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Landau's necessary density conditions for LCA groups ☆

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Abstract

We derive necessary conditions for sampling and interpolation of bandlimited functions on a locally compact abelian group in line with the classical results of H. Landau for bandlimited functions on $\mathbb{R}^d$. Our conditions are phrased as comparison principles involving a certain canonical lattice.

Keywords: Beurling density; Sampling; Interpolation; Homogeneous approximation property; Locally compact abelian group

1. Introduction

H. Landau's necessary density conditions for sampling and interpolation [12] may be viewed as a general principle resting on a basic fact of Fourier analysis: the complex exponentials e^{ikx}

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(k in $\mathbb{Z}$) constitute an orthogonal basis for $L^2([-\pi,\pi])$. The present paper extends Landau's conditions to the setting of locally compact abelian (LCA) groups, relying in an analogous way on the basics of Fourier analysis. The technicalities—in either case of an operator theoretic nature—are however quite different. We will base our proofs on the comparison principle of J. Ramanathan and T. Steger [17].

We recall briefly Landau's results, suitably adapted to our approach. Let Ω be a bounded measurable set in $\mathbb{R}^d$ and let $\mathcal{B}_\Omega$ denote the subspace of $L^2(\mathbb{R}^d)$ consisting of those functions whose Fourier transform is supported on Ω. We say that a subset Λ of $\mathbb{R}^d$ is *uniformly discrete* if the distance between any two points exceeds some positive number. A uniformly discrete set Λ is a *set of sampling for* $\mathcal{B}_\Omega$ if there exists a constant C such that $\|f\|_2^2 \leqslant C\sum_{\lambda\in\Lambda}|f(\lambda)|^2$ for every f in $\mathcal{B}_\Omega$, and a *set of interpolation for* $\mathcal{B}_\Omega$ if, for each square-summable sequence $\{a_\lambda\}_{\lambda\in\Lambda}$, there is a solution f in $\mathcal{B}_\Omega$ to the interpolation problem $f(\lambda)=a_\lambda$, λ in Λ.

The canonical case is when Ω is a cube of side length 2π and Λ the integer lattice $\mathbb{Z}^d$. Since the complex exponentials $e^{i\lambda\cdot x}$ (λ in Λ) constitute an orthogonal basis for $L^2(\Omega)$, it is immediate by the Plancherel identity that Λ is both a set of sampling and a set of interpolation for $\mathcal{B}_\Omega$. This result scales in a trivial way: $c\mathbb{Z}^d$ is a set of sampling and a set of interpolation for $\mathcal{B}_\Omega$ when Ω is a cube of side length $c^{-1}2\pi$. If we agree that the density of the integer lattice is 1, then we have that the density of the lattice equals $(2\pi)^{-d}$ times the volume of the spectrum Ω.

Landau's work may be understood as saying that this density result takes the following form for general Ω and uniformly discrete sets Λ:

(S) If Λ is a set of sampling for $\mathcal{B}_\Omega$, then Λ is everywhere at least as dense as the lattice $(2\pi)^{-1}|\Omega|^{1/d}\mathbb{Z}^d$.

(I) If Λ is a set of interpolation for $\mathcal{B}_\Omega$, then Λ is everywhere at least as sparse as the lattice $(2\pi)^{-1}|\Omega|^{1/d}\mathbb{Z}^d$.

Landau gave precise versions of these statements in terms of the following notion of density. For $h>0$ and x a point in $\mathbb{R}^d$, let $Q_h(x)$ denote the closed cube centered at x of side length h. Then the *lower Beurling density* of the uniformly discrete set Λ is defined as

$$\mathcal{D}_B^-(\Lambda)=\liminf_{h\to\infty}\inf_{x\in\mathbb{R}^d}\frac{\operatorname{card}(\Lambda\cap Q_h(x))}{h^d},$$

and its *upper Beurling density* is

$$\mathcal{D}_B^+(\Lambda)=\limsup_{h\to\infty}\sup_{x\in\mathbb{R}^d}\frac{\operatorname{card}(\Lambda\cap Q_h(x))}{h^d}.$$

Landau's result says that a set of sampling Λ for $\mathcal{B}_\Omega$ satisfies $\mathcal{D}_B^-(\Lambda)\geqslant(2\pi)^{-d}|\Omega|$ and a set of interpolation Λ for $\mathcal{B}_\Omega$ satisfies $\mathcal{D}_B^+(\Lambda)\leqslant(2\pi)^{-d}|\Omega|$.

Given two uniformly discrete sets Λ and Λ' and nonnegative numbers α and α', we write $\alpha\Lambda\preccurlyeq\alpha'\Lambda'$ if for every positive ϵ there exists a compact subset K of $\mathbb{R}^d$ such that for every compact subset L we have

$$(1-\epsilon)\alpha\operatorname{card}(\Lambda\cap L)\leqslant\alpha'\operatorname{card}\bigl(\Lambda'\cap(K+L)\bigr). \tag{1}$$

With this notation, we have the following equivalent way[4] of expressing Landau's density conditions:

(S) If Λ is a set of sampling for $\mathcal{B}_\Omega$, then $(2\pi)^{-d}|\Omega|\mathbb{Z}^d \preccurlyeq \Lambda$.
(I) If Λ is a set of interpolation for $\mathcal{B}_\Omega$, then $\Lambda \preccurlyeq (2\pi)^{-d}|\Omega|\mathbb{Z}^d$.

The latter formulation may look less appealing than that given by Landau, but it has the advantage of presenting Landau's density conditions as a *comparison principle*; we note that this version does not require the use of dilations of cubes, which in general LCA groups make no sense.

We take as our starting point this reformulation of Landau's results, and in brief our plan is as follows. We need to identify, in the general setting of LCA groups, a canonical case to be used for comparison. Then, besides comparing Λ with a suitable canonical lattice, we also need to compare spectra. We will do this by estimating a general spectrum Ω in terms of a disjoint union of small "cubes." A nontrivial point will be to clarify what are the right "cubes" and what are the "lattices" associated with such sets. The technicalities of the comparison will in fact take place at this "atomic" level, and it is here that the Ramanathan–Steger comparison lemma will play a crucial role.

While our approach leads to a best possible asymptotic result in the more general setting of LCA groups, we lose a subtle level of precision compared to Landau's work, which is based on estimates for the eigenvalues of a certain concentration operator. For Ω a finite union of real intervals, Landau obtained sharp bounds for the number of points from a set of sampling or of interpolation to be found in a large interval I. In these bounds appears an additional term of order $\log |I|$, and—as shown in [16]—this can be seen as a manifestation of the John–Nirenberg theorem.

It is worth noting that one may encounter situations in which no obvious analogue of a lattice is available. An interesting example is that of the unit sphere in $\mathbb{R}^d$. In a recent paper [13], J. Marzo managed to employ Landau's method in this setting without any explicit comparison between uniformly discrete sets. In our setting, the group of p-adic numbers is an example of an LCA group that fails to contain a lattice. Our approach will be to restrict to a discrete quotient on which a meaningful comparison with a lattice can be made.

The ideas of Ramanathan and Steger have been employed by many authors. We would in particular like to mention the basic theory developed by R. Balan, P. Casazza, C. Heil, and Z. Landau in [1]. That paper introduces a notion of density for frames parameterized by discrete abelian groups, such as Gabor frames. The present paper is however only loosely related to [1]; we will require a more general notion of density, since we will be dealing with uniformly discrete sets in general LCA groups rather than discrete abelian groups.

2. Landau's density theorem for LCA groups

We start by recalling some basic facts about locally compact abelian (LCA) groups. For more information we refer to the books [5] and [9].

Let G be a locally compact abelian group; to avoid trivialities, we assume that G is *non-compact*. The group multiplication will be written multiplicatively as xy, and we will use the notation $xK = \{y \in G\colon\ y = xk, k \in K\}$ and $KL = \{y \in G\colon\ y = kl, k \in K, l \in L\}$ for $x \in G$

[4] A proof of the equivalence is given in Section 7 of this paper.

and K, L being subsets of G. A locally compact group G is always equipped with a Haar measure, which in the following will be denoted by μ_G. We follow the convention that the Haar measure of a compact (sub)group is normalized to be a probability measure.

Let $\widehat{G}$ be the dual group of G. We write the action of a character $\omega \in \widehat{G}$ on $x \in G$ by $\langle \omega, x\rangle$. The *annihilator* $H^\perp \subseteq \widehat{G}$ of a subgroup $H \subseteq G$ is defined as $H^\perp = \{\chi \in \widehat{G}\colon\ H \subseteq \ker\chi\}$. By Pontrjagin duality, we can identify G with $\widehat{\widehat{G}}$, and we will frequently use that $\widehat{H} \simeq \widehat{G}/H^\perp$ and $(G/H)\widehat{\ } \simeq H^\perp$.

The Fourier transform $\mathcal{F}$ is defined by

$$\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_G f(x)\overline{\langle \omega, x\rangle}\, d\mu_G(x), \quad \omega \in \widehat{G}.$$

We assume that the Haar measures on G and $\widehat{G}$ have been chosen such that $\mathcal{F}$ is a unitary map from $L^2(G)$ onto $L^2(\widehat{G})$, in accordance with Plancherel's theorem. If $\Omega \subseteq \widehat{G}$ is a measurable set of positive measure,

$$\mathcal{B}_\Omega = \left\{f \in L^2(G)\colon \operatorname{supp} \widehat{f} \subseteq \Omega\right\}$$

is the space of "band-limited" functions with spectrum in Ω.

A subset Λ of G is called *uniformly discrete* if there exists an open set U such that the sets λU (λ in Λ) are pairwise disjoint. The definition of sets of sampling and interpolation given in the introduction extends without any change to the setting of LCA groups. We are interested in such sets for the space $\mathcal{B}_\Omega$.

We will assume that the dual group $\widehat{G}$ is compactly generated. This may seem a rather severe restriction and means that for instance p-adic groups are excluded from our consideration. However, if the spectrum is relatively compact, we may assume without loss of generality that $\widehat{G}$ is compactly generated. For a clarification of this point, we refer to Section 8. By the structure theory of LCA groups [9], $\widehat{G}$ is then isomorphic to $\mathbb{R}^d \times \mathbb{Z}^m \times K_0$ for a compact group K_0. Consequently, G is of the form $G = \mathbb{R}^d \times \mathbb{T}^m \times D_0$ with D_0 a (countable) discrete group. We then select the uniformly discrete subset $\Gamma_0 = \mathbb{Z}^d \times \{e\} \times D_0$ as the canonical lattice to be used for comparison, where e is the identity element in $\mathbb{T}^m$. We assume that the Haar measure $\mu_{\widehat{G}}$ is normalized so that $\mu_{\widehat{G}}([-\pi,\pi]^d \times \{e\} \times K_0) = 1$.

We define the relation '$\preccurlyeq$' for uniformly discrete subsets of G as we did in (1): Given two uniformly discrete sets Λ and Λ' and nonnegative numbers α and α', we write $\alpha\Lambda \preccurlyeq \alpha'\Lambda'$ if for every positive ϵ there exists a compact subset K of G such that for every compact subset L we have

$$(1-\epsilon)\alpha \operatorname{card}(\Lambda \cap L) \leqslant \alpha' \operatorname{card}(\Lambda' \cap KL). \tag{2}$$

It is immediate from the definition given by (2) that the relation '$\preccurlyeq$' is transitive, a fact that will be used repeatedly in what follows.

With this notation, we may state Landau's necessary conditions for sampling and interpolation in the context of a general LCA group as follows.

Theorem 1. *Suppose Λ is a uniformly discrete subset of the LCA group G and Ω is a relatively compact subset of $\widehat{G}$.*

(S) *If Λ is a set of sampling for $\mathcal{B}_\Omega$, then $\mu_{\widehat{G}}(\Omega)\Gamma_0 \preccurlyeq \Lambda$.*
(I) *If Λ is a set of interpolation for $\mathcal{B}_\Omega$, then $\Lambda \preccurlyeq \mu_{\widehat{G}}(\Omega)\Gamma_0$.*

One may think of $\mu_{\widehat{G}}(\Omega)$ as the "Nyquist density." Indeed, the relation '$\preccurlyeq$' gives us a way of defining densities of a uniformly discrete set: The *lower uniform density* of Λ is defined as

$$\mathcal{D}^-(\Lambda) = \sup\{\alpha\colon\ \alpha\Gamma_0 \preccurlyeq \Lambda\},$$

and its *upper uniform density* is

$$\mathcal{D}^+(\Lambda) = \inf\{\alpha\colon\ \Lambda \preccurlyeq \alpha\Gamma_0\},$$

with the understanding that $\mathcal{D}^+(\Lambda) = \infty$ if the set on the right-hand side is empty. We will later show that both densities are always finite, and so the infimum in the definition of $\mathcal{D}^-(\Lambda)$ is in fact a minimum, and the supremum in the definition of $\mathcal{D}^+(\Lambda)$ is a maximum. With these definitions, Theorem 1 can be reformulated in the following classical way.

Theorem 1′. *Suppose Λ is a uniformly discrete subset of the LCA group G and Ω is a relatively compact subset of $\widehat{G}$.*

(S) *If Λ is a set of sampling for $\mathcal{B}_\Omega$, then $\mathcal{D}^-(\Lambda) \geqslant \mu_{\widehat{G}}(\Omega)$.*
(I) *If Λ is a set of interpolation for $\mathcal{B}_\Omega$, then $\mathcal{D}^+(\Lambda) \leqslant \mu_{\widehat{G}}(\Omega)$.*

We will show below (Lemma 8) that when $G = \mathbb{R}^d$, $\mathcal{D}^-(\Lambda)$ and $\mathcal{D}^+(\Lambda)$ reduce to the usual Beurling densities. Indeed, we will see that an "intermediate" formulation of the densities, valid for any LCA group G, may be obtained by replacing the counting measure of Γ_0 by the Haar measure μ_G. We will also show that, in general, $\mathcal{D}^-(\Lambda) \leqslant \mathcal{D}^+(\Lambda) < \infty$. A particular consequence of this bound is that $\mathcal{D}^-(\Gamma_0) = \mathcal{D}^+(\Gamma_0) = 1$, because the transitivity of the relation '$\preccurlyeq$' implies that either $\mathcal{D}^-(\Gamma_0) = \mathcal{D}^+(\Gamma_0) = 1$ or $\mathcal{D}^-(\Gamma_0) = \mathcal{D}^+(\Gamma_0) = \infty$.

We will return to this discussion of uniform densities in Section 7, after the proof of Theorem 1. That proof requires some preparation, to be presented in the next three paragraphs. The most significant ingredients are the Fourier bases for small "cubes," given in Section 4, and the Ramanathan–Steger comparison principle, treated in Section 5. The actual proof of Theorem 1 is given in Section 6.

After a consideration of the case when $\widehat{G}$ is not compactly generated in Section 8, we close in Section 9 with some additional remarks pertaining to Theorem 1.

3. Square sums of point evaluations at uniformly discrete sets

The purpose of this section is mainly to show that our a priori assumption that Λ be a uniformly discrete set implies no loss of generality. However, one piece of this discussion will be needed in the proof of Theorem 1. This is Lemma 2 below, which says that uniformly discrete sets generate Carleson measures in a natural way.

We may of course remove the a priori assumption that a set of interpolation be uniformly discrete, but it is easy to see that, at any rate, a set of interpolation will be uniformly discrete. The argument is standard. We first note that we can always solve the interpolation problem with control of norms. This means that if Λ is a set of interpolation, then there exists a constant M such that the interpolation problem $f(\lambda) = a_\lambda$ can be solved with f in $\mathcal{B}_\Omega$ in such a way that

$$\|f\|_2^2 \leqslant M \sum_\lambda |a_\lambda|^2$$

for every square-summable sequence $\{a_\lambda\}_{\lambda\in\Lambda}$. This well-known fact is a consequence of the open mapping theorem. Now assume that for every open set U in G there are points λ_1 and λ_2 in Λ such that $\lambda_1^{-1}\lambda_2$ is in U. Solving the problem $f(\lambda_1) = 1$ and $f(\lambda) = 0$ for every other λ in Λ, we get that $\|f\|_2 \leqslant M$ and

$$\begin{aligned} 1 = \left|f(\lambda_1) - f(\lambda_2)\right| &\leqslant \int_\Omega \left|\widehat{f}(\omega)\right|\left|\langle\omega,\lambda_1\rangle - \langle\omega,\lambda_2\rangle\right| d\mu_{\hat{G}}(\omega) \\ &\leqslant M\mu_{\widehat{G}}(\Omega)^{1/2} \sup_{\omega\in\Omega}\left|1 - \left\langle\omega,\lambda_1^{-1}\lambda_2\right\rangle\right|, \end{aligned}$$

which cannot hold for arbitrary U when Ω is relatively compact.

The reduction from a more general definition of sets of sampling follows the same pattern as in [19, pp. 140, 141]. We will therefore be brief and only mention a few technical modifications. We begin with the following result on Carleson measures.

Lemma 2. *Let Λ be a uniformly discrete subset of G, and assume Ω is a relatively compact subset of $\widehat{G}$. Then there is a positive constant C such that*

$$\sum_{\lambda\in\Lambda} \left|f(\lambda)\right|^2 \leqslant C\|f\|_2^2$$

holds for every f in $\mathcal{B}_\Omega$.

Proof. The proof is identical to that of Lemma 1 in [6]. Choose a function g in $L^1(G)$ so that $\widehat{g}(\omega) = 1$ for $\omega \in \overline{\Omega}$ and such that for any (symmetric) compact neighborhood U of e, the function $g^\sharp(x) = \sup_{u\in U}|g(xu)|$ is also in $L^1(G)$. (Such a function exists by [18].) If f is in $\mathcal{B}_\Omega$, then $f = f * g$ and $f^\sharp(x) \leqslant (|f| * g^\sharp)(x)$ for all $x \in G$. Consequently, $\|f^\sharp\|_2 \leqslant \|f\|_2\|g^\sharp\|_1$ for all f in $\mathcal{B}_\Omega$. Clearly, $|f(\lambda)| \leqslant f^\sharp(x)$ whenever $x \in \lambda U$. Since Λ is uniformly discrete, we may choose U such that

$$\begin{aligned} \sum_{\lambda\in\Lambda}\left|f(\lambda)\right|^2 &= \sum_{\lambda\in\Lambda}\frac{1}{\mu_G(U)}\int_{\lambda U}\left|f(\lambda)\right|^2 d\mu_G(x) \\ &\leqslant \sum_{\lambda\in\Lambda}\frac{1}{\mu_G(U)}\int_{\lambda U}\left|f^\sharp(x)\right|^2 d\mu_G(x) \\ &\leqslant \frac{1}{\mu_G(U)}\int_G\left|f^\sharp(x)\right|^2 d\mu_G(x) \leqslant \frac{\|g^\sharp\|_1^2}{\mu_G(U)}\|f\|_2^2. \qquad\square \end{aligned} \tag{3}$$

This lemma and the G-invariance of $\mathcal{B}_\Omega$ imply that an inequality of the form

$$\sum_{\lambda\in\Lambda}\left|f(\lambda)\right|^2\leqslant C\|f\|_2^2,$$

valid for every f in $\mathcal{B}_\Omega$, holds if and only if Λ is a finite union of uniformly discrete sets. The existence of such an inequality is sometimes explicitly required in the definition of a set of sampling.

We may now go one step further and prove that if there are positive constants c and C such that

$$c\|f\|_2^2\leqslant\sum_{\lambda\in\Lambda}\left|f(\lambda)\right|^2\leqslant C\|f\|_2^2$$

holds for every f in $\mathcal{B}_\Omega$, then there are a uniformly discrete subset Λ' of Λ and positive constants c' and C' such that

$$c'\|f\|_2^2\leqslant\sum_{\lambda'\in\Lambda'}\left|f(\lambda')\right|^2\leqslant C'\|f\|_2^2$$

for every f in $\mathcal{B}_\Omega$. The key ingredient in the proof of this result is the following continuity property. Suppose Λ is a uniformly discrete subset of G. Then, for every $\varepsilon>0$, there exists a neighborhood U of the identity e such that if $\lambda\mapsto\lambda'$ is a mapping from Λ to G satisfying $\lambda'\lambda^{-1}\in U$, then we have

$$\sum_{\lambda\in\Lambda}\left|f(\lambda)-f(\lambda')\right|^2\leqslant\varepsilon\|f\|_2^2 \tag{4}$$

for every f in $\mathcal{B}_\Omega$.

We give the short proof of (4) and refer otherwise to [19, Lemma 3.11]. We let g be as in the proof of Lemma 2 and obtain

$$\begin{aligned}
&\sum_{\lambda\in\Lambda}\left|f(\lambda)-f(\lambda')\right|^2\\
&\quad\leqslant\sum_{\lambda\in\Lambda}\biggl(\int_G\left|f(y)\right|\left|g\bigl(\lambda y^{-1}\bigr)-g\bigl(\lambda' y^{-1}\bigr)\right|d\mu_G(y)\biggr)^2\\
&\quad\leqslant\sum_{\lambda\in\Lambda}\int_G\left|f(y)\right|^2\left|g\bigl(\lambda y^{-1}\bigr)-g\bigl(\lambda' y^{-1}\bigr)\right|d\mu_G(y)\int_G\left|g\bigl(\lambda x^{-1}\bigr)-g\bigl(\lambda' x^{-1}\bigr)\right|d\mu_G(x).
\end{aligned}$$

Since the translation operator $g(x)\mapsto g(\xi x)$ is continuous with respect to the L^1-norm, the integral to the right can be made arbitrarily small by a suitable choice of U, which is an estimate that is uniform with respect to λ and λ'. In the integral to the left, we may then interchange the order of summation and integration and essentially repeat the calculation made in (3) with g in place of f. With a suitable choice of U, the resulting estimate is (4).

4. Fourier bases on small "cubes"

We will in what follows rewrite sampling and interpolation properties in terms of the spanning properties of the resulting functions on the Fourier transform side. By Lemma 2, if Ω is relatively compact, then Λ is a set of sampling for $\mathcal{B}_\Omega$ if and only if the system $\{e_\lambda(\omega) = \langle \omega, \lambda \rangle \chi_\Omega(\omega)$: $\lambda \in \Lambda\}$ is a frame for $L^2(\Omega) \subseteq L^2(\widehat{G})$. Likewise, Λ is a set of interpolation for $\mathcal{B}_\Omega$ if and only if $\{e_\lambda(\omega) = \langle \omega, \lambda \rangle \chi_\Omega(\omega)$: $\lambda \in \Lambda\}$ is a Riesz sequence in $L^2(\Omega)$. This means that $\{e_\lambda$: $\lambda \in \Lambda\}$ is a Riesz basis in the closed linear span of the functions $\{e_\lambda\}$.

We may at once apply this observation to the canonical lattice $\Gamma_0 = \mathbb{Z}^d \times \{e\} \times D_0$ of Theorem 1. Indeed, writing as before $\widehat{G} = \mathbb{R}^d \times \mathbb{Z}^m \times K_0$, we note that the characters labelled by Γ_0 and restricted to $\Omega_0 := [-\pi, \pi]^d \times \{e\} \times K_0$ constitute an orthonormal basis for $L^2(\Omega_0)$. Consequently, Γ_0 is both a set sampling and a set of interpolation for $\mathcal{B}_{\Omega_0}$. (See also [10].)

In the classical case when $G = \mathbb{R}^d$, this is all we need, because we can just scale Γ_0 to obtain Fourier bases for arbitrarily small cubes.[5] For general LCA groups, we need a different approach. It is convenient to introduce some notation in order to state the lemma to be used in place of a simple rescaling. We will say that a discrete subgroup Γ of G is a *lattice* if the quotient G/Γ is compact. A uniformly discrete set Γ in G will be said to be a *quasi-lattice* if the following holds. There is a compact subgroup K of $\widehat{G}$ and a lattice Υ in $K^\perp$ such that $\Gamma = \{\widehat{k}\upsilon\}$, where υ ranges over Υ and $\widehat{k} \in G$ ranges over a set of representatives of $G/K^\perp$ in G. We may identify $\{\widehat{k}\}$ with $\widehat{K} \simeq G/K^\perp$, and consequently $\{\langle k, \widehat{k} \rangle\}$ (k in K) is an orthonormal basis for $L^2(K, \mu_K)$.

We note that every lattice Λ is in particular a quasi-lattice; just take $K = \{e\}$ and $\Upsilon = \Lambda$. In $\mathbb{R}^d$, every quasi-lattice is a lattice because there are no nontrivial compact subgroups. In the group $G = \mathbb{R} \times \mathbb{Z}$ the set $\{(k + x_j, j)\}$ (j, k in $\mathbb{Z}$) is a quasi-lattice for any choice of x_j in $\mathbb{R}$, but it is a lattice only when $x_j = qj$ for some rational number q. In this case, we may take $\{0\} \times \mathbb{T} \subseteq \widehat{G}$ as the compact subgroup K, then $K^\perp = \mathbb{R} \times \{0\}$ and $G/K^\perp = \mathbb{Z}$.

Lemma 3. *Let G be an LCA group whose dual group $\widehat{G}$ is compactly generated. For every open neighborhood U of the identity e in $\widehat{G}$ there exists a relatively compact subset C of U and a quasi-lattice Γ in G with the following properties*:

(i) *$L^2(C)$ possesses an orthogonal basis of characters restricted to C and labelled by Γ.*
(ii) *There exists a discrete subset D of $\widehat{G}$ such that the translates dC, $d \in D$, form a partition of $\widehat{G}$.*

Proof. Since $\widehat{G}$ is compactly generated, the structure theory implies that any neighborhood $U \subseteq \widehat{G}$ of e contains a compact subgroup K, such that $H := \widehat{G}/K \simeq \mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{T}^\ell \times F$, where F is a finite group and $d, m, \ell \geqslant 0$. See [9, Theorem 9.6]. Since the canonical projection $\pi : \widehat{G} \to H$ is an open mapping, the image of U in H contains a neighborhood of the form

$$C_0 = [-\epsilon/2, \epsilon/2)^d \times \{0\} \times \left[-\frac{1}{2N}, \frac{1}{2N}\right)^\ell \times \{e\}.$$

[5] We recall from the introduction that the motivation for such a rescaling is that we wish to approximate an arbitrary spectrum by a union of small "cubes."

By construction, C_0 is a fundamental domain for the lattice $\Xi = (\epsilon\mathbb{Z})^d \times \mathbb{Z}^k \times \mathbb{Z}_N^\ell \times F \subseteq H$. Consequently, $L^2(C_0)$ possesses an orthogonal basis consisting of characters restricted to C_0 and labelled by $\Upsilon := \Xi^\perp$.

Since Ξ is a lattice in H, Υ is a lattice in $\widehat{H}$. We may identify Υ with a subgroup of G by $\Upsilon \subseteq \widehat{H} \simeq (\widehat{G}/K)\widehat{\ } = K^\perp \subseteq \widehat{\widehat{G}} \simeq G$. Consequently, by fixing representatives $\widehat{k}$ from the cosets $\widehat{K}$, we obtain a quasi-lattice $\Gamma = \{\widehat{k}\upsilon\}$ in G with $\widehat{k}$ ranging over $\widehat{K}$ and υ over Υ.

Next, set $C = \pi^{-1}(C_0)$ and define for γ in Γ and ω in $\widehat{G}$

$$\psi_\gamma(\omega) = \mu_{\widehat{G}}(C)^{-1/2}\langle\omega,\gamma\rangle\chi_C(\omega) = \mu_{\widehat{G}}(C)^{-1/2}\langle\omega,\gamma\rangle\chi_{C_0}\big(\pi(\omega)\big).$$

We now prove that the functions ψ_γ form an orthonormal basis for $L^2(C)$. We assume as usual that the Haar measure of a compact subgroup K is normalized to be a probability measure and that the Haar measure of $\widehat{G}/K$ is normalized so that the Weil–Bruhat formula [18] $d\mu_{\widehat{G}}(\omega) = d\mu_K(k)\,d\mu_{\widehat{G}/K}(\pi(\omega))$ holds. So we obtain that

$$\begin{aligned}\mu_{\widehat{G}}(C) &= \int_{\widehat{G}} \chi_C(\omega)\,d\mu_{\widehat{G}}(\omega) = \int_H\int_K \chi_C(\omega k)\,d\mu_K(k)\,d\mu_H\big(\pi(\omega)\big)\\ &= \int_H \chi_{C_0}\big(\pi(\omega)\big)\,d\mu_H\big(\pi(\omega)\big) = \mu_H(C_0)\end{aligned}$$

and that $\|\psi_\gamma\|_2 = 1$ for every γ in Γ. If $\gamma = \widehat{k}\upsilon$ and $\gamma' = \widehat{k}'\upsilon'$ are in Γ, then using the Weil–Bruhat formula once more, we obtain that

$$\begin{aligned}&\int_{\widehat{G}} \psi_\gamma(\omega)\overline{\psi_{\gamma'}(\omega)}\,d\mu_{\widehat{G}}(\omega)\\ &\quad= \mu_{\widehat{G}}(C)^{-1}\int_H\bigg(\int_K \big\langle\omega k, \widehat{k}\upsilon\widehat{k'}^{-1}(\upsilon')^{-1}\big\rangle\chi_C(\omega k)\,d\mu_K(k)\bigg)\,d\mu_H\big(\pi(\omega)\big)\\ &\quad= \delta_{\widehat{k},\widehat{k}'}\mu_{\widehat{G}}(C)^{-1}\int_H \big\langle\pi(\omega), \upsilon(\upsilon')^{-1}\big\rangle\chi_{C_0}\big(\pi(\omega)\big)\,d\mu_H\big(\pi(\omega)\big) = \delta_{\gamma,\gamma'}.\end{aligned}$$

Here we have used that $\langle\omega k, \upsilon(\upsilon')^{-1}\rangle$ is independent of k in K, that $\{\langle k, \widehat{k}\rangle\}$ is an orthonormal basis for $L^2(K)$, and that $\{\langle\pi(\omega),\upsilon\rangle\}_{\upsilon\in\Upsilon}$ is an orthogonal basis for $L^2(C_0)$.

Next we show that the linear span of ψ_γ (γ in Γ) is dense in $L^2(C)$. So assume that for some f in $L^2(C)$ and all γ in Γ we have

$$\begin{aligned}0 &= \int_{\widehat{G}} f(\omega)\overline{\psi_\gamma(\omega)}\,d\mu_{\widehat{G}}(\omega)\\ &= \mu_{\widehat{G}}(C)^{-1/2}\int_H\bigg(\int_K f(\omega k)\overline{\langle\omega k,\widehat{k}\rangle}\,d\mu_K(k)\bigg)\overline{\langle\pi(\omega),\upsilon\rangle}\chi_{C_0}\big(\pi(\omega)\big)\,d\mu_H\big(\pi(\omega)\big).\end{aligned}$$

Since $\{\langle\pi(\omega), \upsilon\rangle\}$ is an orthogonal basis for $L^2(C_0)$, we find that

$$\int_K f(\omega k)\overline{\langle \omega k, \widehat{k}\rangle}\, d\mu_K(k) = 0$$

for almost all $\pi(\omega)$ in $\widehat{G}/K$ and all $\widehat{k}$ in $\hat{K}$. We infer that $f(\omega k) = 0$ for almost all ω in C and k in K, since $\{\langle k, \widehat{k}\rangle\}$ is an orthonormal basis for $L^2(K)$. Thus the functions ψ_γ form an orthonormal basis for $L^2(C)$.

To show (ii) we choose a pre-image D of Ξ in $\widehat{G}$, i.e., for each λ in Ξ, $D \cap \pi^{-1}(\lambda)$ contains exactly one element. Then $\pi(D) = \Xi$. If $dC \cap d'C \neq \emptyset$ for $d \neq d'$ (d, d' in D), then $\pi(d)\pi(C) \cap \pi(d')\pi(C) = \lambda C_0 \cap \lambda' C_0 \neq \emptyset$ for $\lambda \neq \lambda'$ (λ, λ' in Ξ). Since C_0 is a fundamental domain for the lattice Ξ, we conclude that $\lambda = \lambda'$. By choice of D we also have $d = d'$, a contradiction. Thus the translates dC (d in D) form a partition of $\widehat{G}$, and (ii) is proved. □

5. The Ramanathan–Steger comparison principle

The following lemma is a variation of an argument invented by Ramanathan and Steger [17]. Their decisive idea has been investigated quite intensively in recent years. See [1,3,6,8,11] for a sample of references and [7] for an excellent survey. We follow the early paper [6]. In what follows, $\mathcal{H}$ is a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

Lemma 4. *Let Γ and Λ be uniformly discrete subsets of G. Suppose that the sequence $\{g_\gamma\colon \gamma \in \Gamma\}$ is a Riesz sequence in $\mathcal{H}$ and that there exists a sequence $\{h_\lambda\colon \lambda \in \Lambda\}$ so that, for fixed $\epsilon > 0$ and a compact set $K \subseteq G$,*

$$\operatorname{dist}_{\mathcal{H}}\bigl(g_\gamma, \operatorname{span}\{h_\lambda\colon \lambda \in \Lambda \cap \gamma K\}\bigr) < \epsilon \tag{5}$$

for every $\gamma \in \Gamma$. Then for every compact set $L \subseteq G$ we have

$$(1 - c\epsilon)\operatorname{card}(\Gamma \cap L) \leqslant \operatorname{card}(\Lambda \cap LK). \tag{6}$$

The constant $c > 0$ depends only on $\{g_\gamma\}$. In particular, $c = 1$ if the g_γ constitute an orthonormal set.

Proof. Fix a compact set $L \subseteq G$ and set

$$\mathcal{H}_0 = \overline{\operatorname{span}\{g_\gamma\colon \gamma \in \Gamma\}}.$$

Then $\{g_\gamma\colon \gamma \in \Gamma\}$ is a Riesz basis for $\mathcal{H}_0$ with dual basis $\{\tilde{g}_\gamma\colon \gamma \in \Gamma\} \subseteq \mathcal{H}_0$, say. Since $\{\tilde{g}_\gamma\}$ is also a Riesz basis, it is bounded, and so

$$c = \sup_{\gamma\in\Gamma} \|\tilde{g}_\gamma\| < \infty. \tag{7}$$

If $\{g_\gamma\colon \gamma \in \Gamma\}$ is an orthonormal basis, then $\tilde{g}_\gamma = g_\gamma$ and $c = 1$.

Let $W_r(L) = \operatorname{span}\{g_\gamma\colon \gamma \in \Gamma \cap L\}$ and $W_f(KL) = \operatorname{span}\{h_\lambda\colon \lambda \in \Lambda \cap KL\}$. Let P_{W_r} denote the orthogonal projection onto $W_r(L)$ and Q_{W_f} the orthogonal projection onto $W_f(LK)$.

Using these projections, we can recast assumption (5) as $\|(I - Q_{W_f})g_\gamma\| < \epsilon$ provided that $\gamma \in \Gamma \cap L$ (because in this case $\Lambda \cap \gamma K \subseteq \Lambda \cap KL$). Consequently, we also have

$$\left\|(I - P_{W_r} Q_{W_f})g_\gamma\right\| = \left\|P_{W_r}(I - Q_{W_f})P_{W_r} g_\gamma\right\| < \epsilon \quad \text{for all } \gamma \in \Gamma \cap L. \tag{8}$$

The proof is done by estimating the trace of $T = P_{W_r} Q_{W_f} P_{W_r} : \mathcal{H}_0 \to \mathcal{H}_0$ in two different ways. First, since all eigenvalues ν_k of T satisfy $0 \leqslant \nu_k \leqslant 1$, we have

$$\operatorname{tr}(T) \leqslant \operatorname{rank} T \leqslant \dim\bigl(W_f(LK)\bigr) \leqslant \operatorname{card}(\Lambda \cap LK). \tag{9}$$

On the other hand, using (7) and (8), we find that

$$\begin{aligned}
\operatorname{tr}(T) &= \sum_{\gamma \in \Gamma \cap L} \langle T g_\gamma, \tilde{g}_\gamma \rangle \\
&= \sum_{\gamma \in \Gamma \cap L} \bigl(\langle g_\gamma, \tilde{g}_\gamma \rangle - \bigl\langle (I - T) g_\gamma, \tilde{g}_\gamma \bigr\rangle\bigr) \\
&\geqslant \sum_{\gamma \in \Gamma \cap L} 1 - \sum_{\gamma \in \Gamma \cap L} c\epsilon \\
&= (1 - c\epsilon) \operatorname{card}(\Gamma \cap L).
\end{aligned} \tag{10}$$

The claim (6) now follows from (9) and the above. □

In the proof of our main theorem, we will use an orthonormal basis with the property that N functions are associated to each point γ in Γ. In this case we have to count each γ in the final estimate (10) with multiplicity N. This modification yields the following statement.

Lemma 5. *Let Γ and Λ be uniformly discrete subsets of G. Suppose that the sequence $\{g_{\gamma,j}\colon \gamma \in \Gamma, j = 1, \dots, N\}$ is a Riesz sequence in $\mathcal{H}$ and that there exists a sequence $\{h_\lambda\colon \lambda \in \Lambda\}$ so that, for fixed $\epsilon > 0$ and a compact set $K \subseteq G$,*

$$\operatorname{dist}_{\mathcal{H}}\bigl(g_{\gamma,j}, \operatorname{span}\{h_\lambda\colon \lambda \in \Lambda \cap \gamma K\}\bigr) < \epsilon$$

for every $\gamma \in \Gamma$ and $j = 1, \dots, N$. Then for every compact set $L \subseteq G$ we have

$$(1 - c\epsilon) N \operatorname{card}(\Gamma \cap L) \leqslant \operatorname{card}(\Lambda \cap LK).$$

The constant $c > 0$ depends only on $\{g_{\gamma,j}\}$, and $c = 1$ if $\{g_{\gamma,j}\}$ is an orthonormal set.

Our application of the Ramanathan–Steger comparison lemma will require an estimate usually called the *homogeneous approximation property*. To state it, we introduce the following notation. Let M_x be the modulation operator defined by $M_x f(\omega) = \langle \omega, x \rangle f(\omega)$ for $f \in L^2(\widehat{G})$, $x \in G$, $\omega \in \widehat{G}$.

Lemma 6. *Let $\widehat{G}$ be compactly generated, and assume that $\{e_\lambda = M_\lambda g\colon \lambda \in \Lambda\}$, with g in $L^\infty(\Omega)$, is a frame for $L^2(\Omega)$ with dual frame $\{h_\lambda\colon \lambda \in \Lambda\}$. Then for every f in $L^2(\Omega)$ and $\epsilon > 0$ there is a compact set $K \subseteq G$ (depending on f and ϵ) such that*

$$\operatorname{dist}_{\mathcal{H}}\bigl(M_x f, \operatorname{span}\{h_\nu\colon \nu \in \Lambda \cap xK\}\bigr) < \epsilon \tag{11}$$

for every $x \in G$.

Proof. The proof is identical to the proof of Lemma 2 in [6]. Using the frame expansion of $f \in L^2(\Omega)$, we write

$$M_x f = \sum_{\lambda \in \Lambda} \langle M_x f, M_\lambda g\rangle h_\lambda.$$

Let $P_{x,K}$ denote the orthogonal projection from $L^2(\Omega)$ onto $\operatorname{span}\{h_\lambda\colon \lambda \in \Lambda \cap xK\}$. Since $\sum_{\lambda \in \Lambda \cap xK} \langle M_x f, M_\lambda g\rangle h_\lambda$ is some approximation of f in $P_{x,K} L^2$, the square of the distance in (11) is at most

$$\begin{aligned}
\|M_x f - P_{x,K} f\|_2^2 &\leqslant \Bigl\| \sum_{\lambda \notin xK} \langle M_x f, M_\lambda g\rangle h_\lambda \Bigr\|_2^2 \\
&\leqslant C \sum_{\lambda \notin xK} \bigl|\langle M_x f, M_\lambda g\rangle\bigr|^2 \\
&= C \sum_{\lambda \notin xK} \bigl|\langle f, M_{x^{-1}\lambda} g\rangle\bigr|^2.
\end{aligned}$$

Set $F(x) = \int_\Omega f(\omega)\overline{g(\omega)}\overline{\langle \omega, x\rangle}\, d\omega = \mathcal{F}^{-1}(f\bar{g})(x^{-1})$. Then $F \in \mathcal{B}_\Omega$, and the latter expression equals $C \sum_{\lambda \notin xK} |F(x^{-1}\lambda)|^2$. If $\lambda \notin xK$, then $x^{-1}\lambda \notin K$, and so we obtain as in the estimate (3) in the proof of Lemma 2 that

$$\begin{aligned}
\|M_x f - P_{x,K} f\|_2^2 &\leqslant \sum_{\lambda \notin xK} \frac{1}{\mu_G(U)} \int_{x^{-1}\lambda U} \bigl|F^\sharp(t)\bigr|^2\, d\mu_G(t) \\
&= \sum_{x^{-1}\lambda \notin K} \frac{1}{\mu_G(U)} \int_{x^{-1}\lambda U} \bigl|F^\sharp(t)\bigr|^2\, d\mu_G(t) \\
&\leqslant \frac{1}{\mu_G(U)} \int_{K^{\mathrm{c}} U} \bigl|F^\sharp(t)\bigr|^2\, d\mu_G(t),
\end{aligned}$$

with U depending only on Λ, but not on $x \in G$. Since $F^\sharp$ is in $L^2(G)$, we may choose K so large that the expression on the right becomes less than ϵ, and this bound holds uniformly in x. □

6. Proof of Theorem 1

For the proof we will use the connection between sets of sampling (sets of interpolation) in $\mathcal{B}_\Omega$ and frames (Riesz sequences) of characters in $L^2(\Omega)$, mentioned in Section 4. We will prove the following statement that is equivalent to Theorem 1.

Theorem 1″. *Suppose Λ is a uniformly discrete subset of the LCA group G and Ω is a relatively compact subset of $\widehat{G}$.*

(S) *If the system $\{e_\lambda(\omega) = \langle\omega, \lambda\rangle\chi_\Omega(\omega)\colon \lambda \in \Lambda\}$ is a frame for $L^2(\Omega) \subseteq L^2(\widehat{G})$, then $\mu_{\widehat{G}}(\Omega)\Gamma_0 \preccurlyeq \Lambda$.*

(I) *If $\{e_\lambda(\omega) = \langle\omega, \lambda\rangle\chi_\Omega(\omega)\colon \lambda \in \Lambda\}$ is a Riesz sequence in $L^2(\Omega)$, then $\Lambda \preccurlyeq \mu_{\widehat{G}}(\Omega)\Gamma_0$.*

The proof becomes slightly simpler if we replace Γ_0 by

$$\Gamma_0' = \left(\mu_{\widehat{G}}(\Omega)^{1/d}\mathbb{Z}\right)^d \times \{e\} \times D_0.$$

This replacement can be made because it is plain that $\Gamma_0' \preccurlyeq \mu_{\widehat{G}}(\Omega)\Gamma_0$ as well as $\mu_{\widehat{G}}(\Omega)\Gamma_0 \preccurlyeq \Gamma_0'$. Thus, by transitivity of the relation '$\preccurlyeq$,' it suffices to prove that if the uniformly discrete set Λ is a set of sampling for $\mathcal{B}_\Omega$, then $\Gamma_0' \preccurlyeq \Lambda$, and if Λ is a set of interpolation for $\mathcal{B}_\Omega$, then $\Lambda \preccurlyeq \Gamma_0'$.

The body of the proof is an intermediate step in which we compare Λ with an integer multiple of one of the quasi-lattices of Lemma 3. Incidentally, this analysis applies to Γ_0' as well, with

$$\Omega' := \left[-\pi\mu_{\widehat{G}}(\Omega)^{1/d}, \pi\mu_{\widehat{G}}(\Omega)^{1/d}\right]^d \times \{e\} \times K.$$

This observation will enable us to eliminate the quasi-lattices. In this part of the proof, Γ_0' will play a "complementary" role to Λ; Γ_0' is treated as a set of interpolation for $\mathcal{B}_{\Omega'}$ when Λ is a set of sampling for $\mathcal{B}_\Omega$, and vice versa.

We begin by covering Ω by an open set Ω_0 such that $\mu_{\widehat{G}}(\Omega_0 \setminus \Omega) < \epsilon^2/4$. We then take a neighborhood basis $\{V\}$ of e in $\widehat{G}$ and construct the corresponding cubes C_V and discrete sets $D_V \subseteq \widehat{G}$ according to Lemma 3. It is easy to see that the collection $\bigcup_V \{d_V C_V\colon d_V \in D_V\}$ generates the Borel sets in $\widehat{G}$.

By taking V small enough, we may choose a cube $C_0 = C_V$ and a finite number of pairwise disjoint translates $d_j C_0$, $d_j \in D$, $j = 1, \ldots, N$, such that

$$\Omega_* = \bigcup_{j=1}^{N} \Omega_j \subseteq \Omega_0 \quad \text{and} \quad \mu_{\widehat{G}}(\Omega \setminus \Omega_*) < \frac{1}{4}\epsilon^2\mu_{\widehat{G}}(\Omega_*).$$

This is possible because the Haar measure is regular. We may even assume that N is of the form $N = 2^n$ for a positive integer n because the possibly discrete set of permissible values for $\mu_{\widehat{G}}(C_0)$ is sufficiently dense. More precisely, for arbitrary $c > 1$, every interval of the form $(\delta, c\delta)$ will contain a permissible value for $\mu_{\widehat{G}}(C_0)$ provided that δ is sufficiently small.

By Lemma 3, $L^2(d_jC_0)$ possesses an orthonormal basis $\{\psi_\gamma\colon \gamma \in \Gamma\}$ that is labelled by a quasi-lattice Γ in G. Consequently, $L^2(\Omega_*)$ contains an orthonormal basis of the form $\{\psi_{\gamma,j}, \gamma \in \Gamma, j = 1, \ldots, N\}$ where $\psi_{\gamma,j}$ is given explicitly by

$$\psi_{\gamma,j}(\omega) = \mu_{\widehat{G}}(C_0)^{-1/2}\langle \omega, \gamma\rangle \chi_{d_jC_0}\bigl(\pi(\omega)\bigr) \quad \text{for } \gamma \in \Gamma.$$

We now construct another orthonormal basis for $L^2(\Omega_*)$ of the form

$$\phi_{\gamma,j}(\omega) = \mu_{\widehat{G}}(\Omega_*)^{-1/2}\langle \omega, \gamma\rangle g_j(\omega)$$

for $\gamma \in \Gamma$, where g_j is a real function such that $|g_j| = \chi_{\Omega_*}$. We obtain g_j in the following way. Let $U = (u_{kl})$, $k, l = 1, \ldots, N$ be a Hadamard matrix, i.e., U has entries ± 1 and is a multiple of an orthogonal matrix. (Such a matrix exists because $N = 2^n$.) We set

$$\phi_{\gamma,j}(\omega) = \mu_{\widehat{G}}(\Omega_*)^{-1/2}\langle \omega, \gamma\rangle \sum_{k=1}^{N} u_{jk}\chi_{d_kC}(\omega). \tag{12}$$

Then $\{\phi_{\gamma,j}\colon \gamma \in \Gamma, j = 1, \ldots, N\}$ is an orthonormal basis for $L^2(\Omega_*)$ with $\|\phi_{\gamma,j}\|_\infty = \mu_{\widehat{G}}(\Omega_*)^{-1/2}$. Thus

$$\begin{aligned}\operatorname{dist}_{L^2}\bigl(\phi_{\gamma,j}, L^2(\Omega)\bigr) &= \|\phi_{\gamma,j} - \phi_{\gamma,j}\chi_\Omega\|_2 \\ &= \|\phi_{\gamma,j}\|_\infty \|\chi_{\Omega_*} - \chi_\Omega\|_2 = \mu_{\widehat{G}}(\Omega_*)^{-1/2}\mu_{\widehat{G}}(\Omega_*\Delta\Omega)^{1/2} < \frac{\epsilon}{2}.\end{aligned}$$

Let us first assume that Λ is a set of sampling for $\mathcal{B}_\Omega$. We then apply the homogeneous approximation property (Lemma 6) to the frame $e_\lambda = M_\lambda\chi_\Omega$, $\lambda \in \Lambda$, with dual frame h_λ, and each of the functions $g_j\chi_\Omega$. We then obtain a compact set K such that

$$\operatorname{dist}_{L^2(\widehat{G})}\bigl(M_\gamma g_j\chi_\Omega, \operatorname{span}\{h_\lambda \in \Lambda \cap \gamma K\}\bigr) < \frac{\epsilon}{2}$$

for $j = 1, \ldots, N$. Therefore,

$$\operatorname{dist}_{L^2(\widehat{G})}\bigl(\phi_{\gamma,j}, \operatorname{span}\{h_\lambda \in \Lambda \cap \gamma K\}\bigr) < \epsilon.$$

This is exactly the hypothesis of Lemma 5, and we have therefore shown that, for every compact set L, we have

$$(1-\epsilon)N \operatorname{card}(\Gamma \cap L) \leqslant \operatorname{card}(\Lambda \cap KL). \tag{13}$$

If Λ is a set of interpolation, we argue similarly. The only difference is that now the functions $\phi_{\gamma,j} = M_\gamma g_j$ are viewed as a frame, and the functions e_λ constitute a Riesz sequence. We apply again the homogeneous approximation property and use Lemma 4 to get

$$(1-c\epsilon)\operatorname{card}(\Lambda \cap L) \leqslant N \operatorname{card}(\Gamma \cap KL) \tag{14}$$

for every compact set L, where K is the compact set given by Lemma 6.

We have now what we need to finish the proof. To prove part (S) of Theorem 1, we use that Γ_0' is a set of interpolation for $\mathcal{B}_{\Omega'}$. Hence, by (14), there exists a compact set K, such that

$$(1 - c\epsilon)\,\mathrm{card}(\Gamma_0' \cap L) \leqslant N\,\mathrm{card}(\Gamma \cap KL) \tag{15}$$

holds for every compact set L; we may of course adjust ϵ and the approximation of Ω so that the Γ also suits the approximation of Ω'. If Λ is a set of sampling, then combining (13) with (15), we obtain that

$$(1 - c\epsilon)\,\mathrm{card}(\Gamma_0' \cap L) \leqslant \frac{1}{1-\epsilon}\,\mathrm{card}\bigl(\Lambda \cap K^2 L\bigr),$$

from which the desired relation $\Gamma_0' \preccurlyeq \Lambda$ follows.

Reversing the roles of Λ and Γ_0', we obtain similarly $\Lambda \preccurlyeq \Gamma_0'$ when Λ is a set of interpolation for $\mathcal{B}_{\Omega}$.

7. Properties of uniform densities

We return to some basic questions about uniform densities that were raised in Section 2.

Lemma 7. *For every uniformly discrete subset Λ of an LCA group G, we have $\mathcal{D}^-(\Lambda) \leqslant \mathcal{D}^+(\Lambda) < \infty$.*

Proof. It is sufficient to prove that both $\mathcal{D}^+(\Lambda) < \infty$ and $\mathcal{D}^-(\Lambda) < \infty$. Indeed, if $\mathcal{D}^-(\Lambda) > \mathcal{D}^+(\Lambda)$, then $\Lambda \preccurlyeq \delta\Lambda$ for some $\delta < 1$. By the transitivity of the relation '$\preccurlyeq$,' this can only happen if $\mathcal{D}^-(\Lambda) = 0$ or $\mathcal{D}^-(\Lambda) = \infty$.

We first prove that $\mathcal{D}^+(\Lambda) < \infty$. We need to show that there exists a positive number α such that $\Lambda \preccurlyeq \alpha\Gamma_0$. Let L be a compact subset of Λ. Since Λ is uniformly discrete, there is a uniform bound, say M, on the number of points from $L \cap \Lambda$ to be found in each set γK, where $K := [-1/2, 1/2]^d \times \mathbb{T}^m \times \{e\}$ and γ is an element in Γ_0. Therefore,

$$\mathrm{card}(\Lambda \cap L) \leqslant M\,\mathrm{card}(\Gamma_0 \cap KL),$$

and so $\Lambda \preccurlyeq M\Gamma_0$.

We next prove that $\mathcal{D}^-(\Lambda) < \infty$. Let us assume that we have $\alpha\Gamma_0 \preccurlyeq \Lambda$ for some α. Then for every positive ϵ there exists a compact set K such that

$$(1 - \epsilon)\alpha\,\mathrm{card}(\Gamma_0 \cap L) \leqslant \mathrm{card}(\Lambda \cap KL) \tag{16}$$

for every compact set L. We may assume that $K = B \times \mathbb{T}^m \times F$, where B is a ball in $\mathbb{R}^d$ centered at the origin and F is a finite subset of D_0 such $F^{-1} = F$. Then $\bigcup_{n=1}^{\infty} F^n$ is a finitely generated subgroup of D_0, which has the structure $\mathbb{Z}^l \times E$ with E a finite group. (See [9, p. 451].) To simplify the argument, we may assume that F is just $B' \times E$, with B' a ball in $\mathbb{Z}^l$ centered at the origin. We choose $L = K^n$ and note that for sufficiently large n we have

$$\mathrm{card}(\Gamma_0 \cap L) \geqslant (1 - \epsilon)\mu_G(L). \tag{17}$$

On the other hand, if $U \subseteq K$ is an open set such that the sets λU (λ in Λ) are pairwise disjoint, we obtain

$$\operatorname{card}(\Lambda \cap KL) \leqslant \mu_G(U)^{-1} \mu_G\big(K^{n+2}\big) \leqslant (1+\epsilon)\mu_G(U)^{-1}\mu_G(L) \tag{18}$$

whenever n is sufficiently large. Combining (16)–(18), we obtain that for $\epsilon > 0$

$$\alpha \leqslant \frac{1+\epsilon}{(1-\epsilon)^2}\mu_G(U)^{-1},$$

and thus $\mathcal{D}^-(\Lambda) \leqslant \mu_G(U)^{-1}$. $\square$

The relation '$\preccurlyeq$' may be viewed as a relation between discrete measures. Since the canonical lattice Γ_0 has a highly regular distribution, it should come as no surprise that we may replace the discrete measure associated with Γ_0 by the Haar measure μ_G. Interpreting a uniformly discrete set as a sum of point masses located at the points λ of the set, we may generalize the relation '$\preccurlyeq$' to arbitrary nonnegative measures on G. Thus, if ν and τ are two such measures on G, we write $\nu \preccurlyeq \tau$ if for every $\epsilon > 0$ there exists a compact set K in G such that

$$(1-\epsilon)\nu(L) \leqslant \tau(LK)$$

for every compact set L in G. If we set again

$$K = [-1/2, 1/2]^d \times \mathbb{T}^m \times \{e\},$$

then it is immediate that

$$\mu_G(L) \leqslant \operatorname{card}(\Gamma_0 \cap KL) \quad \text{and} \quad \operatorname{card}(\Gamma_0 \cap L) \leqslant \mu_G(KL)$$

for every compact set L. This implies that $\mu_G \preccurlyeq \Gamma_0$ and $\Gamma_0 \preccurlyeq \mu_G$, so that Theorem 1 can be restated in the following form.

Theorem 1$'''$. *Suppose Λ is a uniformly discrete subset of the LCA group G and Ω is a relatively compact subset of $\widehat{G}$.*

(S) *If Λ is a set of sampling for $\mathcal{B}_\Omega$, then $\mu_{\widehat{G}}(\Omega)\mu_G \preccurlyeq \Lambda$.*
(I) *If Λ is a set of interpolation for $\mathcal{B}_\Omega$, then $\Lambda \preccurlyeq \mu_{\widehat{G}}(\Omega)\mu_G$.*

We finally show that, in $\mathbb{R}^d$, our uniform densities coincide with the classical Beurling densities. In $\mathbb{R}^d$ we use the standard additive notation $x + y$ and $K + L$ instead of the multiplicative notation on arbitrary LCA groups, and we write $|U|$ for the Lebesgue (Haar) measure of $U \subseteq \mathbb{R}^d$.

Lemma 8. *If $G = \mathbb{R}^d$, then $\mathcal{D}^-(\Lambda) = \mathcal{D}_B^-(\Lambda)$ and $\mathcal{D}^+(\Lambda) = \mathcal{D}_B^+(\Lambda)$ for every uniformly discrete set Λ.*

Proof. Let Λ be a uniformly discrete subset of $\mathbb{R}^d$. Then, for every $\epsilon > 0$, there exists a compact set $K = Q_R(0) = [-R/2, R/2]^d$ such that

$$(1-\epsilon)\mathcal{D}^-(\Lambda)\,\mathrm{card}\big(\mathbb{Z}^d \cap L\big) \leqslant \mathrm{card}\big(\Lambda \cap \big(L + Q_R(0)\big)\big)$$

for every compact set L. Specializing to cubes $L = Q_h(y)$, $y \in \mathbb{R}^d$, we get that

$$(1-\epsilon)\mathcal{D}^-(\Lambda) \inf_{y\in\mathbb{R}^d} \frac{\mathrm{card}(\mathbb{Z}^d \cap Q_h(y))}{(h+R)^d} \leqslant \inf_{y\in\mathbb{R}^d} \frac{\mathrm{card}(\Lambda \cap Q_{h+R}(y))}{(h+R)^d}.$$

Taking the limit $h \to \infty$, we obtain $(1-\epsilon)\mathcal{D}^-(\Lambda) \leqslant \mathcal{D}_B^-(\Lambda)$, and so $\mathcal{D}^-(\Lambda) \leqslant \mathcal{D}_B^-(\Lambda)$ since the inequality holds for every positive ϵ.

Conversely, for any given $\epsilon > 0$ we may find $h_0 > 0$ such that

$$\frac{\mathrm{card}(\Lambda \cap Q_h(y))}{h^d} \geqslant (1-\epsilon)\mathcal{D}_B^-(\Lambda) \tag{19}$$

for every point y in $\mathbb{R}^d$ and $h > h_0$. Now partition $\mathbb{R}^d$ into cubes $Q_h(hk)$, $k \in \mathbb{Z}^d$, whose interiors are disjoint. Given a compact set $L \subseteq \mathbb{R}^d$, there exist finitely many $k_j \in \mathbb{Z}^d$, $j = 1, \ldots, J$, such that

$$L \subset \bigcup_{j=1}^{J} Q_h(hk_j) \subset L + Q_{2h}(0).$$

Then by (19)

$$\mathrm{card}\big(\Lambda \cap \big(L + Q_{2h}(0)\big)\big) \geqslant \sum_{j=1}^{J} \mathrm{card}\big(\Lambda \cap Q_h(hk_j)\big) \geqslant h^d(1-\epsilon)J\mathcal{D}_B^-(\Lambda).$$

Since $(h+1)^d \geqslant \mathrm{card}(\mathbb{Z}^d \cap Q_h(hk_j))$, it follows that

$$\mathrm{card}\big(\Lambda \cap \big(L + Q_{2h}(0)\big)\big) \geqslant (1-\epsilon)(1+1/h)^{-d}\mathcal{D}_B^-(\Lambda)\,\mathrm{card}\big(\mathbb{Z}^d \cap L\big).$$

We may choose ϵ arbitrarily small and h arbitrarily large, and hence $\mathcal{D}^-(\Lambda) \geqslant \mathcal{D}_B^-(\Lambda)$.

The identity $\mathcal{D}^+(\Lambda) = \mathcal{D}_B^+(\Lambda)$ is proved similarly. □

8. Arbitrary LCA groups

So far we have assumed that $\widehat{G}$ is compactly generated. This is not a serious restriction, as shown by the following lemma. (See also [4].)

Lemma 9. *Assume that $\Omega \subseteq \widehat{G}$ is relatively compact and let H be the open subgroup generated by $\Omega \subseteq \widehat{G}$. Then H is compactly generated and there exists a compact subgroup $K \subseteq G$ such that every $f \in \mathcal{B}_\Omega$ is K-periodic.*

Furthermore, the quotient G/K factors as $G/K \simeq \mathbb{R}^d \times \mathbb{T}^k \times D_0$ for some countable discrete abelian group D_0 and $(G/K)\widehat{\ } = H$, where H is the open subgroup of $\widehat{G}$ that is generated by the spectrum Ω.

Proof. Choose an open, relatively compact neighborhood V of the spectrum $\Omega \subseteq \widehat{G}$, and let H be the *open* subgroup of $\widehat{G}$ that is generated by V. Then $\widehat{G}/H$ is discrete, and thus the group $(\widehat{G}/H)\widehat{\ }$ is compact. We claim that $K := H^\perp$ is the subgroup we are looking for. Let $f \in \mathcal{B}_\Omega$, $x \in G, k \in K$, then by the inversion formula

$$\begin{aligned} f(xk) &= \int_\Omega \widehat{f}(\omega)\omega(xk)\,d\mu_{\widehat{G}}(\omega) \\ &= \int_\Omega \widehat{f}(\omega)\omega(x)\omega(k)\,d\mu_{\widehat{G}}(\omega) \\ &= \int_\Omega \widehat{f}(\omega)\omega(x)\,d\mu_{\widehat{G}}(\omega) = f(x) \end{aligned}$$

since $k \in H^\perp$ and $\Omega \subseteq H$.

Since H is compactly generated, H is isomorphic to a group $H \simeq \mathbb{R}^d \times \mathbb{Z}^k \times L$ for some compact group L by the structure theorem for LCA groups [9, Theorem 9.8]. Consequently,

$$\widehat{H} \simeq \widehat{\widehat{G}}/H^\perp \simeq G/K \simeq \mathbb{R}^d \times \mathbb{T}^k \times D_0,$$

where $D_0 = \widehat{L}$ is a discrete group. $\square$

Consequently, every bandlimited function $f \in \mathcal{B}_\Omega$ lives on a quotient G/K and may be identified with a function $\tilde{f} \in L^2(G/K)$.

Example. Let $\mathbb{Q}_p$ be the group of p-adic numbers [9] with dual group isomorphic to $\mathbb{Q}_p$. The p-adic numbers possess a "quasi-metric" $|\cdot|_p$ such that $|x+y|_p \leqslant \max(|x|_p, |y|_p)$ for all x, y in $\mathbb{Q}_p$. Moreover, for each $n \in \mathbb{Z}$, $K_n := \{x \in \mathbb{Q}_p\colon\ |x|_p \leqslant n\}$ is a compact-open subgroup of $\mathbb{Q}_p$. As a consequence, every relatively compact set $\Omega \subseteq \mathbb{Q}_p$ generates a compact group H contained in some K_n. In particular, $\mathbb{Q}_p$ does not contain any lattice.

It seems that our main theorem does not say anything about sampling in p-adic groups. However, Lemma 9 says that we may assume without loss of generality that $\widehat{G}$ is one of the K_n's where K_n contains the group H generated by the spectrum Ω. Furthermore, all functions in $\mathcal{B}_\Omega$ are $H^\perp$-periodic and thus live on the discrete group $\mathbb{Q}_p/H^\perp$. Thus we may apply Theorem 1 to the pair $G = \mathbb{Q}_p/H^\perp$ and $H \subseteq K_n$.

9. Closing remarks

(1) In his paper [12], Landau made a slightly weaker assumption on Ω when considering sets of sampling. Instead of taking Ω to be relatively compact, he assumed that Ω had positive measure. It is clear that we may similarly take Ω to have positive Haar measure in part (S) of Theorem 1 because such Ω can be approximated by compact sets contained in Ω. Note that this relaxation cannot be made in part (I) of Theorem 1.

(2) Landau used his results in [12] to prove a conjecture of A. Beurling concerning the lower uniform density of sets in $\mathbb{R}^d$ for which so-called balayage is possible. We do not wish to go into detail about Beurling's problem, but we would like to point out that, using our notion of density, we may extend Landau's result concerning balayage. The restriction we have to make is that the group G be of the form $G = \mathbb{R}^d \times \mathbb{Z}^m \times K_0$ with $d \geqslant 1$. Theorem 5 in [12] extends from the setting of $\mathbb{R}^d$ to such groups, under the same regularity conditions on the spectrum. The details needed to carry out this extension can be found in [2, pp. 341–350] and in Landau's paper [12].

(3) In his thesis [14], Marzo proved that for every relatively compact set Ω in $\mathbb{R}^d$ we can find sets of sampling and sets of interpolation for $\mathcal{B}_\Omega$ of Beurling densities arbitrarily close to those given by Landau's theorem. It would be interesting to know if, similarly, our density conditions are optimal for every relatively compact set in a general LCA group.

(4) In Section 2, we excluded the case of compact groups. Our result is certainly of no interest for compact groups, but for such groups one can state closely related and nontrivial problems. An example is the recent work of J. Ortega-Cerdà and J. Saludes on Marcinkiewicz–Zygmund inequalities [15]. Their work deals with the group $G = \mathbb{T}$ and the asymptotic behavior of sets of sampling and interpolation when the size of the spectrum grows and we require uniform bounds on the norms. Another, probably much more difficult problem, is to describe similarly asymptotic density conditions when $G = \mathbb{T}^m$ and both the spectrum and m grow.

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