{\Psi \left( \Gamma_1, \Gamma_2; \bar{\rho}, \bar{\delta}_1, \bar{\delta}_2 \right)}{n!} S_0^2 \]

\[ = \sum_{n=0}^{\infty} \left[ e^{-\int_{0}^{T} \lambda^*_t \, dt} \left( 1 - v_1 \right)^n S_0^1 \Psi \left( \Gamma_1, \Gamma_2; \rho, \delta_1, \delta_2 \right) + e^{-\int_{0}^{T} \lambda^*_t \, dt} \left( 1 - v_2 \right)^n S_0^2 \Psi \left( \Gamma_1, \Gamma_2; \bar{\rho}, \bar{\delta}_1, \bar{\delta}_2 \right) \right] \]

where \( p^*_{n,T} = e^{-\int_{0}^{T} \lambda^*_t \, dt} \left( \int_{0}^{T} \lambda^*_t \, dt \right)^n \) are the probabilities of a non-homogeneous Poisson distribution with intensity \( \lambda^*_t \).