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On the base sequence conjecture

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ABSTRACT

Let $BS(m, n)$ denote the set of base sequences $(A; B; C; D)$, with A and B of length m and C and D of length n. The base sequence conjecture (BSC) asserts that $BS(n + 1, n)$ exist (i.e., are non-empty) for all n. This is known to be true for $n \leq 36$ and when n is a Golay number. We show that it is also true for $n = 37$ and $n = 38$. It is worth pointing out that BSC is stronger than the famous Hadamard matrix conjecture.

In order to demonstrate the abundance of base sequences, we have previously attached to $BS(n + 1, n)$ a graph Γ_n and computed the Γ_n for $n \leq 27$. We now extend these computations and determine the Γ_n for $28 \leq n \leq 35$. We also propose a conjecture describing these graphs in general.

1. Introduction

By *binary* respectively *ternary sequence* we mean a sequence $A = a_1, a_2, \ldots, a_m$ whose terms belong to $\{\pm 1\}$ respectively $\{0, \pm 1\}$. To such a sequence we associate the polynomial $A(z) = a_1 + a_2 z + \cdots + a_m z^{m-1}$. We refer to the Laurent polynomial $N(A) = A(z)A(z^{-1})$ as the *norm* of A. *Base sequences* $(A; B; C; D)$ are quadruples of binary sequences, with A and B of length m and C and D of length n, and such that

$$N(A) + N(B) + N(C) + N(D) = 2(m + n). \tag{1.1}$$

(The last condition is equivalent to the vanishing of the sum of the aperiodic auto-correlation functions of A, B, C and D.) We denote the set of such base sequences by $BS(m, n)$. Base sequences, and their special cases such as normal and near-normal sequences, play an important role in the construction of Hadamard matrices [5,11,12]. For instance, the recent discovery of a Hadamard matrix of order 428 [6] used a $BS(71, 36)$, constructed specially for that purpose.

As explained in [1], we can view the normal sequences $NS(n)$ and near-normal sequences $NN(n)$ as subsets of $BS(n+1, n)$. For normal sequences $2n$ must be a sum of three squares, and for near-normal sequences n must be even or 1. The base sequences $(A; B; C; D) \in BS(n+1, n)$ are *normal* respectively *near-normal* if $b_i = a_i$ respectively $b_i = (-1)^{i-1} a_i$ for all $i \leq n$.

The *base sequence conjecture* (BSC), first proposed explicitly in [1] (see also [5]), asserts that the $BS(n + 1, n)$ exist for all integers $n \geq 0$. Implicitly, it appears in earlier papers of Seberry and Yang. So far, BSC has been verified for all $n \leq 36$ and it is also well known that it holds when n is a *Golay number*, i.e., when $n = 2^a 10^b 26^c$ where a, b, c are nonnegative integers. For the cases $n \leq 32$ and references to previous work by other authors see [1,7–9,11]. For the cases $n = 33, 34, 35$ see [10] or Tables 7–9, and for $n = 36$ see [4] or the next section and Table 10.

T-sequences are quadruples of ternary sequences, $(X; Y; Z; W)$, all of the same length n such that for each index i exactly one of the terms x_i, y_i, z_i, w_i is nonzero, and

$$N(X) + N(Y) + N(Z) + N(W) = n.$$

We denote by $TS(n)$ the set of all T-sequences of length n. The *T-sequence conjecture* (TSC) asserts that $TS(n)$ exist for all integers $n \geq 1$.

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In Section 2 we show that BSC is also valid for $n = 37$ and $n = 38$. Our example for $n = 38$ consists of near-normal sequences. Consequently, the number 77 is a Yang number. We recall that *Yang numbers* are odd integers $2s + 1$ for which $NS(s)$ or $NN(s)$ is not empty. We also update the status of the TSC.

Let $\alpha = (A; B; C; D) \in BS(m, n)$ and let $a = A(1)$, $b = B(1)$, $c = C(1)$, $d = D(1)$ and $a^* = A(-1)$, $b^* = B(-1)$, $c^* = C(-1)$, $d^* = D(-1)$. By setting $z = 1$ in the norm identity (1.1), we see that the squares a^2, b^2, c^2, d^2, arranged in decreasing order, form a partition of $2(m + n)$. The same is true for the squares of a^*, b^*, c^*, d^*. We denote the former partition by p_α and the latter by p^*_α.

In the early searches for base sequences $BS(n + 1, n)$ the objective apparently was to construct, for each partition p of $2(2n + 1)$ into four squares, base sequences $\alpha = (A; B; C; D) \in BS(n + 1, n)$ such that $p_\alpha = p$. For this we refer the reader to the paper [8] and its references. A more ambitious program to construct base sequences $\alpha = (A; B; C; D) \in BS(n + 1, n)$ with p_α and p^*_α specified was initiated in our paper [1]. For that purpose, we have defined there the graphs Γ_n, $n \geq 0$.

In Section 3 we recall the definition of the Γ_n. These are undirected graphs with loops allowed but no multiple edges. They were determined for $n \leq 27$ by means of extensive computations of base sequences. We extend these computations to cover the cases $n = 28, 29, \ldots, 35$. The base sequences that we need are listed in Tables 2–9 in Appendix. On the basis of these computations, we propose a conjecture about the isomorphism types of the graphs Γ_n and show that the conjecture is valid for $n \leq 35$.

In Section 4 we describe briefly our algorithm for exhaustive search of the base sequences $BS(n + 1, n)$.

In Section 5 we report the results of our recent searches for $NS(n)$ and $NN(n)$. We also describe what is currently known about the existence of Yang numbers.

2. Current status of BSC

At the time when BSC was formulated in [1], it was known that it holds for $n \leq 32$. This was extended to $n \leq 35$ by Kounias and Sotirakoglou [10]. The examples of $BSC(n + 1, n)$ for $n = 36$ and $n = 38$ were constructed in the course of our exhaustive searches for near-normal sequences [4,3]. We have recently constructed an example for $n = 37$. These three examples will be given below.

Proposition 2.1. *The base sequences $BS(n + 1, n)$ exist for $n \leq 38$ (and for all Golay numbers n).*

As explained above, it suffices to give examples of $BS(n + 1, n)$ for $n = 36$, 37 and 38. To avoid possible errors, we shall give all base sequences in encoded compact form which is used in our computer program. Although this encoding scheme has been described in several of our previous papers, we shall give the details once again for the convenience of the reader.

Let $(A; B; C; D) \in BS(n+1, n)$. We encode the pairs $(A; B)$ and $(C; D)$ separately by using the same scheme. We decompose the pair $(A; B)$ into quads

$$\begin{bmatrix} a_i & a_{n+2-i} \\ b_i & b_{n+2-i} \end{bmatrix}, \quad i = 1, 2, \ldots, \left[\frac{n+1}{2}\right],$$

and, if $n = 2m$ is even, the central column $\begin{bmatrix} a_{m+1} \\ b_{m+1} \end{bmatrix}$. We can assume (and we do) that the first quad of $(A; B)$ is $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$. We attach to this particular quad the label 0. The other quads in $(A; B)$ and all the quads of the pair $(C; D)$, shown with their labels, must be one of the following:

$$1 = \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \quad 2 = \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \quad 3 = \begin{bmatrix} - & + \\ - & + \end{bmatrix}, \quad 4 = \begin{bmatrix} + & - \\ - & + \end{bmatrix},$$

$$5 = \begin{bmatrix} - & + \\ + & - \end{bmatrix}, \quad 6 = \begin{bmatrix} + & - \\ + & - \end{bmatrix}, \quad 7 = \begin{bmatrix} - & - \\ + & + \end{bmatrix}, \quad 8 = \begin{bmatrix} - & - \\ - & - \end{bmatrix}.$$

The central column (if present) is encoded as

$$0 = \begin{bmatrix} + \\ + \end{bmatrix}, \quad 1 = \begin{bmatrix} + \\ - \end{bmatrix}, \quad 2 = \begin{bmatrix} - \\ + \end{bmatrix}, \quad 3 = \begin{bmatrix} - \\ - \end{bmatrix}.$$

If $n = 2m$ is even, the pair $(A; B)$ is encoded as the sequence $q_1 q_2 \cdots q_m q_{m+1}$, where q_i, $1 \leq i \leq m$, is the label of the ith quad and q_{m+1} is the label of the central column. If $n = 2m - 1$ is odd, then $(A; B)$ is encoded by $q_1 q_2 \cdots q_m$, where q_i is the label of the ith quad for each i. We use the same recipe to encode the pair $(C; D)$.

As an example, the base sequences

$$A = +, +, +, +, -, -, +, -, +$$
$$B = +, +, +, -, +, +, +, -, -$$
$$C = +, +, -, -, +, -, -, +$$
$$D = +, +, +, +, -, +, -, +$$

are encoded as 06 142; 1675.

Table 1
The numbers ν_0, ν_1 and ν.

n	ν_0	ν_1	n	ν	n	ν_0	ν_1	n	ν
0	1	0	1	1	20	5	2	21	5
2	1	1	3	1	22	5	4	23	4
4	2	1	5	2	24	4	3	25	7
6	2	1	7	2	26	4	3	27	6
8	3	1	9	3	28	5	3	29	7
10	2	2	11	2	30	4	4	31	8
12	4	1	13	5	32	6	4	33	7
14	2	3	15	3	34	5	5	35	5
16	4	2	17	5	36	6	3	37	12
18	3	2	19	4	38	5	6	39	6
					40	9	4		

With this notation, the three promised base sequences $BS(n+1, n)$ are

$n = 36$: 0764841234846532153; 165154775335162126

$n = 37$: 06862878461535243 26; 1153175814738523732

$n = 38$: 076412378285158562 81; 1782612553714317675.

Those for $n = 36, 38$ are in fact near-normal.

It is well known that there exist maps $BS(m, n) \to TS(m+n)$ and $TS(n) \to TS(2n)$. By using the Proposition 2.1 and taking into account the [5, Remark V.8.47], we obtain

Corollary 2.2. *Apart from the two undecided cases $n = 79, 97$, the T-sequences $TS(n)$ exist for all $n \leq 100$.*

3. The Γ-conjecture

We begin by recalling the definition of the graph Γ_n. Its vertex set is the set of all partitions of $4n+2$ into four squares (including 0 and with repetitions allowed). We postulate that Γ_n may have loops but we do not permit multiple edges. There is a loop at a vertex p if and only if there exist base sequences $\alpha \in BS(n+1, n)$ such that $p_\alpha = p_\alpha^* = p$. If p and q are two distinct vertices, then $\{p, q\}$ is an edge of Γ_n if and only if there exist base sequences $\beta \in BS(n+1, n)$ such that $\{p_\beta, p_\beta^*\} = \{p, q\}$. This completes the definition of Γ_n. We refer to any $\alpha \in BS(n+1, n)$ as a witness for the edge $\{p_\alpha, p_\alpha^*\}$ of Γ_n.

While BSC simply asserts that each $BS(n+1, n)$ is non-empty, we shall propose a new conjecture which gives the description of the graphs Γ_n.

To state this new conjecture, we need some more notation. Let α be as above and assume that n is fixed. Note that $a \equiv b \equiv n+1 \pmod 2$ and $c \equiv d \equiv n \pmod 2$. Thus exactly two of the integers a, b, c, d are even. If n is even, one can show (see [1]) that these two even integers are congruent to each other modulo 4. In that case we say that the vertex α is *even* respectively *odd* if they are congruent to 0 respectively 2 modulo 4. Thus, for even n, the vertex set is partitioned into even and odd vertices.

Let ν denote the number of vertices of Γ_n. If n is even, let ν_0 respectively ν_1 denote the number of even respectively odd vertices of Γ_n. Of course, we have $\nu_0 + \nu_1 = \nu$ when n is even. In Table 1 we give, for $0 \leq n \leq 40$, the value of ν for odd n and the values of ν_0 and ν_1 for even n.

Let K_m denote the complete graph on m vertices. Any two distinct vertices are joined by a single edge. However, K_m has no loops. If we enlarge K_m by attaching a loop at each vertex, we obtain the graph K_m^0. By $K_{m,n}$ we denote the complete bipartite graph with m respectively n vertices in the first respectively second part. The disjoint union of two graphs will be written as a sum.

Γ-conjecture. *Γ_n is isomorphic to*
(a) K_ν^0 *if* n *is odd;*
(b) K_{ν_0,ν_1} *if* $n \equiv 2 \pmod 4$*;*
(c) $K_{\nu_0}^0 + K_{\nu_1}^0$ *if* $n \equiv 0 \pmod 4$ *except for* $n = 4, 8, 12$.

The graphs Γ_n for $n = 4, 8, 12$ are described in [1]. Since we always have $\nu \geq 1$, BSC is a consequence of the Γ-conjecture if $n \not\equiv 2 \pmod 4$. This would also be true when $n \equiv 2 \pmod 4$ provided that one can show that both ν_0 and ν_1 are nonzero. We can formulate this as the following number-theoretical question.

Question. Let $S = \{k^2 : k \in \mathbf{Z}\}$ respectively $T = \{k(k+1)/2 : k \in \mathbf{Z}\}$ be the set of squares respectively triangular numbers. Let $S_2 = \{x+y : x, y \in S\}$ and $T_2 = \{x+y : x, y \in T\}$. Does the set $\{4x+y : x, y \in T_2\}$ respectively $\{2x+y : x \in S_2, y \in T_2\}$ contain all even respectively odd nonnegative integers?

(The BSC implies that the answer is affirmative in both cases.)

We give now the current status of the Γ-conjecture.

Proposition 3.1. *The Γ-conjecture is valid for $n \leq 35$.*

Table 2
$BS(29, 28)$.

Edge	A & B; C & D	a, b, c, d	a^*, b^*, c^*, d^*
1–1	076413275222630; 12875373652226	9, −1, 4, −4	9, −1, 4, 4
1–2	076514146435673; 12566715632821	1, 7, 8, 0	9, −1, 4, 4
1–3	076412161284762; 12876155137475	5, 5, 0, 8	1, 9, 4, 4
1–4	078482447637733; 12858753246321	−9, 1, 4, −4	5, −3, 4, 8
1–5	078451311636611; 12838752334113	7, 7, 4, 0	−1, −9, 4, 4
2–2	078461443688572; 12848552856354	−7, 1, 0, −8	1, −7, 0, 8
2–3	078457641147620; 12856747141347	1, 7, 0, 8	5, −5, 8, 0
2–4	051782353215153; 17678365277211	7, 1, 0, 8	3, 5, 8, 4
2–5	078485628682111; 12845558724283	1, −7, 0, −8	−7, −7, 0, 4
3–3	077658617271583; 12852541333416	−5, 5, 8, 0	−5, 5, 0, −8
3–4	078466512613430; 12862352528373	5, 3, 4, −8	5, −5, 0, 8
3–5	078517356737323; 12747162866717	−5, 5, 0, 8	7, −7, 0, 4
4–4	078458231755712; 12835732236261	−3, 5, 8, −4	5, −3, 4, 8
4–5	078475657853170; 12876165548382	−7, 7, 0, −4	5, 3, 4, 8
5–5	078321422423580; 12887533734554	7, −7, −4, 0	7, −7, 4, 0
6–6	078582621567150; 12456332286115	3, 1, 10, −2	3, 1, 10, −2
6–7	078467557578650; 12836515766382	−9, 5, 2, −2	3, 1, 10, 2
6–8	078416634842140; 12882758538342	3, 1, −2, −10	7, 5, 2, 6
7–7	076443181762112; 12868357554116	5, 9, 2, 2	9, 5, −2, −2
7–8	076411216766222; 12875653427313	9, 5, 2, 2	5, −7, 2, 6
8–8	078436621518110; 12886731231325	7, 5, 6, −2	7, 5, 6, 2

We have to construct witnesses of all hypothetical edges of Γ_n. This was accomplished in [1] for $n \leq 27$, while for $n = 28$ two witnesses were missing. Tables 2–9 of the Appendix confirm the Γ-conjecture for $n = 28, 29, \ldots, 35$ as they contain witnesses for all hypothetical edges of Γ_n.

We have partial results for $n = 36$. Hypothetically, Γ_{36} has 27 edges. We list the witnesses for 19 of them in Table 10.

If n is odd, we use the (decreasing) lexicographic order of partitions to enumerate the vertices of Γ_n. If n is even, we enumerate first the even and then the odd vertices and arrange them (separately) in the lexicographic order. If $n \equiv 2 \pmod 4$ then Γ_n is bipartite (and there are no loops). The symbol $i - j$ in the first column of the tables below denotes the edge joining the ith and the jth vertex. If $i = j$, it refers to the loop at the ith vertex.

For instance, if $n = 28$ then there are eight vertices:

(1)	$(9^2, 4^2, 4^2, 1)$	(2)	$(8^2, 7^2, 1, 0)$	(3)	$(8^2, 5^2, 5^2, 0)$
(4)	$(8^2, 5^2, 4^2, 3^2)$	(5)	$(7^2, 7^2, 4^2, 0)$		
(6)	$(10^2, 3^2, 2^2, 1)$	(7)	$(9^2, 5^2, 2^2, 2^2)$	(8)	$(7^2, 6^2, 5^2, 2^2)$

Since 8, 4, 0 are all $\equiv 0 \pmod 4$, the first five vertices are even. Since 10, 6, 2 are all $\equiv 2 \pmod 4$, the remaining three vertices are odd. The graph Γ_{28} is a disjoint union of K_5^0 on even vertices and K_3^0 on odd ones. The first fifteen base sequences in Table 2 are witnesses for the edges of the "even" component K_5^0, and the next six are witnesses for the edges of the "odd" component K_3^0.

For a witness $\alpha \in BS(n + 1, n)$, the integers a, b, c, d determine the vertex p_α as the partition of $4n + 2$ with parts a^2, b^2, c^2, d^2. Similarly, a^*, b^*, c^*, d^* determine the vertex p_α^*.

4. Sketch of the algorithm

Our computer program is designed for exhaustive search of base sequences $BS(n + 1, n)$ for $n \geq 7$. The search is divided into 18 cases by fixing the first three quads of the pair $(A; B)$ and the first two quads of $(C; D)$. The choice of these cases depends on the parity of n.

Cases for n odd					
(1) 065; 11	(2) 066; 11	(3) 068; 11	(4) 061; 12	(5) 063; 12	(6) 064; 12
(7) 061; 16	(8) 063; 16	(9) 064; 16	(10) 016; 61	(11) 017; 61	(12) 018; 61
(13) 016; 64	(14) 017; 64	(15) 018; 64	(16) 011; 66	(17) 012; 66	(18) 013; 66
Cases for n even					
(1) 076; 12	(2) 077; 12	(3) 078; 12	(4) 076; 16	(5) 077; 16	(6) 078; 16
(7) 071; 18	(8) 072; 18	(9) 073; 18	(10) 065; 11	(11) 066; 11	(12) 068; 11
(13) 061; 12	(14) 063; 12	(15) 064; 12	(16) 061; 16	(17) 063; 16	(18) 064; 16

Each of the 18 cases is treated separately. The first quad of the pair $(A; B)$ is always 0. Thus the nth auto-correlation of $(A; B; C; D)$ is 0. The other four starting quads are chosen so that the $(n - 1)$th and $(n - 2)$th auto-correlation is 0.

Table 3
$BS(30, 29)$.

Edge	A & B; C & D	a, b, c, d	a^*, b^*, c^*, d^*
1–1	068362252723438; 118624666538452	4, −10, 1, −1	10, 4, 1, −1
1–2	068385638777645; 118722343573530	−10, 0, 3, 3	4, 10, −1, −1
1–3	066247531158121; 117543585724280	10, 4, 1, 1	0, 6, 9, 1
1–4	066217723624145; 117432416826461	8, 2, 7, 1	−10, 4, −1, 1
1–5	066417145712627; 117653654785220	6, 8, 3, 3	−4, 10, −1, −1
1–6	068385545252336; 118567425535130	2, −8, 5, 5	−4, 10, 1, 1
1–7	066427711368186; 117726654641520	2, 4, 7, 7	4, 10, −1, −1
2–2	066227632114544; 117768627585431	10, 0, −3, 3	0, 10, −3, 3
2–3	066244127461835; 117687675413252	6, 0, −1, 9	0, 10, 3, −3
2–4	066221154863181; 117586785628152	10, 0, −3, 3	8, 2, 1, 7
2–5	068246422128374; 118657526217580	6, −8, 3, 3	0, −10, 3, 3
2–6	068248487512863; 118768327622521	−2, −8, 5, −5	0, −10, −3, 3
2–7	066325474783574; 117876865367552	−4, 2, −7, 7	10, 0, −3, 3
3–3	066425872412617; 117661785545180	6, 0, 1, 9	0, 6, 1, 9
3–4	066225637518271; 117654433817272	6, 0, −1, 9	8, 2, 7, 1
3–5	066357474847817; 117581625334633	−8, 6, 3, 3	6, 0, −9, −1
3–6	066213624581187; 117683252526141	6, 0, 9, −1	8, 2, 5, −5
3–7	066416178423476; 117671223663613	2, 4, 7, 7	0, 6, −9, −1
4–4	066415721365525; 117726281652351	8, 2, 7, 1	2, 8, 7, 1
4–5	066417218511536; 117644754226580	8, 6, 3, 3	2, 8, −1, 7
4–6	066242378263856; 117685486122122	2, −8, 7, 1	8, 2, −5, 5
4–7	066415277854231; 117662161546363	4, 2, 7, 7	−2, 8, −1, 7
5–5	016186616313366; 641515851514853	8, 6, 3, 3	6, 8, 3, 3
5–6	066424271211847; 117681267525360	8, 2, 5, 5	−6, 8, −3, −3
5–7	066227415141467; 117628153854530	8, 6, 3, 3	2, −4, 7, 7
6–6	068427113134776; 118653736872672	2, 8, −5, 5	−8, 2, −5, 5
6–7	066425635118187; 117765384785371	4, 2, −7, 7	2, 8, 5, −5
7–7	066347444712723; 117823654415150	2, 4, 7, 7	4, −2, 7, 7

Table 4
$BS(31, 30)$.

Edge	A & B; C & D	a, b, c, d	a^*, b^*, c^*, d^*
1–5	0784614381231342; 128685615224114	3, 3, 10, −2	1, −11, 0, 0
1–6	0776853138438782; 128665371865672	−11, 1, 0, 0	−1, −9, 6, −2
1–7	0784216352512611; 128863554766615	11, −1, 0, 0	1, −7, 6, 6
1–8	0784864477847431; 128574476353272	−11, 1, 0, 0	−5, −5, 6, 6
2–5	0776162345126151; 128868657542531	9, 5, 0, −4	3, 3, 10, 2
2–6	0778853587261780; 128558541366151	−9, 1, 6, 2	5, −9, 0, 4
2–7	0778511521651532; 128588623471636	5, 9, 0, −4	7, −1, 6, 6
2–8	0784216213317131; 128863657667445	9, 5, −4, 0	−5, −5, 6, 6
3–5	0764411241717863; 128763613567478	3, 9, −4, 4	−3, 3, 10, 2
3–6	0776261117545653; 128813253753652	3, 9, 4, −4	9, −1, 6, 2
3–7	0784162254551610; 128865236166725	9, 3, 4, −4	7, 1, 6, 6
3–8	0778565314743723; 128563665117166	−5, 5, 6, 6	9, 3, 4, 4
4–5	0778586368314251; 128566641315214	−3, −3, 10, 2	3, 7, 8, 0
4–6	0512656235371531; 165711846213678	9, 1, 2, 6	3, 7, 0, 8
4–7	0764323438577832; 128767756347465	−7, 1, −6, 6	7, 3, 8, 0
4–8	0564376515151581; 118772615545132	5, 5, 6, 6,	7, 3, 8, 0

We proceed by selecting the 4th quad of $(A; B)$ and the 3rd quad of $(C; D)$ so that the $(n - 3)$th auto-correlation vanishes. We continue this procedure as far as possible. If no selection is possible, we backtrack. If we succeed in finding all the quads and the central column, then we test whether all the remaining auto-correlations vanish. If not, we backtrack. Otherwise we record the base sequences that we found. Note that this algorithm does not use any information about the possible sums a, b, c, d of the four constituent sequences. Thus we do not know in advance what these sums will turn out to be.

In order to handle the large values of n, say $n > 31$, we modify the program by breaking it into two phases. The first (easy) phase is to collect into a file the initial segments of quads, say of length 8 for $(A; B)$ and length 7 for $(C; D)$. Such a file has several millions of rows (subcases). It takes only several minutes to generate this file. In the second phase we use a random number generator to select a row in this file as the entry point for our program. The program then completes the computation for a fixed number, say r, of consecutive rows starting from the chosen entry point. We may repeat this subroutine, say s times. In our runs, the product rs was either 10 000 or just 1000. Usually we do not run the program to completion as this would require a prohibitively long time. We collect all base sequences that the program finds, and stop it after 5–6 days.

Table 5
BS(32, 31).

Edge	*A* & *B*; *C* & *D*	*a*, *b*, *c*, *d*	a^*, b^*, c^*, d^*
1–1	0653276646881415; 1187615124567762	2, 0, 1, 11	−2, 0, −1, −11
1–2	0653477313582724; 1186645576741711	0, 2, 1, 11	0, 10, −1, 5
1–3	0664286361577533; 1177645461752362	0, 2, 1, 11	4, 10, −1, −3
1–4	0664283673814787; 1176554618357311	−6, 0, 3, 9	−2, 0, 1, 11
1–5	0664483763412781; 1176755458517611	0, 2, 1, 11	4, 2, −5, 9
1–6	0663151725347817; 1178525218466222	2, 8, 7, −3	2, 0, 1, 11
1–7	0664457618863416; 1176713544647422	0, 2, 1, 11	−8, −6, −5, 1
1–8	0664286477134572; 1177653127576363	0, 2, −1, 11	4, 6, 5, −7
2–2	0663151774538174; 1178566324188782	0, 10, −5, 1	0, 10, 5, −1
2–3	0664134763185177; 1176835434253213	0, 10, 5, 1	4, 10, −1, 3
2–4	0663684725887517; 1177658146162731	−6, 0, 3, 9	10, 0, 1, −5
2–5	0663554568171527; 1177877658254711	2, 4, −5, 9	10, 0, 5, −1
2–6	0663284811641421; 1178625647454413	10, 0, 1, 5	2, 8, 3, 7
2–7	0653257763411145; 1187716272282540	6, 8, 5, 1	10, 0, −5, −1
2–8	0664463374577363; 1177365857427533	−4, 6, −5, 7	0, 10, 1, 5
3–3	0653271351241777; 1186637254782520	4, 10, 3, −1	−4, 10, −3, 1
3–4	0653485354761371; 1186373522521312	0, 6, 9, 3	−4, 10, 3, 1
3–5	0663174726216214; 1178327566525642	10, 4, 1, 3	2, 4, −5, 9
3–6	0664256357162313; 1176615635414833	8, 2, 3, 7	4, 10, −3, 1
3–7	0653461761515422; 1187654414627381	10, 4, 1, 3	−6, 8, −5, 1
3–8	0653182153651377; 1186554317646211	4, 6, 7, 5	4, 10, 1, 3
4–4	0664287241436146; 1177658653747250	6, 0, −3, 9	6, 0, −9, 3
4–5	0664452175768367; 1175561631427380	−2, 4, 5, 9	6, 0, −9, 3
4–6	0664271564363774; 1176735233364512	0, 6, 3, 9	−8, 2, −3, 7
4–7	0664463272861135; 1176513423426451	6, 0, 9, 3	−6, −8, −5, 1
4–8	0664151272416748; 1176758485764363	6, 4, −7, 5	6, 0, −9, 3
5–5	0653172153254877; 1186526273422531	2, 4, 9, −5	2, 4, −9, 5
5–6	0664463478185727; 1177653314631253	−4, 2, 5, 9	−8, 2, 3, 7
5–7	0653487153721511; 1186322757876711	6, 8, −1, 5	2, 4, 5, −9
5–8	0664161572851367; 1177726542361461	4, 6, 5, 7	4, 2, −5, 9
6–6	0653151463787817; 1187652534475472	−2, 8, −3, 7	2, 8, 3, −7
6–7	0664277581637113; 1177664515243633	2, 8, 3, 7	6, 8, 5, 1
6–8	0664475465821113; 1177546132182722	6, 4, 5, 7	2, 8, 7, −3
7–7	0664172363751142; 1176525365342831	8, 6, 5, −1	8, 6, −1, 5
7–8	0664475185416311; 1175416238735363	6, 8, 1, 5	6, 4, −5, 7
8–8	0664453177857275; 1176553815132731	−4, 6, 5, 7	4, −6, 7, 5

If necessary, we repeat this process several times, using different cases, until we find the witnesses for all edges of Γ_n.

As an example, we mention that the construction of Table 6 took in total about 1423 days of CPU time. For this table, we ran the parallelized version of our program on two machines at the same time, one used 128 processors at 3.0 GHz and the other 64 processors at 2.2 GHz. The program constructed in total 2640 different base sequences *BS*(33, 32).

5. Recent results on normal and near-normal sequences

We give here a brief summary of our recent results on these two types of sequences and on Yang numbers. Let us begin by quoting Theorem V.8.38 from the recent handbook [5].

Theorem 5.1. *There is no* $NS(n)$ *for* $n = 6, 14, 17, 21, 22, 23, 24, 27, 28, 30$ *(all other orders of* $n < 31$ *exist).* $NS(31)$ *is the first unknown case.*

We have carried out exhaustive searches for $NS(n)$ for $n = 31, 33, 34, 35, 36, 37, 38, 39$ and did not find any such sequences. As 32 and 40 are Golay numbers, we therefore have the following improvement.

Proposition 5.2. *For* $n \leq 40$, $NS(n) = \emptyset$ *if and only if*

$$n \in \{6, 14, 17, 21, 22, 23, 24, 27, 28, 30, 31, 33, 34, 35, 36, 37, 38, 39\}.$$

The first unknown case is $n = 41$.

Yang conjecture (see [5, Conjecture V.8.39]) asserts that $NN(n)$ exist for all even integers n. This has been known to be true when $n \leq 30$ (see [1] and [5, Remark V.8.40]). Complete classification of near-normal sequences has been carried out recently in our notes [2,4,3] for all even $n \leq 40$. It turns out that they exist for all even $n \leq 40$. Thus Yang conjecture remains open.

Consequently, we have the following result about Yang numbers (compare with [5, Theorem V.8.42.1]).

Table 6
BS(33, 32).

Edge	*A* & *B*; *C* & *D*	*a*, *b*, *c*, *d*	a^*, b^*, c^*, d^*
1–1	07643661131422181; 1286331583848171	11, 3, 0, 0	11, 3, 0, 0
1–2	07644347541711811; 1287716551833826	3, 11, 0, 0	7, −9, 0, 0
1–3	07642434354781830; 1286715346831111	−1, 1, 8, 8	3, −11, 0, 0
1–4	07642414351367712; 1284656553724755	3, 11, 0, 0	7, −1, 8, 4
1–5	07643151228512711; 1287676581466462	11, 3, 0, 0	−5, −5, 8, 4
1–6	07641116654178182; 1283857157633244	3, 11, 0, 0	7, 7, 4, 4
2–2	07841512343414140; 1663752642548557	9, 7, 0, 0	9, 7, 0, 0
2–3	06613883181363680; 1166661118633681	1, −1, 8, 8	9, 7, 0, 0
2–4	07641411467215623; 1287676534628461	9, 7, 0, 0	1, 7, 4, 8
2–5	07644776741834562; 1283561165748383	−7, 9, 0, 0	5, 5, 4, 8
2–6	07642431513713560; 1283556571663853	7, 9, 0, 0	7, −7, 4, 4
3–3	07237773326362331; 1863661181633311	1, 1, 8, 8	1, 1, 8, 8
3–4	07641562387182580; 1285614117616664	1, −1, 8, 8	1, 7, 8, 4
3–5	07644814118241362; 1284626522431467	5, 5, 8, −4	1, 1, 8, 8
3–6	07786885528463431; 1286525731546371	−7, −7, 4, 4	1, 1, −8, 8
4–4	07632712148552560; 1283745543432111	7, 1, 8, 4	7, 1, 8, 4
4–5	07643457175562810; 1283871353112172	1, 7, 8, 4	5, −5, 4, 8
4–6	07644123143216771; 1287661715652463	7, 7, 4, 4	7, −1, 8, 4
5–5	07641561751648621; 1285616124737125	5, 5, 8, 4	5, 5, 8, 4
5–6	07642753664476473; 1285131344465413	−5, 5, 8, 4	7, −7, 4, 4
6–6	07842423683125320; 1288686675821473	7, −7, −4, −4	7, −7, 4, 4
7–7	07632833612216140; 1285664844541762	11, 1, 2, −2	11, 1, 2, 2
7–8	07632578175158550; 1283617225865111	−1, 5, 10, 2	11, 1, 2, 2
7–9	07644318776114630; 1284842363371533	1, 11, 2, −2	9, 3, 2, 6
7–10	07645728186111662; 1281566243114774	3, 7, 6, 6	11, −1, 2, 2
8–8	07641431563668731; 1287676571651331	1, 5, 2, 10	5, 1, 10, 2
8–9	07632612542858710; 1285326563571112	5, −1, 10, 2	9, 3, 6, 2
8–10	07786157654765620; 1287671165413323	−3, 7, 6, 6	5, −1, 10, 2
9–9	07643428324116160; 1287761355637215	9, 3, 2, 6	9, 3, 6, 2
9–10	07644764313231670; 1282876155416351	3, 9, 6, 2	3, −7, 6, 6
10–10	07786231134327142; 1287335713121563	3, 7, 6, 6	7, 3, 6, 6

Table 7
BS(34, 33).

Edge	*A* & *B*; *C* & *D*	*a*, *b*, *c*, *d*	a^*, b^*, c^*, d^*
1–1	01643272281847733; 64437112182612640	2, 0, 11, 3	0, 2, 3, 11
1–2	06426183724377472; 16715585714616133	0, 2, 3, 11	10, 0, −5, 3
1–3	01714352388163846; 64462212371615313	2, 0, 11, 3	4, 10, 3, 3
1–4	06444714358667236; 16771235115272541	0, 2, 9, 7	2, 0, −3, 11
1–5	01644816586568712; 64715715371825472	2, 0, −3, 11	4, 6, 9, −1
1–6	01716725382367832; 64164177317682140	2, 0, 3, 11	8, 6, −5, 3
1–7	01235326158287371; 66571275124148160	6, 0, 7, 7	0, 2, 3, 11
2–2	06437211421686264; 16748465174364121	10, 0, 3, 5	0, −10, 3, 5
2–3	01846171746522125; 64311276482826751	10, 4, 3, −3	0, 10, 3, 5
2–4	01644135625178262; 64186575647548281	10, 0, −5, −3	0, 2, 7, 9
2–5	06552354172663716; 11746533381536372	6, 4, −1, 9	0, 10, 3, 5
2–6	06175464158337517; 12653715652536332	0, 10, 5, 3	−6, 8, 5, 3
2–7	01745856378115355; 64775443215125130	0, 6, 7, 7	−10, 0, 3, −5
3–3	06632247171213645; 11382325857554252	10, 4, 3, −3	4, 10, 3, −3
3–4	01738165165617653; 64433237582218162	4, 10, 3, −3	2, 0, 7, 9
3–5	06176424834163271; 12441571842462262	6, 4, 9, −1	4, 10, −3, 3
3–6	06441362614513772; 16776763822163151	8, 6, 3, 5	10, 4, 3, −3
3–7	06482612536431236; 12461662575778260	10, −4, 3, 3	0, 6, 7, 7
4–4	01848235737566316; 61242628662324763	0, 2, 7, −9	2, 0, 7, −9
4–5	06864765526373544; 11471612568726141	−2, 0, 9, 7	4, 6, 9, −1
4–6	01644614754247125; 64187352131157381	8, 6, 3, 5	2, 0, 7, 9
4–7	06862467734722615; 11675136251536272	2, 0, 7, 9	0, 6, 7, −7
5–5	06178545552317721; 16157375762546143	4, 6, 1, 9	6, −4, 1, 9
5–6	01837321432341743; 64381477511564642	6, 4, −1, 9	8, 6, 3, 5
5–7	01782525345315536; 61216753588111360	6, 0, 7, 7	4, 6, −1, −9
6–6	06554252236661836; 11257264681477341	8, −6, 3, 5	6, 8, 3, 5
6–7	01176167385241254; 66482625745862150	8, 6, 3, −5	6, 0, 7, 7
7–7	01653673337281734; 61473278766448712	0, 6, −7, 7	6, 0, −7, 7

Table 8
BS(35, 34).

Edge	*A* & *B*; *C* & *D*	*a*, *b*, *c*, *d*	*a**, *b**, *c**, *d**
1–6	076761387537518140; 12564355586883173	−3, 11, −2, −2	−1, −11, 0, 4
1–7	076544121368256783; 12564782516511413	1, −1, 10, 6	11, 1, 0, −4
1–8	076813248638665463; 12447117137868357	−3, −5, −2, 10	11, 1, 4, 0
1–9	076518224314621143; 12526647176348573	11, 1, 0, 4	9, 7, 2, 2
1–10	076813687885775451; 12534322471376448	−11, 1, 4, 0	7, 7, 6, 2
2–6	076813155217615621; 12534846527651176	9, 5, 4, 4	11, 3, −2, 2
2–7	076813446761268643; 12534882516521623	−1, 1, 10, −6	9, −5, 4, 4
2–8	076823855753834630; 12441164836225723	−5, −3, 10, −2	5, −9, 4, 4
2–9	076541256821114362; 12663152258827675	9, 5, 4, −4	7, −9, −2, 2
2–10	076814553215115552; 12534517187266537	7, 7, 2, 6	9, 5, 4, 4
3–6	076423483282237882; 12837164638247415	−3, −11, 2, 2	3, −1, 8, 8
3–7	053765656464871261; 17765746348615187	1, 1, −6, 10	3, −1, −8, −8
3–8	076544215376333280; 12662553656248264	3, 1, 8, −8	−3, −5, 10, −2
3–9	076541326141144653; 12565462867178642	9, 7, 2, −2	−1, −3, −8, 8
3–10	076541313864244753; 12565532682263655	1, 3, 8, −8	7, −7, 2, 6
4–6	076821154786531510; 12441254686615465	5, 7, 8, 0	11, −3, −2, 2
4–7	076542388881587133; 12634625571747754	−7, −5, 0, 8	−1, 1, 10, −6
4–8	076821421676434513; 12441325771765766	5, 3, 2, 10	7, 5, 8, 0
4–9	076535878535141762; 17677852174231455	−5, 7, 0, 8	9, −7, 2, −2
4–10	076764325821511142; 12563712335271855	7, 7, 6, 2	5, −7, −8, 0
5–6	076541434617337753; 12565287475625713	−3, 11, 2, 2	7, −3, −8, −4
5–7	076531753465353411; 12456864615253117	3, 7, 8, 4	1, 1, 6, 10
5–8	076543211437821351; 12664184625565624	7, 3, 8, −4	5, −3, 10, 2
5–9	076542443567112150; 12634755737233827	9, 7, −2, 2	3, −7, 8, 4
5–10	076532871428885871; 12455284637661614	−7, −7, 6, 2	7, 3, 8, 4

Table 9
BS(36, 35).

Edge	*A* & *B*; *C* & *D*	*a*, *b*, *c*, *d*	*a**, *b**, *c**, *d**
1–1	066128524558167276; 115512428681612272	4, −2, 11, 1	−4, −2, −11, −1
1–2	061752175573814614; 123367131555842723	4, 10, 5, 1	−4, 2, 11, −1
1–3	066224581257478141; 114662461732721433	6, 0, 9, 5	−2, 4, −1, 11
1–4	065532351825386471; 116425237255181721	4, −2, 11, 1	−4, −6, 9, 3
1–5	061754416162673578; 123357118657181721	2, 8, 5, 7	2, 4, 11, 1
2–2	061751252386515416; 123355426257165781	10, 4, 5, −1	−10, 4, −1, 5
2–3	016414317335677244; 616264816227573640	4, 10, 5, 1	0, 6, −5, −9
2–4	016815552241223875; 611817576227536871	10, −4, −1, 5	6, −4, 9, 3
2–5	016622578364127651; 615115726753232751	8, 2, 7, 5	−4, 10, 1, −5
3–3	016735472553122818; 617256517744612640	6, 0, 5, 9	6, 0, −9, −5
3–4	065532881558483613; 116421432717756380	0, −6, 5, 9	−4, −6, 3, −9
3–5	064713642432155468; 123473144616418230	6, 0, 9, 5	−2, 8, −5, 7
4–4	068753558343827566; 116258162734771362	−6, −4, 3, 9	6, 4, 9, 3
4–5	063828821524645187; 128664764835184882	2, −8, −7, −5	−6, 4, 3, 9
5–5	016567812318227135; 611513566817626551	8, 2, 7, 5	8, 2, 5, 7

Proposition 5.3. *For odd integers $n \le 81$, n is a Yang number if and only if*

$$n \notin \{35, 43, 47, 55, 63, 67, 71, 75, 79\}.$$

The first unknown case is $n = 83$.

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Appendix. Lists of base sequences

See Tables 2–10.

Table 10
$BS(37, 36)$.

Edge	A & B; C & D	a, b, c, d	a^*, b^*, c^*, d^*
1–1	0642483723773112832; 162444213616245723	1, 1, 12, 0	1, 1, −12, 0
1–4	0781647583615324282; 167557211545523777	−1, −1, 0, 12	−1, −9, 0, −8
1–5	0876588628114455150; 161427612285272431	1, −1, 12, 0	9, 7, −4, 0
1–6	0764841234846532153; 165154775335162126	3, −3, 8, 8	−1, 1, −12, 0
2–5	0767144683434761771; 124873577128343623	−5, 11, 0, 0	7, −9, −4, 0
2–6	0785618342468563210; 126551157157241538	3, −3, 8, 8	−5, −11, 0, 0
3–4	0764214143622153442; 164323881543744174	11, 3, 0, 4	−1, −9, 8, 0
3–5	0616123851727712413; 123473518825755738	9, 7, −4, 0	−3, 11, −4, 0
3–6	0785641356385516141; 126547124474373121	3, 3, 8, 8	11, 3, 0, −4
4–5	0616123816854524630; 123473466224661443	9, −1, 8, 0	9, 7, 0, 4
4–6	0865126576744588551; 161271417534556246	−3, −3, 8, 8	1, 9, 0, −8
5–5	0717855753413764382; 186871154644661856	−7, 9, 0, 4	9, −7, 0, −4
5–6	0615512388414537671; 126528535625368412	3, 3, 8, −8	7, −9, −4, 0
7–7	0864743671415823362; 163242244661482565	−1, 3, 10, −6	−1, 3, 6, −10
7–8	0778285253655118732; 128814612256532345	−3, 1, 10, −6	5, 9, 2, 6
7–9	0764367614152248343; 162525438825532618	3, 1, 6, −10	7, 5, 6, −6
8–8	0762165645151374421; 162127434137824455	9, 5, 6, 2	5, 9, 2, −6
8–9	0868124566444233641; 161842144127757326	5, −7, 6, 6	5, 9, 2, −6
9–9	0637461414423752660; 128647367258131611	7, 5, 6, 6	7, 5, −6, −6

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