GLOBAL MATRIX FORMULATION OF WAVE PHENOMENA IN PLANE LAYERED MEDIA

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ABSTRACT

A direct global matrix formulation for computing the multilayer Green's impulse response function in media with arbitrary depth-dependent elastic or fluid properties is reviewed, after recent work by Schmidt. The global Green's function matrix for solving the resulting linear system of elastodynamic equations is formally analogous to the global stiffness matrix in the direct Finite Element Method and proves both computationally efficient and numerically stable. This analogy and the matrix solution are discussed, as embodied in the general applications code, "SAFARI" ("SAclant FAst Field Program for Range-Independent Environments"), developed by one of us (H. Schmidt). Cross-disciplinary applications of this direct global matrix algorithm are considered for three pertinent areas of current research; namely underwater acoustic propagation, ultrasonic scattering, and crustal seismic modeling. Results obtained show that the SAFARI model correctly reproduces the complete wavefield response in plane-layered, welded, homogeneous media. Arbitrary source/receiver numbers, frequencies, locations, and geometries can also be considered over much larger-scale multilayered media models than previously feasible with extant propagator matrix-based codes. The physical simplicity, CPU savings, and flexible performance of the direct global matrix algorithm suggests that many more full wavefield phenomena can now be simulated without excessively complicated, specialized, or expensive modeling schemes.

1. INTRODUCTION

In analyzing the continuous or transient response of a system comprising arbitrary homogeneous isotropic layers in welded contact, closed-form analytic solution for the multilayer/multisource Green's function and its transform have traditionally not been available. For this reason, a direct numerical solution of wide generality has been repeatedly sought. Since the first systematic studies of wave propagation in layered media (Ewing et al., 1957), repeated efforts have been accorded to the problem of efficiently computing the correct and complete (fullwave) Green's impulse response characterizing the medium. In theoretical and applied mechanics, underwater acoustics, and seismology, several popular local recursive propagator matrix methods have been computerized for specific applications (DiNapoli, 1971; Kutschale, 1971; Raspit et al., 1985; Franssens, 1984; Kennett, 1974). When solving the monofrequency impulse response of a very small-layer (Fig. 1) system, for single source and receiver configurations, the Thomson-Haskell propagator formulation has been widely applied. However, as is now well known (Chin et al., 1984), for large numbers of elastic-layers where thicknesses exceed source wavelength, numeric instabilities in the matricant exponentials require more complex computational schemes. Furthermore, for multisource and multifrequency pulse modeling, numeric implementation of these integral Green's function solutions becomes computationally prohibitive, since the recursion must be repeated for each layer, frequency, and/or receiver depth (Fig. 2).

As an improved alternative, recently Chin et al. (in seismology) have formally outlined a nonrecursive global matrix approach to Green's function computations. In this paper, we review a previous and independently derived direct global matrix technique similar to, but simpler and more general than that proposed by Chin, as actually implemented by Schmidt. The new direct global approaches are shown to be a more general and flexible numeric implementation of the computational possibilities inherent in the original Green's function solutions of Ewing et al. (1957). Now, however, efficient numerical techniques adopted from modern finite element algorithms are employed.

2. DERIVATION OF FIELD EQUATIONS

The integral transform field representation used by Schmidt closely follows the classic presentation given by Ewing et al. (1957). The medium is assumed to be plane-stratified isotropic and homogeneous viscoelastic continua with Lame' constants $\lambda_i$ and $\mu_i$, and density $\rho_i$, where the index subscript $i$ refers to layer number (Fig. 3).
The field equations are given here in axisymmetric cylindrical geometry of \( \{r, \theta, \phi\}\). In the absence of body forces, the equations of motion are satisfied if the displacement components \(\{u, v, w\}\) in each layer are expressed via scalar displacement potentials \(\{\phi, \psi\}\) as

\[
u(r, \theta, \phi) = \frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial \theta}
\]

\[
u(r, \theta, \phi) = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial \phi}
\]

with the potentials satisfying the Helmholtz wave equations

\[
\nabla^2 - \frac{1}{\nu^2} \frac{\partial^2}{\partial \theta^2} \phi_j = 0 : \quad \psi_j = \sqrt{\lambda_j + 2 \mu_j} / \alpha_j .
\]

We assume a compressional point source. The source or sources are assumed harmonic with frequency \(\omega\); the steady state term \(\exp(-\mathrm{i}\omega t)\) common to all field variables is omitted. Viscoelastic attenuation is introduced via complex velocities. The wave equations above now become

\[
\nabla^2 + k_p^2 \phi_j = 0 \quad \nabla^2 + k_s^2 \psi_j = 0.
\]

where \(k_p\) and \(k_s\) are compressional and shear wave numbers

\[
k_p^2 = \frac{\omega^2 \alpha_j}{\lambda_j + 2 \mu_j} \quad k_s^2 = \frac{\omega^2 \beta_j}{\mu_j}.
\]

Applying Fourier transforms to the above yields the following general integral transform solutions for layer \(j\)

\[
\phi_j(r, \theta, \phi) = \int \left[ A_r^j(k_r) \exp(-\mathrm{i}k_r \rho) + \lambda_r^j(k_r) \exp(\mathrm{i}k_r \rho) \right] J_\nu(k_r) \, dk_r,
\]

\[
\psi_j(r, \theta, \phi) = \int \left[ B_r^j(k_r) \exp(-\mathrm{i}k_r \rho) + B_r^j(k_r) \exp(\mathrm{i}k_r \rho) \right] J_\nu(k_r) \, dk_r,
\]

where \(A_r^j\) and \(B_r^j\) are the arbitrary degrees of freedom in the horizontal wavenumber \(k_r\) to be solved for, with

\[
\alpha_j(k_r) = \sqrt{k_r^2 - k_p^2} \quad \beta_j(k_r) = \sqrt{k_r^2 - k_s^2}.
\]

By substituting the \(P\) and \(S\) scalar potentials into the displacement equations of motion, and by applying Hooke's law, the following integral representations are obtained for the field parameters involved in the boundary conditions (Fig. 4):

\[
\begin{bmatrix}
\{u(r, \theta, \phi)\}
\{v(r, \theta, \phi)\}
\{\alpha_{\phi}(r, \theta, \phi)\}
\{\sigma_{\phi}(r, \theta, \phi)\}
\end{bmatrix}_j = \int_0^\infty [C(k_r) j] \{a(k_r)\} \, dk_r
\]

Here, \(\{a(k_r)\}\) is a diagonal matrix with Bessel function \(J_\nu\) in diagonal elements corresponding to \(W\) and \(\alpha_{\phi}\), and \(j_1\) for \(u\) and \(\alpha_{\phi}\), \([C(k_r)]_j\) is the depth-dependent coefficient matrix, which can be factored as

\[
[C(k_r)]_j = [d(k_r)] \{a(k_r)c_r(k_r)\}_j.
\]

\([d(k_r)]_j\) contains only simple functions of horizontal wavenumber, and \([a(k_r)c_r(k_r)]_j\) is a diagonal matrix containing the four transfer exponentials in the scalar potentials. \(\{a(k_r)c_r(k_r)\}_j\) is the vector containing the four unknown \(P\) and \(S\) wavefield amplitudes for layer \(j\) (Fig. 5).
\[ 
\begin{align*}
\psi^*(z) &= \int_0^\infty \frac{e^{-|z-z_0|/\alpha_j}}{\alpha_j} J_0(kr) \, r \, dk \\
\text{SO THAT SOURCE TERMS ARE} \\
\begin{bmatrix}
W_{\text{TOTAL}} \\
U_{\text{TOTAL}} \\
\sigma_{\text{TOTAL}} \end{bmatrix} &= \int_0^\infty \begin{bmatrix}
\hat{V}(k,z) \\
\hat{J}(k,r) \\
\end{bmatrix} \begin{bmatrix}
\sigma \omega \alpha \\
\end{bmatrix} \, r \, dk \\
\text{(MULTIPLE SOURCES OBTAINED BY SUMMATION)} \\
\text{TOTAL FIELD} &= \text{HOMOGENEOUS} + \text{SOURCE CONTRIBUTIONS} \\
W_{\text{TOTAL}} &= W_{\text{HOMOGENEOUS}} + W_{\text{SOURCE}} \\
U_{\text{TOTAL}} &= U_{\text{HOMOGENEOUS}} + U_{\text{SOURCE}} \\
\sigma_{\text{TOTAL}} &= \sigma_{\text{HOMOGENEOUS}} + \sigma_{\text{SOURCE}} \\
\end{align*}
\]

To obtain expressions for the total field in layer \( j \), the inhomogeneous field produced by one (or more) sources within each layer must be superposed. The Sommerfeld-Weyl source representation here is

\[ 
\phi^*(r,z) = \int_{-\infty}^{\infty} \frac{e^{-|z-z_0|/\alpha_j}}{\alpha_j} J_0(kr) \, dk, \quad \psi^*(r,z) = 0
\]

where \( z_j \) is the source depth(s) in layer \( j \). For multiple sources in one or more layers the source terms are replaced by a summation over all sources. The wavefield at each interface \( j \) now has two distinct integral representations, from the interface above and below. Satisfying the classical boundary conditions of continuity for all interfaces simultaneously leads directly to a linear system of \( 4N-2 \) equations in the \( 4N-2 \) unknown wave amplitude functions \( A \pm \) and \( B \pm \). As noted, this system can, of course, be solved analytically, as was historically done for particular few-layered cases. However, the transforms do not have analytic closed forms; moreover, when treating a general multilayer/multisource case, numerical solutions are more convenient. The numeric solution and subsequent integral evaluation are treated in Sections 3 and 4.

3. NUMERICAL SOLUTION TECHNIQUE; DIRECT GLOBAL MATRIX METHOD

As in all synthetic seismic methods based on full wave integral transform techniques, solution for the complete wavefield is divided into two parts. First, the unknown wave amplitude functions are found at discrete horizontal wavenumbers from the (global) system of equations expressing the boundary conditions. Second, the field quantities are found at desired depths and ranges by numerically evaluating the integral transforms. The initial Green's function computations here (and traditionally) have been the most time consuming. This latter is precisely where the present direct global matrix approach differs from propagator matrix methods.

As outlined, one can avoid repetitive application of difficult and time-consuming recursive propagators, and obtain solutions for the com-

Figure 4. Boundary conditions and total wavefield solution by superposition.

Figure 5. Direct global coefficient matrix structure and solution.

Figure 6. Sparsity pattern due to analogy of finite element connectivity with direct global matrix continuity conditions.
GLOBAL MATRIX FORMULATION

complete wavefield in all layers in only one computational pass, via simple simultaneous solution by Gaussian elimination. This is possible both because of the particular (sparse diagonally banded) form of the global coefficient matrix (Figs. 5–6) and of the rapid, numerically stable assembly and solution algorithms now available for use on minicomputers and/or attached array processors. This matrix form and solution is completely analogous to the utilization of numeric Green’s functions global stiffness matrices in the Finite Element Method (Figs. 6 and 7).

If the Green’s function kernels involved in the boundary conditions at interface $j-1$ are expressed in vector form as $\{v\}_j$, then to assemble the $N-1$ local wave equations into one global Green’s function matrix, the following unique mapping is employed:

$$\{a\}_j = \{S\}_j \{A\}_j$$

$$\{V\}_j = \sum_{i=1}^{N-1} \{T\}_i \{v\}_{i+1}$$

Here, $\{A\}_j$ and $\{V\}_j$ are global vectors containing all degrees of freedom and field parameters involved in the continuity conditions. The mapping matrices $\{S\}_j$ and $\{T\}_i$ are extremely sparse and contain only zeroes and ones. As Schmidt and Jensen (1984) have shown, these matrices are exactly analogous in form and function to the nodal topology or connectivity matrices in the Finite Element Method. As is the case there, this mapping is extremely efficient, since these matrices are operationally replaced by an equivalent set of indicial pointers. Discussions of these data structures and pertinent programming features of the direct global matrix algorithm are given elsewhere (Schmidt and Tango, 1985).

Substitution of these mappings into the homogeneous and inhomogeneous wavefield continuity vector yields the following global system of equations

$$\{C\}_j \{A\}_j = \{F\}_j$$

coefficient solutions sources, Direct Global Matrix Method.

**BOUNDARY CONDITIONS AND RESULTING MATRIX GLOBAL**

SATISFYING ABOVE BOUNDARY CONDITIONS ACROSS ALL $N+1$ INTERFACES SIMULTANEOUSLY LEADS DIRECTLY TO A LINEAR

SYSTEM OF $4N^2$ EQUATIONS IN THE $4N^2$ UNKNOWN WAVE

AMPLITUDE FUNCTIONS $A^2 + B^2$.

BECAUSE HANKEL TRANSFORMS FOR FIELD QUANTITIES DO NOT

HAVE CLOSED FORMS, AND TO TREAT THE MOST GENERAL MULTI-

LAYER CASE, SEEK A NUMERICAL SOLUTION

IN THE DIRECT GLOBAL MATRIX METHOD, THE RESULTING SYSTEM IS

$$\{C\}_j \{A\}_j = \{F\}_j$$

**THIS IS PRECISELY ANALOGOUS TO THE CANONIC SYSTEM IN THE

FINITE ELEMENT METHOD, WHERE

$$\{K\}_j \{D\}_j = \{F\}_j$$

**corresponding exactly to the global stiffness and forcing matrices in the Finite Element Method.**

The pointers defined above depend only on interfacial continuity conditions and can therefore be determined a priori. The calculations needed at each horizontal wavenumber are then limited to creating local coefficient matrices and right-hand source sides, and to integrating the global Green’s function kernels over desired depths and ranges.

With the present technique, one can obtain unconditionally stable solutions by making other than scaled Gaussian elimination with partial pivoting, as demonstrated by Schmidt and Jensen (1984). Use of a local coordinate system, and lack of any explicit z-dependence other than exponential transfer functions allows rapid field evaluation at any number and location of depth stations. The fact that no special numerical efforts except scaling and pivoting are needed to ensure stability is a major reason for the SAFARI direct global matrix algorithm's improvement in computational speed over modified Thomson-Haskell propagators. This advantage is true even for canonical one-layer cases, and increases with number of layers. At present, up to 250 solid layers have been examined on a VAX 11/780 minicomputer at frequencies in excess of 10 MHz with no difficulties. Another important advantage is that almost all operations are readily vectorized, which makes SAFARI extremely well suited for implementation on parallel array processors, and yields another factor of 15 over serial computers (Schmidt and Tango, 1985).

Having obtained the mulitlayered Green’s function to obtain the total CW or pulse response, it is necessary to evaluate one or two numeric integrations. It is well known that for guided wave propagation in a lossless multilayered waveguide, the kernels in previous integral solutions have real axis poles (corresponding to normal modes and surface waves). In these cases analytic integration is usually inconvenient, and a direct numerical real axis integration is generally employed (either by steepest descents or pole shifting via attenuation (Kennett, 1983)). This latter option has long been utilized in underwater acoustics and seismology and is quickly implemented via an FFT, which is the method employed here. However, any other quadrature, Filon, or adaptive integration could also be used. Thus, the only present limitations in the FFT integration in SAFARI is that of sampling, since depth-dependent Green’s functions in general are highly oscillatory (Fig. 8).

**4. INTERDISCIPLINARY COMPUTATIONAL EXAMPLES**

The general applications direct global matrix algorithm, SAFARI, has been applied to a wide variety of time-harmonic and transient wave propagation problems in underwater and atmospheric acoustics (Tango et al., 1985), crustal (Tango, 1985) and exploration (Tango and Schmidt, 1985b) seismology, ultrasonic scattering (Schmidt and Jensen, 1985), and nondestructive evaluation (Schmidt et al., 1985). As such, it can function as a common mathematical model linking diverse disciplines investigating wave propagation. Here, for illustration, we
briefly present the results of a selected subset of three recent studies, which involve mono- and multifrequency propagation in multilayered ocean, solid earth, and engineering materials, for which reference numerical and analytic solutions are available.

5. CONCLUSIONS

Numerical modeling efforts in wave propagation have been heavily impacted by the advent of faster and larger-memory computers that permit correct and complete mathematical models for environmental and experimental configurations of interest. This evolution in the direction of increased physical correctness and completeness need not necessarily lead to a computationally more complex and specialized implementation, when an appropriate compromise between a generalized physical model, and computational algorithm and hardware can be achieved. In this paper, we have reviewed a new technique for determining the depth-dependent Green's impulse response function in an arbitrary plane-layered medium. The direct global matrix method presented consists of a return to the original boundary-value problem formulation of guided propagation in the fluid/solid/mixed waveguide of Pekeris (1948), Ewing et al. (1957), and Budden. The governing integral transform equations are obtained by imposing classical continuity conditions to scalar potential wave solutions and expressing the homogeneous and source fields via a summation of plane waves.

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PEKERIS WAVE GUIDE PROBLEM

PROBLEM: FIND THE COMPLEX PRESSURE FIELD AT ALL POINTS IN RANGE & DEPTH FROM A CW POINT SOURCE

(Pekeris, 1948; Geol. Soc. Am. Mem. 26)

STRATIFIED SHALLOW WATER ENVIRONMENT FOR (RANGE-INDEPENDENT) CW PROPAGATION MODELING

Figure 9. Ocean waveguide model and resulting pressure loss field (contoured in dB loss vs. range and depth; after Pekeris, 1948, and NORDA Test Case 313).

MCMECHAN STRUCTURED LVL PROFILE

VELOCITY PROFILE

Figure 11. Deep crustal velocity-depth function for the eastern basin and range and corresponding fullwave synthetic time series (as a function of receiver range).
COMPARISON OF SYNTHETIC SEISMOGRAMS FOR MCMECHAN N/Z MODEL:

REFLECTIVITY SOLUTION (FUCHS-MÜLLER)

Figure 12. Deep crustal velocity-depth function for the eastern basin and range and corresponding fullwave synthetic time series (as a function of receiver range).

Figure 13. Schematic of experimental scenario for ultrasonic scattering from a fluid/solid interface.

Figure 14. Results of DGM numerically computed Gaussian beam reflection loss, clearly showing Goos-Hänchen interference null.