Journal of Number Theory 131 (2011) 1258–1275

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Galois deformation theory for norm fields and flat deformation rings

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A R T I C L E   I N F O

Article history:
Received 26 June 2010
Revised 13 January 2011
Accepted 13 January 2011
Available online 21 March 2011
Communicated by Mark Kisin

MSC:
11S20

Keywords:
Kisin theory
Local Galois deformation theory
Equi-characteristic analogue of Fontaine's theory

A B S T R A C T

Let K be a finite extension of $\mathbb{Q}_p$, and choose a uniformizer $\pi \in K$, and put $K_\infty := K(\sqrt[p^\infty]{\pi})$. We introduce a new technique using restriction to $\mathrm{Gal}(\overline{K}/K_\infty)$ to study flat deformation rings. We show the existence of deformation rings for $\mathrm{Gal}(\overline{K}/K_\infty)$-representations "of height $\leqslant h$" for any positive integer h, and prove that when $h = 1$ they are isomorphic to "flat deformation rings". This $\mathrm{Gal}(\overline{K}/K_\infty)$-deformation theory has a good positive characteristics analogue of crystalline representations in the sense of Genestier–Lafforgue. In particular, we obtain a positive characteristic analogue of crystalline deformation rings, and can analyze their local structure.

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2011.01.008

0. Introduction

Since the pioneering work of Wiles on the modularity of semi-stable elliptic curves over $\mathbb{Q}$, there has been huge progress on modularity lifting. Notably, Kisin [21,20] (later improved by Gee [8,9]) proved a very general modularity lifting theorem for potentially Barsotti–Tate representations, which had enormous impacts on this subject. (For the precise statement of the theorem, see the aforementioned references.) An important technical step is the study of certain local deformation rings (namely, flat deformation rings).[1] The main purpose of this paper is to develop another technique to study flat deformation rings.

To explain, let K be a finite extension of $\mathbb{Q}_p$, $K_\infty = K(\sqrt[p^\infty]{\pi})$ for a chosen uniformizer $\pi \in K$, $\mathcal{G}_K := \mathrm{Gal}(\overline{K}/K)$ and $\mathcal{G}_{K_\infty} := \mathrm{Gal}(\overline{K}/K_\infty)$. We study a special kind of p-adic $\mathcal{G}_{K_\infty}$-representations (i.e., representations of height $\leqslant h$, which is defined in Section 1.2) which contains the $\mathcal{G}_{K_\infty}$-restrictions of semistable $\mathcal{G}_K$-representations with Hodge–Tate weights in a certain range[2] [17, Proposition 2.1.5]. Kisin used them to construct potentially semi-stable (respectively, potentially crystalline) $\mathcal{G}_K$-deformation rings [19], and $\mathcal{G}_{K_\infty}$-representations of height $\leqslant 1$ plays an important role in classifying finite flat group schemes and p-divisible groups over $\mathscr{O}_K$ (*e.g.*, [17, Section 2.2, Section 2.3], [20, Section 1], and [16]).

In this paper, we construct a deformation ring of $\mathcal{G}_{K_\infty}$-representations whose generic fiber classifies all $\mathcal{G}_{K_\infty}$-lifts of height $\leqslant h$. Let $\bar\rho_\infty : \mathcal{G}_{K_\infty} \to \mathrm{GL}_d(\mathbb{F})$ be a representation, where $\mathbb{F}$ is a finite field of characteristic p. For any finite algebra A over $\mathrm{Frac}\, W(\mathbb{F})$, we say a representation $\rho_A : \mathcal{G}_{K_\infty} \to \mathrm{GL}_d(A)$ is an *A-lift* of $\bar\rho_\infty$ if there exists a finite flat $W(\mathbb{F})$-subalgebra $A^\circ \subset A$ such that ρ_A factors through $\mathrm{GL}_d(A^\circ)$ and this lifts $\bar\rho_\infty \otimes_{\mathbb{F}} (A^\circ/\mathrm{rad}\, A^\circ)$ as defined by Mazur [25,26]. The following theorem is the main result of this paper:

Theorem *(Theorem 1.3). Let h be a non-negative integer. There is a complete local noetherian $W(\mathbb{F})$-algebra $R_\infty^{\square,\leqslant h}$ such that for any finite algebra A over* $\mathrm{Frac}\, W(\mathbb{F})$ *there exists a natural bijection between* $\mathrm{Hom}_{W(\mathbb{F})}(R_\infty^{\square,\leqslant h}, A)$ *and the set of A-lifts of $\bar\rho_\infty$ which are of height $\leqslant h$ as p-adic representations* (*Definition* 1.1.3)*.*

We indeed prove slightly more: the ring $R_\infty^{\square,\leqslant h}$ represents some functor on the artin local $W(\mathbb{F})$-algebras. Note that we cannot apply Mazur's theorem [25,26] to (unrestricted) $\mathcal{G}_{K_\infty}$-deformation functors, as $\mathcal{G}_{K_\infty}$ does not satisfy "p-finiteness". See Section 1.4 for more discussions.

The case of interest is when $\bar\rho_\infty$ is obtained as a $\mathcal{G}_{K_\infty}$-restriction of a $\mathcal{G}_K$-representation $\bar\rho$. In this case, one obtains, almost by construction, a natural map from a framed $\mathcal{G}_{K_\infty}$-deformation ring of $\bar\rho_\infty$ with height $\leqslant h$ to framed semi-stable and crystalline $\mathcal{G}_K$-deformation rings of $\bar\rho$ (with suitable Hodge–Tate weight bounds) defined by viewing $\mathcal{G}_K$-deformations as $\mathcal{G}_{K_\infty}$-deformations (Lemma 2.1.3). This induces a natural isomorphism between a $\mathcal{G}_{K_\infty}$-deformation ring of $\bar\rho_\infty$ with height $\leqslant 1$ and flat deformation ring of $\bar\rho$ (Corollary 2.2.1), which follows from the Breuil–Kisin classification of finite flat group schemes.[3]

Kisin used his construction of moduli of finite flat group schemes to prove irreducibility of certain loci in flat deformation spaces, which is the crucial local argument in proving modularity lifting for potentially Barsotti–Tate representations. (For the precise statement, see [21, Corollary 2.5.16], [8], and [14].) In this paper, we observe that exactly the same linear algebra argument as in [21, Section 2] improved by [8] and [14] proves the same irreducibility statement for some $\mathcal{G}_{K_\infty}$-deformation rings with height $\leqslant 1$ (Proposition 3.3). It seems not too implausible that one can prove its mild generalization when the height bound h could be bigger than 1, though we have not succeeded here for the reason explained in Remark 3.4.

1 See [18, Corollary 1.4] for the list of sufficient conditions on local deformation rings to prove modularity lifting.

2 More precisely, we mean semistable $\mathcal{G}_K$-representation V such that $\mathrm{gr}^w D^*_{\mathrm{dR}}(V) = 0$ for all $w \notin [0, h]$.

3 When $p > 2$ this is proved by Kisin [17, Theorem 2.3.5], and when $p = 2$ this is proved by Kisin [20, Section 1] under connectedness assumption which was removed by the author [16], and independently by Eike Lau [23,22] with a different approach.

A harder but more interesting question would be whether one can give a meaningful and workable description of the (rigid analytic) subspace of crystalline lifts inside a $\mathcal{G}_{K_\infty}$-deformation space of $\bar{\rho}_\infty$ with height $\leqslant h$.

We point out that our technique is motivated by the author's study of positive characteristic analogue of crystalline deformation rings (using the theory of Genestier and Lafforgue [10] and Hartl [13, 12]). We include a section (Section 4) to sketch this positive characteristic deformation theory.

0.1. Structure and overview of the paper

In Section 1 we prove the existence of "$\mathcal{G}_{K_\infty}$-deformation ring of height $\leqslant h$". In Section 2 we discuss the relation with semi-stable and crystalline deformation rings, and show that a "$\mathcal{G}_{K_\infty}$-deformation ring of height $\leqslant 1$" is isomorphic to a flat deformation ring. In Section 3 we interpret moduli of finite flat group schemes [21, Section 2] as a kind of "resolution" of $\mathcal{G}_{K_\infty}$-deformation space of height $\leqslant 1$. In Section 4, we explain the positive characteristic analogue of this deformation theory.

1. Deformation rings of height $\leqslant h$

Let k be a perfect field of characteristic $p > 0$ (which will be assumed to be a finite field for the most of this paper[4]), $W(k)$ its ring of Witt vectors, and $K_0 := W(k)[\frac{1}{p}]$. Let K be a finite totally ramified extension of K_0 and let us fix its algebraic closure $\overline{K}$. We *fix* a uniformizer $\pi \in K$, and choose $\pi^{(n)} \in \overline{K}$ so that $(\pi^{(n+1)})^p = \pi^{(n)}$ and $\pi^{(0)} = \pi$. Put $K_\infty := \bigcup_n K(\pi^{(n)})$, $\mathcal{G}_K := \mathrm{Gal}(\overline{K}/K)$, and $\mathcal{G}_{K_\infty} := \mathrm{Gal}(\overline{K}/K_\infty)$. We refer to [17] for the motivation of considering K_∞.

1.1. Étale φ-modules and Kisin modules

Let us consider a ring R equipped with an endomorphism $\sigma : R \to R$. (We will often assume that σ is finite flat.) By *(φ, R)-module* (often abbreviated as a *φ-module*, if R is understood), we mean a finitely presented R-module M together with an R-linear morphism $\varphi_M : \sigma^* M \to M$, where σ^* denotes the scalar extension by σ. A *morphism* between two (φ, R)-modules is a φ-compatible R-linear map. For any R-algebra R' equipped with an endomorphism σ' over σ, the "scalar extension" $M \otimes_R R'$ has a natural (φ, R')-module structure.

Let $\mathfrak{S} := W(k)[[u]]$ where u is a formal variable. Let $\mathscr{O}_\mathcal{E}$ be the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$, and $\mathcal{E} := \mathscr{O}_\mathcal{E}[\frac{1}{p}]$. Note that $\mathscr{O}_\mathcal{E}$ is a complete discrete valuation ring with uniformizer p and $\mathscr{O}_\mathcal{E}/(p) \cong k((u))$.[5] We extend the Witt vectors Frobenius to $\mathfrak{S}$, $\mathscr{O}_\mathcal{E}$, and $\mathcal{E}$ by sending u to u^p, and denote them by σ. (We write $\sigma_\mathfrak{S}$ instead, if we need to specify that it is an endomorphism on $\mathfrak{S}$, for example.) Note that σ is finite and flat. We denote by $\sigma^*(\cdot)$ the scalar extension by σ. We fix an Eisenstein polynomial $\mathcal{P}(u) \in W(k)[u]$ with $\mathcal{P}(\pi) = 0$ and $\mathcal{P}(0) = p$, and view it as an element of $\mathfrak{S}$.

Definition 1.1.1. An *étale φ-module* is a $(\varphi, \mathscr{O}_\mathcal{E})$-module (M, φ_M) such that $\varphi_M : \sigma^* M \xrightarrow{\sim} M$ is an isomorphism. We say an étale φ-module M is *free* (respectively, *torsion*) if the underlying $\mathscr{O}_\mathcal{E}$-module is free (respectively, p^∞-torsion).

For a non-negative integer h, a *φ-module of height $\leqslant h$* is a $(\varphi, \mathfrak{S})$-module such that the underlying $\mathfrak{S}$-module $\mathfrak{M}$ is free and $\mathrm{coker}(\varphi_\mathfrak{M})$ is killed by $\mathcal{P}(u)^h$. A *torsion φ-module of height $\leqslant h$* is a $(\varphi, \mathfrak{S})$-module such that the underlying $\mathfrak{S}$-module $\mathfrak{M}$ is p^∞-torsion with no non-zero u-torsion and $\mathrm{coker}(\varphi_\mathfrak{M})$ is killed by $\mathcal{P}(u)^h$.

Note that a non-zero p^∞-torsion $\mathfrak{S}$-module is of projective dimension $\leqslant 1$ if and only if it has no non-zero u-torsion, and by [17, Lemma 2.3.4] any torsion φ-module of height $\leqslant h$ is obtained as a φ-equivariant quotient of φ-module of height $\leqslant h$.

[4] All the deformation theory results require k to be finite, while for Proposition 2.2 it suffices to assume k is perfect.
[5] We should view the residue field $k((u))$ as the norm field for the extension K_∞/K. See [29] for more details.

Let $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$ denote the p-adic completion of strict henselization of $\mathscr{O}_{\mathcal{E}}$. By the universal property of strict henselization, $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$ has a natural $\mathcal{G}_{K_\infty}$-action and a ring endomorphism σ. For a finitely generated $\mathbb{Z}_p$-module T with continuous $\mathcal{G}_{K_\infty}$-action, define

$$\underline{D}_{\mathcal{E}}(T) := (T \otimes_{\mathscr{O}_{\mathcal{E}}} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{\mathcal{G}_{K_\infty}}, \tag{1.1.2a}$$

equipped with the φ-structure induced from σ on $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$. For an étale φ-module M, define

$$\underline{T}_{\mathcal{E}}(M) := (M \otimes_{\mathscr{O}_{\mathcal{E}}} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{\varphi=1}, \tag{1.1.2b}$$

viewed as a $\mathcal{G}_{K_\infty}$-module via its natural action on $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$.

By Fontaine [7, Section A1.2], $\underline{T}_{\mathcal{E}}$ and $\underline{D}_{\mathcal{E}}$ define quasi-inverse exact equivalences of categories between the categories of étale φ-modules and the category of finitely generated $\mathbb{Z}_p$-module with continuous $\mathcal{G}_{K_\infty}$-action, which respects all the natural operations and preserves ranks and lengths whenever applicable.

Let $\mathfrak{M}$ denote either a φ-module of height $\leqslant h$ or a torsion φ-module of height $\leqslant h$. Then $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}}$ is an étale φ-module,[6] so we may associate $\mathcal{G}_{K_\infty}$-representation to such $\mathfrak{M}$ as follows:

$$\underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}) := \underline{T}_{\mathcal{E}}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}})(h), \tag{1.1.2c}$$

where $T(h)$ denotes the "Tate twist"; i.e., twisting the $\mathcal{G}_{K_\infty}$-action on T by $\chi^h_{\mathrm{cyc}}|_{\mathcal{G}_{K_\infty}}$. It is a non-trivial theorem of Kisin [17, Proposition 2.1.12] that this functor $\underline{T}^{\leqslant h}_{\mathfrak{S}}$ from the category of φ-module of height $\leqslant h$ to the category of $\mathcal{G}_{K_\infty}$-representations is fully faithful. Note that $\underline{T}^{\leqslant h}_{\mathfrak{S}}$ is not in general fully faithful on the category of *torsion* φ-module of height $\leqslant h$.

Definition 1.1.3. A $\mathbb{Z}_p$-lattice $\mathcal{G}_{K_\infty}$-representation[7] T is *of height* $\leqslant h$ if there exists a φ-module $\mathfrak{M}$ of height $\leqslant h$ such that $T \cong \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M})$.

A p-adic $\mathcal{G}_{K_\infty}$-representation V is *of height* $\leqslant h$ if there exists a $\mathcal{G}_{K_\infty}$-stable $\mathbb{Z}_p$-lattice which is of height $\leqslant h$ (or equivalently by [17, Lemma 2.1.15],[8] if any $\mathcal{G}_{K_\infty}$-stable $\mathbb{Z}_p$-lattice which is of height $\leqslant h$).

A torsion $\mathcal{G}_{K_\infty}$-representation[9] T is *of height* $\leqslant h$ if there exists a torsion φ-module $\mathfrak{M}$ of height $\leqslant h$ such that $T \cong \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M})$.

The motivation of this definition is Kisin's theorem [17, Proposition 2.1.5] which asserts that the restriction to $\mathcal{G}_{K_\infty}$ of a p-adic semi-stable $\mathcal{G}_K$-representation with Hodge–Tate weights in $[0, h]$ is of height $\leqslant h$.

Lemma 1.1.4. *A torsion $\mathcal{G}_{K_\infty}$-representation T is of height $\leqslant h$ if and only if $T \cong \widetilde{T}/\widetilde{T}'$ for some $\widetilde{T}$ and $\widetilde{T}'$ which are $\mathbb{Z}_p$-lattice $\mathcal{G}_{K_\infty}$-representations of height $\leqslant h$.*

Proof. Note that for any exact sequence $(\dagger): 0 \to \widetilde{\mathfrak{M}}' \to \widetilde{\mathfrak{M}} \to \mathfrak{M} \to 0$ with $\widetilde{\mathfrak{M}}, \widetilde{\mathfrak{M}}' \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$ and $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, the sequence $\underline{T}^{\leqslant h}_{\mathfrak{S}}(\dagger)$ is exact. This is a consequence of the exactness of $\underline{T}_{\mathcal{E}}$ defined in (1.1.2b), which is proved in [7, Section A1.2]. The "if" direction now follows from this.

6 Note that $\mathcal{P}(u)$ is a unit in $\mathscr{O}_{\mathcal{E}}$.

7 I.e., a finite free $\mathbb{Z}_p$-module with continuous $\mathcal{G}_{K_\infty}$-action.

8 In fact, we need a slight refinement of [17, Lemma 2.1.15]; namely, replacing "finite height" in the statement by "height $\leqslant h$". The proof can be easily modified to prove this refinement.

9 I.e., a finite torsion $\mathbb{Z}_p$-module with continuous $\mathcal{G}_{K_\infty}$-action.

To show the “only if” direction, one needs to show that any $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ can be put in some exact sequence (†) as above, but this could be done by the essentially same proof of [17, Lemma 2.3.4].[10] □

1.2. Deformations of height $\leqslant h$

From now on, we assume that the residue field k of $\mathcal{O}_K$ is finite unless stated otherwise. Let $\mathbb{F}$ be a finite field of characteristic p, and $\bar{\rho}_\infty : \mathcal{G}_{K_\infty} \to \mathrm{GL}_d(\mathbb{F})$ be a representation. Let $\mathcal{O}$ be a p-adic discrete valuation ring with residue field $\mathbb{F}$. Let $\mathfrak{AR}_{\mathcal{O}}$ be the category of artin local $\mathcal{O}$-algebras A whose residue field is $\mathbb{F}$, and let $\widehat{\mathfrak{AR}}_{\mathcal{O}}$ be the category of complete local noetherian $\mathcal{O}$-algebras with residue field $\mathbb{F}$.

Let $D_\infty, D_\infty^\Box : \widehat{\mathfrak{AR}}_{\mathcal{O}} \to (\mathbf{Sets})$ be the deformation functor and framed deformation functor for $\bar{\rho}_\infty$. For the definition, see the standard references such as [26,25,11]. Contrary to local and global deformation functors we usually consider, these functors *cannot* be represented by complete local noetherian rings since the tangent spaces $D_\infty(\mathbb{F}[\epsilon])$ and $D_\infty^\Box(\mathbb{F}[\epsilon])$ are infinite-dimensional $\mathbb{F}$-vector spaces. See Section 1.4 for more details.

We say that a deformation $\rho_{\infty,A}$ over $A \in \mathfrak{AR}_{\mathcal{O}}$ is *of height $\leqslant h$* if it is a torsion $\mathcal{G}_{K_\infty}$-representation of height $\leqslant h$ as a torsion $\mathbb{Z}_p[\mathcal{G}_{K_\infty}]$-module; or equivalently, if there exists $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ and an isomorphism $\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}) \cong \rho_{\infty,A}$ as $\mathbb{Z}_p[\mathcal{G}_{K_\infty}]$-modules. For $A \in \widehat{\mathfrak{AR}}_{\mathcal{O}}$, we say that $\rho_{\infty,A}$ is *of height $\leqslant h$* if $\rho_{\infty,A} \otimes A/\mathfrak{m}_A^n$ is a deformation of height $\leqslant h$ for $n \gg 1$.[11] When $A \in \mathfrak{AR}_{\mathcal{O}}$, both definitions clearly coincide. When A is finite flat over $\mathbb{Z}_p$, a deformation $\rho_{\infty,A}$ over A is of height $\leqslant h$ if and only if $\rho_{\infty,A}$ is of height $\leqslant h$ as a $\mathbb{Z}_p$-lattice $\mathcal{G}_{K_\infty}$-representation (in the sense of Definition 1.1.3), by [24, Theorem 2.4.1].

Let $D_\infty^{\leqslant h} \subset D_\infty$ and $D_\infty^{\Box,\leqslant h} \subset D_\infty^\Box$ respectively denote subfunctors of deformations and framed deformations of height $\leqslant h$. The following theorem is one of the main result of this paper:

Theorem 1.3. *Assume that k is finite. Then functor $D_\infty^{\leqslant h}$ always has a hull. If furthermore* $\mathrm{End}_{\mathcal{G}_{K_\infty}}(\bar{\rho}_\infty) \cong \mathbb{F}$ *then $D_\infty^{\leqslant h}$ is representable (by $R_\infty^{\leqslant h} \in \widehat{\mathfrak{AR}}_{\mathcal{O}}$). The functor $D_\infty^{\Box,\leqslant h}$ is representable (by $R_\infty^{\Box,\leqslant h} \in \widehat{\mathfrak{AR}}_{\mathcal{O}}$) with no assumption on $\bar{\rho}_\infty$. Furthermore, the natural inclusions $D_\infty^{\leqslant h} \hookrightarrow D_\infty$ and $D_\infty^{\Box,\leqslant h} \hookrightarrow D_\infty^\Box$ of functors are relatively representable by surjective maps in $\widehat{\mathfrak{AR}}_{\mathcal{O}}$.*

We call $R_\infty^{\Box,\leqslant h}$ the *universal framed deformation ring of height $\leqslant h$* and $R_\infty^{\leqslant h}$ the *universal deformation ring of height $\leqslant h$* if it exists. We prove this theorem for the rest of this section beginning from Section 1.4.

Remark 1.3.1. Let A be any finite algebra over $\mathrm{Frac}\,\mathcal{O}$, and $\rho_{\infty,A} : \mathcal{G}_{K_\infty} \to \mathrm{GL}_d(A)$ be any lift of $\bar{\rho}_\infty$; i.e., there exist some finite $\mathcal{O}$-subalgebra $A^\circ \subset A$ and $\mathcal{G}_{K_\infty}$-stable A°-lattice in $\rho_{\infty,A}$ which lifts $\bar{\rho}_\infty$. Then, from [24, Theorem 2.4.1] it follows that $\rho_{A,\infty}$ arises as a pull back of the universal (framed) deformation of height $\leqslant h$ if and only if $\rho_{A,\infty}$ is of height $\leqslant h$ as a $\mathbb{Q}_p$-representation.

Remark 1.3.2. It is not very difficult to show that $R_\infty^{\Box,\leqslant h}[\frac{1}{p}]$ (respectively, $R_\infty^{\leqslant h}[\frac{1}{p}]$, if it exists) is a formally smooth $\mathrm{Frac}\,\mathcal{O}$-algebra; see [15, Theorem 11.2.9] for the proof. Furthermore, one can compute, with harder work, the dimension of the equi-dimensional union of connected components defined by fixing a suitable analogue of Hodge type, and obtain a formula analogous to Kisin's dimension formula for crystalline deformation rings [19, Theorem 3.3.8]. See [15, Corollary 11.3.11] for the precise statement and the proof.

[10] We will only need this result when $h = 1$ which is proved in [17, Lemma 2.3.4]. In general, one just need to modify the proof as follows: using the same notation as in the proof of [17, Lemma 2.3.4], take $\widetilde{L}$ to be a finite free $\mathfrak{S}/\mathcal{P}(u)^h$-module which admits an $\mathfrak{S}/\mathcal{P}(u)^h$-surjection $\widetilde{L} \twoheadrightarrow L := \mathrm{coker}(1 \otimes \varphi_{\mathfrak{M}})$.

[11] By Lemma 1.4.1, it is equivalent to require that $\rho_{\infty,A} \otimes A/\mathfrak{m}_A^n$ be a deformation of height $\leqslant h$ for each n.

1.4. Resume of Mazur's and Ramakrishna's theory

Schlessinger [28, Theorem 2.11] gave a set of criteria (H1)–(H4) for a functor $D : \mathfrak{AR}_{\mathscr{O}} \to (\mathbf{Sets})$ to be representable. For a profinite group Γ and a continuous $\mathbb{F}$-linear Γ-representation $\bar\rho$, Mazur [25, Section 1.2] showed that the framed deformation functor $D^{\square}_{\bar\rho}$ of $\bar\rho$ satisfies all the Schlessinger criteria *except* the finiteness of the "tangent space" $D^{\square}_{\bar\rho}(\mathbb{F}[\epsilon])$, and the same for the deformation functor $D_{\bar\rho}$ if $\mathrm{End}_{\Gamma}(\bar\rho) \cong \mathbb{F}$. When Γ is either an absolute Galois group for a finite extension of $\mathbb{Q}_p$, or a certain quotient of the absolute Galois group of any finite extension of $\mathbb{Q}$, Mazur obtained the finiteness of the tangent space from so-called p-finiteness [25, Section 1.1], but it is very unlikely to hold for more general class of Γ.

Unfortunately, $\mathcal{G}_{K_\infty}$ does not satisfy the p-finiteness, and in fact the tangent space $D_\infty(\mathbb{F}[\epsilon])$ is infinite even when $\bar\rho_\infty$ is 1-dimensional. To see this, note that $D_\infty(\mathbb{F}[\epsilon]) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathcal{G}_{K_\infty}, \mathbb{F})$ when $\bar\rho_\infty$ is 1-dimensional. This is infinite from the norm field isomorphism $\mathcal{G}_{K_\infty} \cong \mathrm{Gal}(k((u))^{\mathrm{sep}}/k((u)))$ and the existence of infinitely many Artin–Schreier cyclic p-extensions of $k((u))$. For a finite-dimensional $\bar\rho_\infty$, one sees that the deformation and framed deformation functors D_∞ and $D^{\square}_\infty$ never satisfy (H3) from deforming the determinant,[12] and in particular these 'unrestricted' deformation functors are never represented by a complete local noetherian ring.

Now, let us look at the subfunctors $D^{\leqslant h}_\infty \subset D_\infty$ and $D^{\square,\leqslant h}_\infty \subset D^{\square}_\infty$ which consist of deformations of height $\leqslant h$ (as defined in Section 1.2). We first state the following lemma:

Lemma 1.4.1. *Any subquotients and direct sums of torsion $\mathcal{G}_{K_\infty}$-representations of height $\leqslant h$ is of height $\leqslant h$.*

Proof. The assertion about direct sums is obvious. Now consider a short exact sequence $0 \to M' \to M \to M'' \to 0$ of p^∞-torsion étale φ-modules and assume that there is a φ-stable $\mathfrak{S}$-submodule $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ in M such that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} = M$. Let $\mathfrak{M}''$ be the image of $\mathfrak{M}$ by $M \twoheadrightarrow M''$ and $\mathfrak{M}'$ the kernel of the natural map $\mathfrak{M} \to \mathfrak{M}''$. One can check that $\mathfrak{M}'$ and $\mathfrak{M}''$ are objects in $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ such that $\mathfrak{M}' \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} = M'$ and $\mathfrak{M}'' \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} = M''$. Now the lemma follows from the exactness of $\underline{D}_{\mathcal{E}}$ and $\underline{T}_{\mathcal{E}}$. $\square$

Lemma 1.4.1 implies that the condition of being of height $\leqslant h$ is closed under fiber products. It immediately follows (*cf.* the proof of [27, Theorem 1.1]) that the functor $D^{\square,\leqslant h}_\infty$ satisfies all the Schlessinger criteria except the finiteness of $D^{\square,\leqslant h}_\infty(\mathbb{F}[\epsilon])$; and the same for $D^{\leqslant h}_\infty$ if we have $\mathrm{End}_{\mathcal{G}_{K_\infty}}(\bar\rho_\infty) \cong \mathbb{F}$. So to prove the representability assertion of Theorem 1.3 it remains to check the finiteness[13] of $D^{\leqslant h}_\infty(\mathbb{F}[\epsilon])$ and $D^{\square,\leqslant h}_\infty(\mathbb{F}[\epsilon])$. Before doing this, let us digress to show the relative representability of the subfunctor $D^{\leqslant h}_\infty \subset D_\infty$, which "essentially" follows from Lemma 1.4.1.

Proposition 1.5. *The subfunctor $D^{\leqslant h}_\infty \subset D_\infty$ is relatively representable by surjective maps in $\widehat{\mathfrak{AR}}_{\mathscr{O}}$. In other words, for any given deformation ρ_A over $A \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$, there exists a universal quotient $A^{\leqslant h}$ of A over which the deformation is of height $\leqslant h$.*

Proof. Consider a functor $\mathfrak{h}_A : \mathfrak{AR}_{\mathscr{O}} \to (\mathbf{Sets})$ defined by $\mathfrak{h}_A(B) := \mathrm{Hom}_{\mathscr{O}}(B, A)$ for $B \in \mathfrak{AR}_{\mathscr{O}}$, and a subfunctor $\mathfrak{h}^{\leqslant h}_A \subset \mathfrak{h}_A$ defined as below:

$$\mathfrak{h}^{\leqslant h}_A(B) := \{ f : B \to A \text{ such that } \rho_A \otimes_{A,f} B \text{ is of height } \leqslant h \},$$

[12] For any $\mathbb{F}[\epsilon]$-deformation $\det(\bar\rho_\infty) + \epsilon \cdot c$ of $\det(\bar\rho_\infty)$ (where $c : \mathcal{G}_K \to F$ is a cocycle), the deformation $\bar\rho_\infty + \epsilon \cdot \tilde c$ with $\tilde c := \begin{pmatrix} c & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}$ has determinant $\det(\bar\rho_\infty) + \epsilon \cdot c$.

[13] Even though $D^{\square}_\infty(\mathbb{F}[\epsilon])$ is infinite, one can hope that the subspace $D^{\square,\leqslant h}_\infty(\mathbb{F}[\epsilon])$ is finite.

where $B \in \mathfrak{AR}_{\mathcal{O}}$. Since $\mathfrak{h}_A$ is prorepresentable and the subfunctor $\mathfrak{h}_A^{\leqslant h}$ is closed under subquotients and direct sums, it follows that $\mathfrak{h}_A^{\leqslant h}$ is prorepresentable, say by a quotient $A^{\leqslant h}$ of A. It is clear that $A^{\leqslant h}$ satisfies the desired properties. (*Cf.* the proof of [27, Theorem 1.1].) □

Now let us verify (H3) for $D_\infty^{\leqslant h}$ and $D_\infty^{\square,\leqslant h}$, thus prove the representability assertion of Theorem 1.3.

Proposition 1.6. *Assume that k is finite. Then the tangent spaces $D_\infty^{\leqslant h}(\mathbb{F}[\epsilon])$ and $D_\infty^{\square,\leqslant h}(\mathbb{F}[\epsilon])$ are finite-dimensional $\mathbb{F}$-vector spaces.*

Proof. Let us first fix the notation.

1.6.1. Notations and definitions

Let A be a p-adically separated and complete topological ring[14] (for example, finite $\mathbb{Z}_p$-algebras or any ring A with $p^N \cdot A = 0$ for some N). Set $\mathfrak{S}_A := \mathfrak{S} \widehat{\otimes}_{\mathbb{Z}_p} A := \varprojlim_\alpha \mathfrak{S} \widehat{\otimes}_{\mathbb{Z}_p} A/I_\alpha$ where $\{I_\alpha\}$ is a basis of open ideals in A. We define a ring endomorphism $\sigma : \mathfrak{S}_A \to \mathfrak{S}_A$ (and call it the *Frobenius endomorphism*) by A-linearly extending the Frobenius endomorphism $\sigma_{\mathfrak{S}}$. We also put $\mathcal{O}_{\mathcal{E},A} := \mathcal{O}_{\mathcal{E}} \widehat{\otimes}_{\mathbb{Z}_p} A := \varprojlim_\alpha \mathcal{O}_{\mathcal{E}} \widehat{\otimes}_{\mathbb{Z}_p} A/I_\alpha$ and similarly define an endomorphism $\sigma : \mathcal{O}_{\mathcal{E},A} \to \mathcal{O}_{\mathcal{E},A}$.

Let $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ be the category of finite free $\mathfrak{S}_A$-modules $\mathfrak{M}_A$ equipped with an $\mathfrak{S}_A$-linear map $\varphi_{\mathfrak{M}_A} : \sigma^*(\mathfrak{M}_A) \to \mathfrak{M}_A$ such that $\mathcal{P}(u)^h$ annihilates $\mathrm{coker}(\varphi_{\mathfrak{M}_A})$. If A is finite artinean $\mathbb{Z}_p$-algebra, then $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ is precisely a torsion $(\varphi, \mathfrak{S})$-module of height $\leqslant h$ equipped with a φ-compatible A-action such that $\mathfrak{M}_A$ is finite free over $\mathfrak{S}_A$.

Let $(\mathrm{ModFI}/\mathcal{O}_{\mathcal{E}})_A^{\text{ét}}$ be the category of finite free $\mathcal{O}_{\mathcal{E},A}$-modules M_A equipped with an $\mathcal{O}_{\mathcal{E},A}$-linear isomorphism $\varphi_{M_A} : \sigma^*(M_A) \xrightarrow{\sim} M_A$. If A is finite artinean $\mathbb{Z}_p$-algebra, then one can check that $\underline{T}_{\mathcal{E}}$ and $\underline{D}_{\mathcal{E}}$, defined in (1.1.2), induce rank-preserving quasi-inverse exact equivalences of categories between the category of A-representations of $\mathcal{G}_{K_\infty}$ and $(\mathrm{ModFI}/\mathcal{O}_{\mathcal{E}})_A^{\text{ét}}$.[15]

Lemma 1.6.2. *Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$, and $\bar{\rho}$ a $\mathcal{G}_{K_\infty}$-representation over $\mathbb{F}$ which is of height $\leqslant h$ as a torsion $\mathcal{G}_{K_\infty}$-representation* (*in the sense of Definition* 1.1.3). *Then there exists $\mathfrak{M}_{\mathbb{F}} \in (\mathrm{ModFI}/\mathfrak{S})_{\mathbb{F}}^{\leqslant h}$ such that $\bar{\rho} \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_{\mathbb{F}})$.*

Proof. Put $M := \underline{D}_{\mathcal{E}}^{\leqslant h}(\bar{\rho}(-h))$ and let $\mathfrak{M}_{\mathbb{F}} := \mathfrak{M}^+ \subset M$ be the maximal $\mathfrak{S}$-submodule of height $\leqslant h$, which exists by [5, Proposition 3.2.3]. Then the φ-compatible $\mathbb{F}$-action on M (induced by the scalar multiplication on $\bar{\rho}$) induces a φ-compatible $\mathbb{F}$-action on $\mathfrak{M}_{\mathbb{F}}$, which makes $\mathfrak{M}_{\mathbb{F}}$ a projective $\mathfrak{S}_{\mathbb{F}}$-module. (Note that $\mathfrak{S}_{\mathbb{F}}$ is a product of copies of a discrete valuation ring.) To show $\mathfrak{M}_{\mathbb{F}} \in (\mathrm{ModFI}/\mathfrak{S})_{\mathbb{F}}^{\leqslant h}$ it is left to show that $\mathfrak{M}_{\mathbb{F}}$ is free over $\mathfrak{S}_{\mathbb{F}}$, but this follows because the endomorphism $\sigma : \mathfrak{S}_{\mathbb{F}} \to \mathfrak{S}_{\mathbb{F}}$ transitively permutes the orthogonal idempotents of $\mathfrak{S}_{\mathbb{F}}$. □

Remark 1.6.3. Lemma 1.6.2 does not fully generalize to a $\mathcal{G}_{K_\infty}$-representation ρ_A of height $\leqslant h$ over a finite artinean $\mathbb{Z}_p$-algebra A. Assume that $he \geqslant p$ where e is the absolute ramification index of K, and consider $\mathbb{F}[\epsilon]$ where $\epsilon^2 = 0$. Let M be a rank-1 free $\mathcal{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-module equipped with $\varphi_M(\sigma^*\mathbf{e}) = (\mathcal{P}(u)^h + \frac{1}{u}\epsilon)\mathbf{e}$ for an $\mathcal{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\mathbf{e} \in M$. Let $\mathfrak{M}$ be an $\mathfrak{S}_{\mathbb{F}}$-span of $\{\mathbf{e}, \frac{1}{u}\epsilon\mathbf{e}\}$ in M. Then $\mathfrak{M} \subset M$ is an $\mathfrak{S}$-submodule of height $\leqslant h$ (using that $he \geqslant p$), but one can check that there cannot exist an $\mathfrak{S}$-submodule of height $\leqslant h$ which is rank-1 free over $\mathfrak{S}_{\mathbb{F}[\epsilon]}$.[16] Note also that $\mathfrak{M}$ above is the maximal $\mathfrak{S}$-submodule of height $\leqslant h$ and has a φ-compatible $\mathbb{F}[\epsilon]$-action induced from M, but $\mathfrak{M}$ is not projective over $\mathfrak{S}_{\mathbb{F}[\epsilon]}$. This is where the proof of Lemma 1.6.2 fails.

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[14] For us topological rings are always linearly topologized. Later we need to consider coefficient rings that are not finite $\mathbb{Z}_p$-algebras such as $A = \mathbb{F}[t]$, especially for analyzing the connected components of the generic fiber of a deformation ring.

[15] The relevant freeness follows from length consideration and Nakayama lemma.

[16] One way to see this is by directly computing the "φ-matrix" for any $\mathcal{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\mathbf{e}' \in M$, and show that it cannot divide $\mathcal{P}(u)^h$.

Now we can begin the proof of Proposition 1.6. Since $D_\infty^{\square,\leqslant h}(\mathbb{F}[\epsilon])$ is a torsor of $\widehat{\mathrm{GL}_d}(\mathbb{F}[\epsilon])/(1+\epsilon\,\mathrm{Ad}(\bar\rho_\infty)^{\mathcal{G}_{K_\infty}})$ over $D_\infty^{\leqslant h}(\mathbb{F}[\epsilon])$ (where $\widehat{\mathrm{GL}_d}$ is the formal completion of GL_d at the identity section), it is enough to show that the set $D_\infty^{\leqslant h}(\mathbb{F}[\epsilon])$ is finite.

1.6.4. Setup

Let $\overline{M} := \underline{D}_{\mathcal{E}}(\bar\rho_\infty(-h))$ and consider (M,ι), where $M \in (\mathrm{ModFI}/\mathscr{O}_{\mathcal{E}})^{\text{ét}}_{\mathbb{F}[\epsilon]}$ and $\iota : \overline{M} \cong M \otimes_{\mathbb{F}[\epsilon]} \mathbb{F}$ is a φ-compatible $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-linear isomorphism. Two such lifts (M,ι) and (M',ι') are *equivalent* if there exists an isomorphism $f : M \xrightarrow{\sim} M'$ with $(f \bmod \epsilon)\circ\iota = \iota'$. Fontaine's theory of étale φ-modules [7, Section A1.2] implies that $\underline{T}_{\mathcal{E}}(\cdot)(h)$ and $\underline{D}_{\mathcal{E}}(\cdot(-h))$ induce inverse bijections between $D_\infty(\mathbb{F}[\epsilon])$ and the set of equivalent classes of (M,ι).

Now assume that there is a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M} \subset M$ of height $\leqslant h$. (Note that we do not require $\mathfrak{M}$ to be an $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-submodule.) By Lemma 1.6.2, the set of equivalence classes of (M,ι) admitting such $\mathfrak{M} \subset M$ exactly corresponds to $D_\infty^{\leqslant h}(\mathbb{F}[\epsilon])$ via the bijections in the previous paragraph. So Proposition 1.6 is equivalent to the following claim:

Claim 1.6.5. *If k is finite, then there exist only finitely many equivalence classes of* (M,ι) *where M admits a* φ*-stable* $\mathfrak{S}_{\mathbb{F}}$*-lattice which is of height* $\leqslant h$*.*

1.6.6. Strategy and outline

One possible approach to prove Claim 1.6.5 is to fix an $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-basis for $\overline{M}$ and a lift to an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis for each deformation M once and for all, and identify M with the "φ-matrix" with respect to the fixed basis and interpret the equivalence relations in terms of the "φ-matrix". Then the problem turns into showing the finiteness of equivalence classes of matrices with some constraints – namely, having some "integral structure"; more precisely, having a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice with height $\leqslant h$ (but not necessarily an $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice; *cf.* Lemma 1.6.2 and Remark 1.6.3). So the fixed basis has to "reflect" the integral structure.

This approach faces the following obstacles. Firstly, the deformations M we consider do not necessarily allow any φ-stable $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice with height $\leqslant h$ as we have seen at Remark 1.6.3. In other words, we cannot expect, in general, to find an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\{e_i\}$ for M in such a way that $\{e_i,\epsilon e_i\}$ generates an $\mathfrak{S}_{\mathbb{F}}$-lattice of height $\leqslant h$. In Sections 1.6.7–1.6.9 we show that a weaker statement is true. Roughly speaking, we show that there is an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\{\mathbf{e}_i\}$ for M so that there exists a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice with height $\leqslant h$ with an $\mathfrak{S}_{\mathbb{F}}$-basis only involving "uniformly" u-adically bounded denominators as coefficients relative to the $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-basis $\{\mathbf{e}_i,\epsilon\cdot\mathbf{e}_i\}$ of M.

Secondly, we may have more than one φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice with height $\leqslant h$ for $\overline{M}$ or for M, especially when he is large. In particular, a fixed $\mathfrak{S}_{\mathbb{F}}$-lattice for $\overline{M}$ may not be nicely related to any φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice with height $\leqslant h$ for some lift $M \in (\mathrm{ModFI}/\mathscr{O}_{\mathcal{E}})^{\text{ét}}_{\mathbb{F}[\epsilon]}$. We get around this issue by varying the basis for $\overline{M}$ among finitely many choices. This step is carried out in Section 1.6.11. In fact, we only need finitely many choices of bases because there are only finitely many $\mathfrak{S}_{\mathbb{F}}$-lattices of height $\leqslant h$ for a fixed $\overline{M}$, thanks to [5, Proposition 3.2.3].

Once we get around these technical problems, we show the finiteness by a σ-conjugacy computation of matrices. This is the key technical step and crucially uses the assumption that the $\mathbb{F}[\epsilon]$-deformations we consider (or rather, the corresponding étale φ-module M) admits a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice in M with height $\leqslant h$. See Claim 1.6.12 for more details.

1.6.7. Let M correspond to some $\mathbb{F}[\epsilon]$-deformation of height $\leqslant h$. Even though there may not exist any φ-stable $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice with height $\leqslant h$ for M, we can find a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M}$ with height $\leqslant h$ such that $\mathfrak{M}$ is stable under multiplication by ϵ.[17] In fact, the maximal $\mathfrak{S}$-submodule $\mathfrak{M}^+ \subset M$ among the ones with height $\leqslant h$ does the job. (The existence of $\mathfrak{M}^+$ is by [5, Proposition 3.2.3].)

[17] This means that $\mathfrak{M}$ is a φ-module over $\mathfrak{S}_{\mathbb{F}[\epsilon]}$ and is projective over $\mathfrak{S}_{\mathbb{F}}$, but $\mathfrak{M}$ does not have to be a projective $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-module. Hence, such $\mathfrak{M}$ may not be an object in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\mathbb{F}[\epsilon]}$. This actually occurs: $\mathfrak{M} \cong \mathfrak{S}_{\mathbb{F}}\cdot\mathbf{e} \oplus \mathfrak{S}_{\mathbb{F}}\cdot(\frac{1}{u}\epsilon\mathbf{e})$ discussed in Remark 1.6.3 is such an example.

1.6.8. For an $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M} \subset M$ of height $\leqslant h$ which is stable under the ϵ-multiplication, we can find an $\mathfrak{S}_{\mathbb{F}}$-basis which can be "nicely" written in terms of some $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis of M, as follows. Let $\overline{\mathfrak{M}}$ be the image of $\mathfrak{M} \to \overline{M}$ induced by the natural projection $M \to \overline{M}$, which is a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice in $\overline{M}$ with height $\leqslant h$. Now, consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{M} & \longrightarrow & \overline{\mathfrak{M}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \epsilon \cdot M & \longrightarrow & M & \longrightarrow & M_{\mathbb{F}} & \longrightarrow & 0, \end{array}$$

where $\mathfrak{N} := \ker[\mathfrak{M} \twoheadrightarrow \overline{\mathfrak{M}}]$ is a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice with height $\leqslant h$ in M. We choose an $\mathfrak{S}_{\mathbb{F}}$-basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\overline{\mathfrak{M}}$. Viewing them as an $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-basis of $\overline{M}$, we lift $\{\mathbf{e}_i\}$ to an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis of M (again denoted by $\{\mathbf{e}_i\}$). By the assumption from the previous step, we have $\bigoplus_{i=1}^n \mathfrak{S}_{\mathbb{F}} \cdot (\epsilon \mathbf{e}_i) \subset \mathfrak{N}$, where both are $\mathfrak{S}_{\mathbb{F}}$-lattices of height $\leqslant h$ for $\epsilon \cdot M$. It follows that $(\frac{1}{u^{r_i}}\epsilon)\mathbf{e}_i$ form an $\mathfrak{S}_{\mathbb{F}}$-basis of $\mathfrak{N}$ for some non-negative integers r_i. Therefore, $\{\mathbf{e}_i, (\frac{1}{u^{r_i}}\epsilon)\mathbf{e}_i\}$ is an $\mathfrak{S}_{\mathbb{F}}$-basis of $\mathfrak{M}$.

1.6.9. In this step, we find an upper bound for the non-negative integers r_i only depending on $\overline{\mathfrak{M}}$ and the choice of $\mathfrak{S}_{\mathbb{F}}$-basis of $\overline{\mathfrak{M}}$. Since $\mathfrak{N}$ is a φ-stable submodule, it contains

$$\varphi_M\left(\sigma^*\left(\frac{1}{u^{r_i}}\epsilon \mathbf{e}_i\right)\right) = \left(\frac{1}{u^{pr_i}}\epsilon\right) \cdot \varphi_{\overline{\mathfrak{M}}}\big(\sigma^* \mathbf{e}_i\big) = \frac{1}{u^{pr_i}}\epsilon \cdot \sum_{j=1}^n \alpha_{ij}\mathbf{e}_j, \tag{1.6.10}$$

where $\alpha_{ij} \in \mathfrak{S}_{\mathbb{F}}$ satisfy $\varphi_{\overline{\mathfrak{M}}}(\sigma^*\mathbf{e}_i) = \sum_{j=1}^n \alpha_{ij}\mathbf{e}_j$. Note that we obtain the first identity because $\varphi_M(\sigma^*\mathbf{e}_i)$ lifts $\varphi_{\overline{\mathfrak{M}}}(\sigma^*\mathbf{e}_i)$ and the ϵ-multiple ambiguity in the lift disappears when we multiply against ϵ. Since any element of $\mathfrak{N}$ is an $\mathfrak{S}_{\mathbb{F}}$-linear combination of $(\frac{1}{u^{r_i}}\epsilon)\mathbf{e}_i$, we obtain inequalities $\mathrm{ord}_u(\alpha_{ij}) - pr_i \geqslant -r_j$ for all i, j from the above equation (1.6.10). Let $r := \max_j\{r_j\}$ and we obtain $pr_i \leqslant r + \min_j\{\mathrm{ord}_u(\alpha_{ij})\}$ for all i. (Note that the right side of the inequality is always finite.) Now, by taking the maximum among all i, we obtain

$$r \leqslant \frac{1}{p-1} \max_i \Big\{ \min_j \big\{\mathrm{ord}_u(\alpha_{ij})\big\}\Big\} < \infty.$$

This shows that the non-negative integers r_i have an upper bound which only depends on the matrices entries for $\varphi_{\overline{\mathfrak{M}}}$ with respect to the $\mathfrak{S}_{\mathbb{F}}$-basis of $\overline{\mathfrak{M}}$.

1.6.11. Recapitulation

Let $\{\overline{\mathfrak{M}}^{(a)}\}$ denote the set of all the $\mathfrak{S}_{\mathbb{F}}$-lattices of height $\leqslant h$ in $\overline{M}$. This is a finite set by [5, Proposition 3.2.3]. For each $\overline{\mathfrak{M}}^{(a)}$, we fix an $\mathfrak{S}_{\mathbb{F}}$-basis $\{\mathbf{e}_i^{(a)}\}$ and let $\alpha^{(a)} = (\alpha_{ij}^{(a)}) \in \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ be the "φ-matrix" with respect to $\{\mathbf{e}_i^{(a)}\}$; i.e., $\varphi_{\overline{\mathfrak{M}}^{(a)}}(\sigma^*\mathbf{e}_i^{(a)}) = \sum_{i=1}^n \alpha_{ij}^{(a)}\mathbf{e}_j^{(a)}$. We also view $\{\mathbf{e}_i^{(a)}\}$ as an $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-basis for $\overline{M}$ and $(\alpha_{ij}^{(a)})$ is the matrix for $\varphi_{\overline{M}}$ with respect to $\{\mathbf{e}_i^{(a)}\}$. Note that $(\alpha_{ij}^{(a)})$ is invertible over $\mathscr{O}_{\mathcal{E},\mathbb{F}}$ since $\overline{M} = \overline{\mathfrak{M}}^{(a)}[\frac{1}{u}]$ is an étale φ-module. We pick an integer $r^{(a)} \geqslant \frac{1}{p-1}\max_i\{\min_j\{\mathrm{ord}_u(\alpha_{ij})\}\}$, for each index a.

For any M which corresponds to a deformation of height $\leqslant h$, we may find an $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M} \subset M$ of height $\leqslant h$ which is stable under ϵ-multiplication. (See Section 1.6.7.) Then the image of $\mathfrak{M}$ inside $\overline{M}$ is equal to one of $\overline{\mathfrak{M}}^{(a)}$. Pick such $\overline{\mathfrak{M}}^{(a)}$, and lift the chosen basis $\{\mathbf{e}_i^{(a)}\}$ to an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis for M. Then $\mathfrak{M}$ admits an $\mathfrak{S}_{\mathbb{F}}$-basis of form $\{\mathbf{e}_i^{(a)}, (\frac{1}{u^{r_i}}\epsilon)\mathbf{e}_i^{(a)}\}$ for some integers $r_i \leqslant r^{(a)}$ (Sections 1.6.8–1.6.9).

Let us consider the matrix representation of φ_M with respect to the basis $\{\mathbf{e}_i^{(a)}\}$. We have $\varphi_M(\mathbf{e}_i^{(a)}) = \sum_i (\alpha_{ij}^{(a)} + \epsilon \beta_{ij}^{(a)})\mathbf{e}_j^{(a)}$ for some $\beta^{(a)} = (\beta_{ij}^{(a)}) \in \mathrm{Mat}_n(\mathscr{O}_{\mathcal{E},\mathbb{F}})$ because φ_M lifts $\varphi_{\overline{M}}$. Furthermore we have that $\beta \in \frac{1}{u^{r(a)}} \cdot \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ since $\mathfrak{M} \subset M$ is φ-stable. We say two such matrices β and β' are *equivalent* if there exists a matrix $X \in \mathrm{Mat}_n(\mathscr{O}_{\mathcal{E},\mathbb{F}})$ such that $\beta' = \beta + (\alpha^{(a)} \cdot \sigma(X) - X \cdot \alpha^{(a)})$. This equation is obtained from the following:

$$\left(\alpha^{(a)} + \epsilon\beta'\right) = (\mathrm{Id}_n + \epsilon X)^{-1} \cdot \left(\alpha^{(a)} + \epsilon\beta\right) \cdot \sigma(\mathrm{Id}_n + \epsilon X),$$

which defines the equivalence of two étale φ-modules whose φ-structures are given by $(\alpha^{(a)} + \epsilon\beta)$ and $(\alpha^{(a)} + \epsilon\beta')$, respectively.

Now, the theorem is reduced to the verification of the following claim: *for each a, there exist only finitely many equivalence classes of matrices* $\beta \in \frac{1}{u^{r(a)}} \cdot \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$. Indeed, by varying both a and the equivalence classes of β, we cover all the possible lifts M of "height $\leqslant h$" up to equivalence, hence the theorem is proved.

From now on, we fix a and suppress the superscript $(\cdot)^{(a)}$ everywhere. For example, $\overline{\mathfrak{M}} := \overline{\mathfrak{M}}^{(a)}$, $r := r^{(a)}$, and $\alpha := \alpha^{(a)}$. Proving the following claim is the last step of the proof.

Claim 1.6.12. *For any $X \in u^c\,\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ with $c > 2he$, the matrices β and $\beta + X$ are equivalent.*[18]

This claim provides a surjective map from $(\frac{1}{u^r} \cdot \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}}))/(u^c \cdot \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}}))$ onto the set of equivalence classes of β's, and the former is a finite set[19] provided that k is finite, thus we conclude the proof of Proposition 1.6.

We prove the claim by "successive approximation". Let $\gamma = u^{he} \cdot \alpha^{-1}$. Note that $\gamma \in \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ since $\overline{\mathfrak{M}}$ is of height $\leqslant h$ and $\mathcal{P}(u)$ has image in $\mathfrak{S}_{\mathbb{F}} \cong (k \otimes_{\mathbb{F}_p} \mathbb{F})[[u]]$ with u-order e. We set $Y^{(1)} := \frac{1}{u^{he}} \cdot (X\gamma)$, which is in $u^{c-he}\,\mathrm{Mat}_n(\mathfrak{S}_F)$ by the assumption on X. Then $\beta + X$ is equivalent to

$$(\beta + X) + \left(\alpha \cdot \sigma\left(Y^{(1)}\right) - Y^{(1)}\alpha\right) = \beta + \alpha \cdot \sigma\left(Y^{(1)}\right) =: \beta + X^{(1)}$$

with $X^{(1)} \in u^{c^{(1)}} \cdot \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$, where $c^{(1)} := p(c - he) > c$. Now for any positive integer i, we recursively define the following

$$Y^{(i)} := \frac{1}{u^{he}} \cdot \left(X^{(i-1)}\gamma\right), \qquad X^{(i)} := \alpha \cdot \sigma\left(Y^{(i)}\right), \qquad c^{(i)} := p\left(c^{(i-1)} - he\right).$$

One can check that $c^{(i)} > c(i-1)$ $(> 2he)$, $X^{(i)} \in u^{c^{(i)}} \cdot \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$, and $Y^{(i)} \in u^{c^{(i-1)}-he}\,\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$. Since $c^{(i)} \to \infty$ as $i \to \infty$, it follows that the infinite sum $Y := \sum_{i=1}^{\infty} Y^{(i)}$ converges and $X^{(i)} \to 0$ as $i \to \infty$. Therefore we see that $\beta + X$ is equivalent to

$$\begin{aligned}(\beta + X) + \left(\alpha \cdot \sigma(Y) - Y \cdot \alpha\right) &= (\beta + X) + \left(\alpha \cdot \sigma\left(\sum_{i=1}^{\infty} Y^{(i)}\right) - \left(\sum_{i=1}^{\infty} Y^{(i)}\right) \cdot \alpha\right)\\ &= \lim_{i\to\infty}\left(\beta + X^{(i)}\right) = \beta,\end{aligned}$$

so we are done. □

[18] The inequality $c > 2he$ is used to ensure $p(c - he) > c$. So $c = 2he$ also works unless $p = 2$.

[19] We crucially used the fact that we can bound the denominator.

2. Comparison to "Barsotti–Tate deformation rings"

In this section, we relate $\mathcal{G}_{K_\infty}$-deformation rings and crystalline deformation rings. We also show that a "Barsotti–Tate deformation ring" is isomorphic to a suitable $\mathcal{G}_{K_\infty}$-deformation ring of height $\leqslant 1$ via the map defined by "viewing $\mathcal{G}_K$-deformations as $\mathcal{G}_{K_\infty}$-deformations" (Corollary 2.2.1). Throughout the section, we continue to assume that k is finite unless stated otherwise.

2.1. Crystalline and semi-stable deformation rings

Let $\mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Q}_p}(\mathcal{G}_K)$ (respectively, $\mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Q}_p}(\mathcal{G}_K)$) denote the category of p-adic crystalline (respectively, semi-stable) $\mathcal{G}_K$-representations V such that $\mathrm{gr}^w\,\underline{D}^*_{\mathrm{dR}}(V)=0$ for $w\notin[0,h]$. Let $\mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Z}_p}(\mathcal{G}_K)$ (respectively, $\mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Z}_p}(\mathcal{G}_K)$) denote the category of $\mathbb{Z}_p$-lattice $\mathcal{G}_K$-representations T such that $T[\frac{1}{p}]\in\mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Q}_p}(\mathcal{G}_K)$ (respectively, $T[\frac{1}{p}]\in\mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Q}_p}(\mathcal{G}_K)$).

Let $\mathrm{Rep}^{[0,h]}_{\mathrm{cris,tor}}(\mathcal{G}_K)$ (respectively, $\mathrm{Rep}^{[0,h]}_{\mathrm{st,tor}}(\mathcal{G}_K)$) denote the category of finite p^∞-torsion $\mathcal{G}_K$-modules T which admit a $\mathcal{G}_K$-equivariant surjection $\widetilde{T}\twoheadrightarrow T$ where $\widetilde{T}\in\mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Z}_p}(\mathcal{G}_K)$ (respectively, $\widetilde{T}\in\mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Z}_p}(\mathcal{G}_K)$). Note that the subcategories $\mathrm{Rep}^{[0,h]}_{\mathrm{cris,tor}}(\mathcal{G}_K)$ and $\mathrm{Rep}^{[0,h]}_{\mathrm{st,tor}}(\mathcal{G}_K)$ are obviously closed under subquotients and direct sums inside the category of all finite p^∞-torsion $\mathcal{G}_K$-modules.

Let $\bar\rho:\mathcal{G}_K\to\mathrm{GL}_d(\mathbb{F})$ be a representation, and define a subfunctor $D^{\square,[0,h]}_{\mathrm{cris}}$ of the framed deformation functor $D^\square$ of $\bar\rho$ by requiring that a framed deformation ρ_A over $A\in\mathfrak{AR}_{\mathscr{O}}$ is in $D^{\square,[0,h]}_{\mathrm{cris}}(A)$ if and only if $\rho_A\in\mathrm{Rep}^{[0,h]}_{\mathrm{cris,tor}}(\mathcal{G}_K)$ as a torsion $\mathbb{Z}_p[\mathcal{G}_K]$-module (i.e., ignoring the A-action). Applying the discussion in Section 1.4, we obtain the universal quotient $R^{\square,[0,h]}_{\mathrm{cris}}$ of the universal framed deformation ring $R^\square$ which represents $D^{\square,[0,h]}_{\mathrm{cris}}$. We can similarly define a subfunctor $D^{\square,[0,h]}_{\mathrm{st}}\subset D^\square$ using $\mathrm{Rep}^{[0,h]}_{\mathrm{st,tor}}(\mathcal{G}_K)$ and obtain the universal quotient $R^{\square,[0,h]}_{\mathrm{st}}$ or $R^\square$. One can also define subfunctors $D^{[0,h]}_{\mathrm{cris}}$, $D^{[0,h]}_{\mathrm{st}}\subset D$, which turn out to be relatively representable.

The following is a difficult theorem of Tong Liu [24]:

Theorem 2.1.1 *(Tong Liu). Let $\xi:R^\square\to A$ be an $\mathscr{O}$-algebra map where A is a finite algebra over* Frac $\mathscr{O}$*, and let $\rho_\xi:=\rho^\square\otimes_{R^\square,\xi}A$ where $\rho^\square$ is the universal framed deformation. Then ξ factors through the quotient $R^{\square,[0,h]}_{\mathrm{cris}}$ (respectively, $R^{\square,[0,h]}_{\mathrm{st}}$) if and only if $\rho_\xi\in\mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Q}_p}(\mathcal{G}_K)$, where ρ_ξ is viewed as a p-adic $\mathcal{G}_K$-representation by forgetting the A-action.*

Remark 2.1.2. If $\rho\in\mathrm{Rep}^{[0,1]}_{\mathrm{cris},\mathbb{Q}_p}(\mathcal{G}_K)$, then there exists a p-divisible group G over $\mathscr{O}_K$ such that $V_p(G)\cong\rho$. (This is proved by Breuil [3, Théorème 1.4] when $p>2$ and by Kisin [17, Corollary 2.2.6] when $p=2$.) Combining T. Liu's theorem (Theorem 2.1.1), we see that $D^{\square,[0,1]}_{\mathrm{cris}}(A)$ is the same as the flat (framed) deformation functor.

Set $\bar\rho_\infty:=\bar\rho|_{\mathcal{G}_{K_\infty}}$. Let $R^{\square,\leqslant h}_\infty$ be the universal framed deformation ring of $\bar\rho_\infty$ with height $\leqslant h$.

Lemma 2.1.3. *Restricting to $\mathcal{G}_{K_\infty}$ induces the following natural morphisms:*

$$\mathrm{res}^{\leqslant h}_{\mathrm{cris}}:R^{\square,\leqslant h}_\infty\to R^{\square,[0,h]}_{\mathrm{cris}},\quad\textit{and}\quad\mathrm{res}^{\leqslant h}_{\mathrm{st}}:R^{\square,\leqslant h}_\infty\to R^{\square,[0,h]}_{\mathrm{st}}.$$

Furthermore, $\mathrm{res}^{\leqslant h}_{\mathrm{cris}}\otimes\mathbb{Q}_p$ induces surjections on the completions at each maximal ideal of $R^{\square,\leqslant h}_\infty\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. The same holds for deformation rings (without framing) if $\mathrm{End}_{\mathcal{G}_{K_\infty}}(\bar\rho_\infty)\cong\mathbb{F}$.

Proof. For the first assertion it is enough to show that the $\mathcal{G}_{K_\infty}$-restriction of any $T\in\mathrm{Rep}^{[0,h]}_{\mathrm{st,tor}}(\mathcal{G}_K)$ is a torsion $\mathcal{G}_{K_\infty}$-representation of height $\leqslant h$. Choose a presentation $T\cong\widetilde{T}/\widetilde{T}'$ such that $\widetilde{T},\widetilde{T}'\in$

$\mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Z}_p}(\mathcal{G}_K)$. Kisin [17, Proposition 2.1.5, Lemma 2.1.15] showed that $\widetilde{T}$ and $\widetilde{T}'$ are of height $\leqslant h$, so it follows from Lemma 1.1.4 that $T|_{\mathcal{G}_{K_\infty}}$ is a torsion representation of height $\leqslant h$. Finally, the assertion on $\mathrm{res}^{\leqslant h}_{\mathrm{cris}} \otimes \mathbb{Q}_p$ directly follows from the full faithfulness of the $\mathcal{G}_{K_\infty}$-restriction on the category of p-adic crystalline $\mathcal{G}_K$-representations [17, Corollary 2.1.14]. □

Let $\mathrm{Rep}^{\leqslant 1}_{\mathrm{tor}}(\mathcal{G}_{K_\infty})$ denote the category of torsion $\mathcal{G}_{K_\infty}$-representations of height $\leqslant 1$ (Definition 1.1.3). For the following theorem alone, we allow k to be any perfect field of characteristic $p > 0$ (not just a finite field).

Proposition 2.2. *Let k be a perfect field of characteristic $p > 0$. Restricting the $\mathcal{G}_K$-action to $\mathcal{G}_{K_\infty}$ induces an equivalence of categories* $\mathrm{Rep}^{[0,1]}_{\mathrm{cris,tor}}(\mathcal{G}_K) \to \mathrm{Rep}^{\leqslant 1}_{\mathrm{tor}}(\mathcal{G}_{K_\infty})$.

Corollary 2.2.1. *Let k be a finite field of characteristic $p > 0$. Then the natural map* $\underline{\mathrm{res}}^{\leqslant 1}_{\mathrm{cris}} : \mathrm{Spec}\, R^{\square,[0,1]}_{\mathrm{cris}} \to \mathrm{Spec}\, R^{\square,\leqslant 1}_{\infty}$ *(defined in Lemma* 2.1.3*) is an isomorphism. The same holds for unframed deformation rings if they exist.*

Proof of Proposition 2.2. This is proved by Breuil [4, Theorem 3.4.3] when $p > 2$ using the Breuil–Kisin classification of finite flat group schemes. The restriction $p > 2$ is now removed since the classification is extended to the case when $p = 2$ independently by the author [16] and Eike Lau [23, 22]. Alternatively, one can deduce Proposition 2.2 from [16, Proposition 3.6], which asserts that any $\mathcal{G}_{K_\infty}$-stable $\mathbb{Z}_p$-lattice in $V \in \mathrm{Rep}^{[0,1]}_{\mathrm{cris},\mathbb{Q}_p}(\mathcal{G}_K)$ is $\mathcal{G}_K$-stable. □

Remark 2.2.2. Note that $\mathrm{res}^{\leqslant h}_{\mathrm{cris}} : \mathrm{Spec}\, R^{\square,[0,h]}_{\mathrm{cris}} \to \mathrm{Spec}\, R^{\square,\leqslant h}_{\infty}$ is *not* in general an isomorphism (even after inverting p); the source and the target have different dimensions at a maximal ideal of $R^{\square,[0,h]}_{\mathrm{cris}}[\frac{1}{p}]$ which corresponds to a lift which has two Hodge–Tate weights that differ at least by 2. See [19, Theorem 3.3.8] and [15, Corollary 11.3.11] for the dimension formulas.

A natural next question (which seems very hard) is whether one can give a meaningful and workable description of the rigid analytic subspace of crystalline lifts inside a $\mathcal{G}_{K_\infty}$-deformation space of $\bar{\rho}_\infty$ with height $\leqslant h$.

3. Moduli of $\mathfrak{S}$-modules

In this section we give a different description of the “moduli of finite flat group schemes” as a “resolution” of a $\mathcal{G}_{K_\infty}$-deformation ring. One can show that this “resolved $\mathcal{G}_{K_\infty}$-deformation ring” is naturally isomorphic to the moduli of finite flat group schemes by the Breuil–Kisin classification of finite flat group schemes.

3.1. *“Hodge-type”* (0, 1)

Let $\bar{\rho} : \mathcal{G}_K \to \mathrm{GL}_2(\mathbb{F})$ be a 2-dimensional representation. We define a subfunctor $D^{\mathbf{v}}$ of the crystalline $\mathcal{G}_K$-deformation functor $D^{[0,1]}_{\mathrm{cris}}$ of $\bar{\rho}$ so that a deformation $\rho_A \in D^{[0,1]}_{\mathrm{cris}}(A)$ (where $A \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$) is in $D^{\mathbf{v}}(A)$ if and only if ρ_A satisfies the following additional condition:

$$\det \rho_A|_{I_K} \sim \chi_{\mathrm{cyc}}|_{I_K}. \tag{3.1.1}$$

We also define a subfunctor $D^{\mathbf{v}}_\infty$ of the $\mathcal{G}_{K_\infty}$-deformation functor $D^{\leqslant 1}_\infty$ with height $\leqslant 1$ by requiring (3.1.1) with I_K replaced by I_{K_∞}.

Let $\rho_R \in D^{\mathbf{v}}_\infty(R)$ for some $R \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$. Put $M_R := \varprojlim M_n$ where $M_n \in (\mathrm{ModFI}/\mathscr{O}_{\mathcal{E}})^{\text{ét}}_{R/\mathfrak{m}_R^n}$ is such that $\underline{T}_{\mathcal{E}}(M_n)(1) \cong \rho_R \otimes_R R/\mathfrak{m}_R^n$ for each n. The proof of the following proposition is identical to the proof of [21, Propositions 2.1.10, 2.4.8] using [20, Lemma 2.3.4].

Proposition 3.2. *Under the notation as above, there exists a projective R-scheme $\mathscr{GR}^{\mathbf{v}}_{\rho_R}$ such that for any R-algebra A with $\mathfrak{m}_R^N \cdot A = 0$ for some N, the set $\mathrm{Hom}_R(\mathrm{Spec}\, A, \mathscr{GR}^{\mathbf{v}}_{\rho_R})$ is in natural bijection with the set of φ-stable $\mathfrak{S}_A$-lattices $\mathfrak{M}_A$ in $M_R \otimes_R A$ such that $\mathrm{im}(\varphi_{\mathfrak{M}_A})/\mathcal{P}(u)\mathfrak{M}_A \subset \mathfrak{M}_A/\mathcal{P}(u)\mathfrak{M}_A$ is a Lagrangian (for the natural symplectic pairing well defined up to unit multiple). Here, $\mathfrak{S}_A := \mathfrak{S} \otimes_{\mathbb{Z}_p} A$ and we view $M_R \otimes_R A$ as a φ-module by A-linearly extending φ_{M_R}.*

Moreover the structure morphism $\mathscr{GR}^{\mathbf{v}}_{\rho_R} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Spec}\, R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism.

Remark 3.2.1. With the notation of [21, (2.4.2)] the "Lagrangian condition" corresponds to the choice $v_\psi = 1$ for all $\psi : K \hookrightarrow \overline{K}$. In fact, we obtain Proposition 3.2 for different choice of $\{v_\psi\}$ (and when $\bar\rho$ is of arbitrary dimension).

Similarly, for any $\mathcal{G}_{K_\infty}$-deformation ρ_R one obtains a projective R-scheme $\mathscr{GR}^{\leqslant 1}_{\rho_R}$ classifying $\mathfrak{S}$-modules as in Proposition 3.2 but without the "Lagrangian condition". This construction can be generalized to the case with "height $\leqslant h$" for $h > 1$, but this cannot be immediately related to crystalline deformation rings as observed in Remark 2.2.2.

Remark 3.2.2. Let $R := R^{\square,\mathbf{v}}_\infty$ and ρ_R be the universal $\mathcal{G}_{K_\infty}$-deformation. By Corollary 2.2.1 and [17, Corollary 2.2.6], the $\mathcal{G}_{K_\infty}$-restriction induces a natural isomorphism $R^{\square,\mathbf{v}}_\infty \xrightarrow{\sim} R^{\square,\mathbf{v}}_{\mathrm{fl}}$ where $R^{\square,\mathbf{v}}_{\mathrm{fl}}$ is the universal quotient of the flat deformation ring of $\bar\rho$ with the condition (3.1.1). By this isomorphism, one can view $\mathscr{GR}^{\mathbf{v}}_{\rho_R}$ as a projective scheme over $R^{\square,\mathbf{v}}_{\mathrm{fl}}$. As a direct consequence of the Breuil–Kisin classification,[20] this scheme $\mathscr{GR}^{\mathbf{v}}_{\rho_R}$ is exactly the moduli of finite flat group schemes $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},\xi}$ defined in [21, (2.4.2)] with ξ the universal framed flat deformation over $R^{\square,\mathbf{v}}_{\mathrm{fl}}$. This statement also holds when $\bar\rho$ is of arbitrary finite dimension and without the "Lagrangian condition" (or making a different choice of $\{v_\psi\}$ in [21, (2.4.2)]).

From Remark 3.2.2, it is no surprise that the following proposition can be proved by the *same* linear algebraic argument[21] as in the proof of [21, Corollary 2.5.16(2)] (with improvements in [8,14]) applied to $\mathscr{GR}^{\mathbf{v}}_{\rho_R}$ defined in Proposition 3.2.

Proposition 3.3. *Assume that the morphism $\mathrm{Spf}\, R \to D^{\mathbf{v}}_\infty$ induced by ρ_R is formally smooth. For finite local $\mathbb{Q}_p$-algebras A and A', consider maps $\xi : R \to A$ and $\xi' : R \to A'$. Let ρ_ξ and $\rho_{\xi'}$ denote the lifts of $\bar\rho$ corresponding to ξ and ξ', respectively. Then ξ and ξ' are supported on the same irreducible component of $\mathrm{Spec}\, R$ if and only if either both ρ_ξ and $\rho_{\xi'}$ do not admit a non-zero unramified quotient (i.e. non-ordinary) or both ρ_ξ and $\rho_{\xi'}$ admit a rank-1 unramified quotient which lift the same (mod p) character.*

Remark 3.4. One may wonder whether Proposition 3.3 can be generalized to a suitable quotient of a $\mathcal{G}_{K_\infty}$-deformation ring with height $\leqslant h$ with $h > 1$ (and still $\bar\rho_\infty$ is 2-dimensional). The first difficulty is that the obvious generalization of the "Lagrangian condition" does not seem to be a correct definition when $h > 1$ and $e > 1$, and therefore we do not know the right generalization of $\mathscr{GR}^{\mathbf{v}}_{\rho_R}$.

4. Positive characteristic analogue of crystalline deformation rings

In this section, we introduce a class of $\mathrm{Gal}(k((u))^{\mathrm{sep}}/k((u)))$-representation with coefficients in some equi-characteristic local field which could be thought of as an analogue of crystalline representations, and develop a deformation theory for them. Such representations are introduced by Genestier and Lafforgue [10], and its torsion version also appeared in Abrashkin [1]. A useful observation is that the linear algebra objects that give rise to such Galois representations have very similar structure to

[20] The classification is proved when $p = 2$ in [16] and independently in [23,22].

[21] Note that the proof of [21, Corollary 2.5.16(2)] and its improvements are some (very elaborate) linear algebra with $\mathfrak{S}$-modules, and the classification of finite flat group schemes is only used to define the moduli of finite flat group schemes. In the proof of [21, Proposition 2.5.15] Kisin used the boundedness of prolongations of finite flat group schemes, but this has a purely linear-algebraic analogue [5, Proposition 3.2.3].

various $(\varphi, \mathfrak{S})$-modules that we saw in Kisin theory. Considering the norm field isomorphism $\mathcal{G}_{K_\infty} \cong \mathrm{Gal}(k((u))^{\mathrm{sep}}/k((u)))$ [29], it is not too surprising that the $\mathcal{G}_{K_\infty}$-deformation theory has an analogue in positive characteristic.

4.1. Notations/definitions

Let $\mathscr{O}_0 := \mathbb{F}_q[[\pi_0]]$ be a complete discrete valuation ring of characteristic p. For this section, let $K := k((u))$ and $\mathscr{O}_K := k[[u]]$ where k is a finite extension of $\mathbb{F}_q$. (So K is just a finite extension of $\mathbb{Q}_p$.) We fix a finite map $\iota : \mathscr{O}_0 \to \mathscr{O}_K$ over $\mathbb{F}_q$. Roughly speaking, $\mathscr{O}_0$ will play the role of $\mathbb{Z}_p$, and $\pi_0 \in \mathscr{O}_0$ will play the role of p.

Put $\mathcal{G}_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. We will study a certain class of $\mathcal{G}_K$-representations over $\mathscr{O}_0$, $\mathrm{Frac}(\mathscr{O}_0)$, or finite algebras thereof. It is defined in terms of linear-algebraic objects called (*effective*) *local shtukas* over $\mathscr{O}_K$, which we introduce below. Local shtukas have many analogous features to $(\varphi, \mathfrak{S})$-modules of finite height in Kisin theory, so we use similar notations to Kisin theory to emphasize the analogy.

Let $\mathfrak{S} := \mathscr{O}_K[[\pi_0]]$ and $\mathscr{O}_{\mathcal{E}} := K[[\pi_0]]$. We define a partial q-Frobenius endomorphism σ for each of these rings so that it acts as the qth power map on K and $\sigma(\pi_0) = \pi_0$. This σ lifts the qth power map modulo π_0, and fixes $\mathscr{O}_0$. We also set $\mathcal{E} := K((\pi_0))$ and extend σ on $\mathcal{E}$. Then σ fixes $\mathrm{Frac}(\mathscr{O}_0)$.

Let $u_0 := \iota(\pi_0) \neq 0$ where $\iota : \mathscr{O}_0 \to \mathscr{O}_K$ is the map we fixed earlier. Put $\mathcal{P}(u) := \pi_0 - u_0 \in \mathfrak{S}$ and let $e := \mathrm{ord}_u(u_0)$. Clearly we have $\mathfrak{S}/(\mathcal{P}(u)) \cong \mathscr{O}_K$, which is a totally ramified ring extension of $k[[\pi_0]]$. This shows that $\mathcal{P}(u)$ is an $\mathfrak{S}^\times$-multiple of some Eisenstein polynomial in $k[[\pi_0]][u]$ with degree e.

4.1.1. An *étale φ-module* is a $(\varphi, \mathscr{O}_{\mathcal{E}})$-module[22] $(\mathcal{M}, \varphi_M)$ such that φ_M is an isomorphism. Similarly to the p-adic case, we have an equivalence of categories between the category of étale φ-modules and the category of $\mathcal{G}_K$-representations, as follows. Let $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}} := K^{\mathrm{sep}}[[\pi_0]]$, and we let $\mathcal{G}_K$ act on it through the coefficients, and define the partial q-Frobenius endomorphism σ so that it acts as the qth power map on K^{sep} and $\sigma(\pi_0) = \pi_0$. For an étale φ-module M we define

$$\underline{T}_{\mathcal{E}}(M) := (M \otimes_{\mathscr{O}_{\mathcal{E}}} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{\varphi=1}. \tag{4.1.2}$$

This induces an exact equivalence of categories between the category of étale φ-modules and the category of finitely generated $\mathscr{O}_0$-module with continuous $\mathcal{G}_K$-action. One can define the quasi-inverse $\underline{D}_{\mathcal{E}}$ in a similar fashion to (1.1.2a). Furthermore, they respect all the natural operations, and they preserve rank and length whenever applicable. The proof is identical to the proof of the p-adic case [7, Section A1.2].[23]

Definition 4.1.3. Consider the following étale φ-module $M_{\mathcal{LT}} := \mathscr{O}_{\mathcal{E}} \cdot \mathbf{e}$ equipped with $\varphi_{M_{\mathcal{LT}}}(\sigma^*\mathbf{e}) = \mathcal{P}(u)^{-1}\mathbf{e}$. Let $\chi_{\mathcal{LT}} : \mathcal{G}_K \to \mathscr{O}_0^\times$ denote the character that defines the $\mathcal{G}_K$-action on $\underline{T}_{\mathcal{E}}(M_{\mathcal{LT}})$. For any $\mathscr{O}_0[\mathcal{G}_K]$-module V, we let $V(n)$ be the $\mathscr{O}_0[\mathcal{G}_K]$-module whose $\mathcal{G}_K$-action is twisted by $\chi_{\mathcal{LT}}^n$.

This character $\chi_{\mathcal{LT}}$ is equivalent to the character obtained from the π_0-adic Tate module of the Lubin–Tate formal $\mathscr{O}_0$-module over $\mathscr{O}_K$. See [2] for the proof. Note that when K is a finite extension of $\mathbb{Q}_p$, we can obtain $\chi_{\mathrm{cyc}}|_{\mathcal{G}_{K_\infty}}$ from the étale φ-module defined analogously as above [20, Lemma 2.3.4].

4.1.4. For a non-negative integer h, an *effective local shtuka* (*over* $\mathscr{O}_K$) *of height* $\leqslant h$ is a finite free $\mathfrak{S}$-module $\mathfrak{M}$ equipped with an $\mathfrak{S}$-linear morphism $\varphi_{\mathfrak{M}} : \sigma^*\mathfrak{M} \to \mathfrak{M}$ such that $\mathrm{coker}(\varphi_{\mathfrak{M}})$ is killed by $\mathcal{P}(u)^h$. The original definition of effective local shtuka (over $\mathscr{O}_K$) requires $\mathrm{coker}(\varphi_{\mathfrak{M}})$ to be flat

[22] The notion of φ-module is defined in Section 1.1. Note that we use $\mathscr{O}_{\mathcal{E}}$ defined in Section 4.1, not the one in Section 1.1.

[23] See [15, Section 5.1] for the full proof, but the positive characteristic version of the theory of étale φ-modules must have been known for a while.

over $\mathscr{O}_K$, but this is automatic because it is a $\mathcal{P}(u)$-power torsion $\mathfrak{S}$-module of projective dimension $\leqslant 1$. Note that effective local shtukas can be defined over any $\mathscr{O}_0$-scheme (not just over $\operatorname{Spf}\mathscr{O}_K$), and there are more general objects called *local shtukas* which are defined by allowing $\varphi_{\mathfrak{M}}$ to have a pole at $\mathcal{P}(u)$. See [10, Definition 0.1] or [13, Definition 2.1.1] for more general definition.

4.1.5. For a non-negative integer h, a *torsion shtuka of height* $\leqslant h$ is a finitely generated π_0^∞-torsion u-torsion-free $\mathfrak{S}$-module $\mathfrak{M}$ equipped with an $\mathfrak{S}$-linear morphism $\varphi_{\mathfrak{M}} : \sigma^*\mathfrak{M} \to \mathfrak{M}$ such that $\operatorname{coker}(\varphi_{\mathfrak{M}})$ is killed by $\mathcal{P}(u)^h$. We let $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ denote the category of torsion shtukas of height $\leqslant h$ with the obvious notion of morphisms. There is a natural analogue of Cartier duality. (See [15, Section 8.3] for more details.)

4.1.6. Let $\mathfrak{M}$ be either effective local shtuka or torsion shtuka of height $\leqslant h$. Since $\mathcal{P}(u)$ is a unit in $\mathscr{O}_{\mathcal{E}}$, the scalar extension $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}}$ is naturally an étale φ-module. So we can associate a $\mathcal{G}_K$-representation to such $\mathfrak{M}$ as follows:

$$\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}) := \underline{T}_{\mathcal{E}}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}})(h). \tag{4.1.7}$$

We state the following fundamental and non-trivial result on this functor $\underline{T}_{\mathfrak{S}}^{\leqslant h}$. Compare with [17, Proposition 2.1.12, Lemma 2.1.15].

Proposition 4.1.8.

(1) *The functor $\underline{T}_{\mathfrak{S}}^{\leqslant h}$ from the category of effective local shtukas of height $\leqslant h$ to the category of $\mathscr{O}_0$-representations of $\mathcal{G}_K$ is fully faithful.*
(2) *Let $V := \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M})[\frac{1}{\pi_0}]$, then for any $\mathcal{G}_K$-stable $\mathscr{O}_0$-lattice $T' \subset V$ there exists an effective local shtuka $\mathfrak{M}'$ of height $\leqslant h$ such that $T' \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}')$.*

Proof. The proof of (1) is very similar to the proof of its p-adic analogue [17, Proposition 2.1.12], except that one needs to work with "weakly admissible isocrystals with Hodge–Pink structure" instead of filtered φ-modules, and apply [10, Théorème 7.3] instead of [17, Lemma 1.3.13]. The detail is worked out in [15, Theorem 5.2.3].

The claim (2) easily follows from [10, Lemme 2.3] by the same way as its p-adic analogue [17, Lemma 2.1.15] is proved. □

A finite free $\mathscr{O}_0$-module equipped with continuous $\mathcal{G}_K$-action is called *$\mathscr{O}_0$-lattice $\mathcal{G}_K$-representation*. A finitely generated π_0^∞-torsion $\mathscr{O}_0$-module equipped with continuous $\mathcal{G}_K$-action is called *π_0^∞-torsion $\mathcal{G}_K$-representation*.

Definition 4.1.9. Let h be a non-negative integer. An $\mathscr{O}_0$-lattice $\mathcal{G}_K$-action T is called *of height* $\leqslant h$ if there exists an effective local shtuka $\mathfrak{M}$ of height $\leqslant h$ such that $T \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M})$.

A continuous $\mathcal{G}_K$-representation V over $\operatorname{Frac}(\mathscr{O}_0)$ is called *of height* $\leqslant h$ if it admits a $\mathcal{G}_K$-stable $\mathscr{O}_0$-lattice $T \subset V$ which is of height $\leqslant h$; or equivalently by Proposition 4.1.8(2), any $\mathcal{G}_K$-stable $\mathscr{O}_0$-lattice $T \subset V$ is of height $\leqslant h$.

A π_0^∞-torsion $\mathcal{G}_K$-representation T is called *of height* $\leqslant h$ if there exists a torsion shtuka $\mathfrak{M}$ with height $\leqslant h$ such that $T \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M})$.

By the essentially same proof of Lemma 1.1.4, one can prove that a π_0^∞-torsion $\mathcal{G}_K$-representation T is of height $\leqslant h$ if and only if $T \cong \widetilde{T}/\widetilde{T}'$ for some $\widetilde{T}$ and $\widetilde{T}'$ which are $\mathscr{O}_0$-lattice $\mathcal{G}_K$-representations of height $\leqslant h$.

It easily follows from Definition 4.1.3 that $\chi_{\mathcal{LT}}^r$ for $0 \leqslant r \leqslant h$ is of height $\leqslant h$. It is not difficult to show that any unramified $\mathcal{G}_K$-representation is of height $\leqslant 0$ (hence, of height $\leqslant h$ for any non-negative h). See, for example, [15, Proposition 5.2.10] for the proof.

Proposition 4.1.8 suggests that $\mathcal{G}_K$-representations of height $\leqslant h$ should enjoy similar properties to those enjoyed by $\mathcal{G}_{K_\infty}$-representation of height $\leqslant h$ in the setting of Kisin theory. On the other hand, $\mathcal{G}_K$-representations of height $\leqslant h$ can also be regarded as a positive characteristic analogue of crystalline representations with Hodge–Tate weights in $[0, h]$, for the following reasons.[24] Effective local shtukas arise naturally by completing global objects at "places of good reduction" such as t-motives, elliptic sheaves, and Drinfeld shtukas. (See [13, Example 2.1.2] for more details.) It has been known for experts that there exists a natural anti-equivalence of categories between the category of effective local shtukas of height $\leqslant 1$ and the category of strict π_0-divisible groups (using the terminology of [6]), and if $\mathfrak{M}$ is the effective local shtuka of height $\leqslant 1$ which corresponds to a strict π_0-divisible group G then $(\underline{T}^{\leqslant 1}_{\mathfrak{S}}(\mathfrak{M}))^*(1)$ is naturally isomorphic to the π_0-adic Tate module of G. This is generalized by Hartl [12, Section 3] to any effective local shtukas.[25] See [15, Section 7.3] for the proof.

4.1.10. We finally remark that the analogue of the "limit theorem" holds; i.e., an $\mathscr{O}_0$-lattice $\mathcal{G}_K$-representation obtained as a limit of π_0^∞-torsion $\mathcal{G}_K$-representation of height $\leqslant h$ is again of height $\leqslant h$ (as an $\mathscr{O}_0$-lattice $\mathcal{G}_K$-representation). The proof is "identical" to the proof of its p-adic analogue [24, Theorem 2.4.1].

4.2. Deformation theory

Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_q$ (which is the residue field of $\mathscr{O}_0$), and $\bar{\rho} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ be a representation. Let $\mathscr{O}$ be a finite extension of $\mathscr{O}_0$ with residue field $\mathbb{F}$. Let $\mathfrak{AR}_{\mathscr{O}}$ be the category of artin local $\mathscr{O}$-algebras A whose residue field is $\mathbb{F}$, and similarly let $\widehat{\mathfrak{AR}}_{\mathscr{O}}$ be the category of complete local noetherian $\mathscr{O}$-algebras with residue field $\mathbb{F}$.

Let $D, D^\square : \widehat{\mathfrak{AR}}_{\mathscr{O}} \to (\mathbf{Sets})$ be the deformation functor and framed deformation functor for $\bar{\rho}$. Since the tangent spaces of these functors are infinite-dimensional (as explained in Section 1.2), they cannot be represented by complete local noetherian $\mathscr{O}$-algebras.

We say that a deformation ρ_A over $A \in \mathfrak{AR}_{\mathscr{O}}$ is *of height* $\leqslant h$ if it is a π_0^∞-torsion $\mathcal{G}_K$-representation of height $\leqslant h$ as a π_0^∞-torsion $\mathcal{G}_K$-representation; or equivalently, if there exist a torsion shtuka $\mathfrak{M}$ with height $\leqslant h$ and an $\mathscr{O}_0[\mathcal{G}_K]$-isomorphism $\rho_A \cong \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M})$. For $A \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$, we say that ρ_A is *of height* $\leqslant h$ if $\rho_A \otimes A/\mathfrak{m}_A^n$ is a deformation of height $\leqslant h$ for each n. When $A \in \mathfrak{AR}_{\mathscr{O}}$, both definitions are compatible because the condition of being height $\leqslant h$ is closed under subquotient. (The proof is the same as that of Lemma 1.4.1.) When A is finite flat over $\mathscr{O}_0$, a deformation ρ_A over A is of height $\leqslant h$ if and only if ρ_A is of height $\leqslant h$ as an $\mathscr{O}_0$-lattice $\mathcal{G}_K$-representation, as remarked in Section 4.1.10.

Let $D^{\leqslant h} \subset D$ and $D^{\square, \leqslant h} \subset D^\square$ respectively denote subfunctors of deformations and framed deformations of height $\leqslant h$. In this setting, we have the analogue of Theorem 1.3:

Theorem 4.2.1. *The functor $D^{\leqslant h}$ has a hull, and if $\mathrm{End}_{\mathcal{G}_K}(\bar{\rho}) \cong \mathbb{F}$ then $D^{\leqslant h}$ is representable (by $R^{\leqslant h} \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$). The functor $D^{\square, \leqslant h}$ is representable (by $R^{\square, \leqslant h} \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$) with no assumption on $\bar{\rho}$. Furthermore, the natural inclusions $D^{\leqslant h} \hookrightarrow D$ and $D^{\square, \leqslant h} \hookrightarrow D^\square$ of functors are relatively representable.*

We call $R^{\square, \leqslant h}$ the *universal framed deformation ring of height* $\leqslant h$ and $R^{\leqslant h}$ the *universal deformation ring of height* $\leqslant h$ if it exists.

The proof of Theorem 1.3 can easily be adapted. The main step is to show the finiteness of the tangent space, but the same proof of Proposition 1.6 works if we replace φ-modules over $\mathfrak{S}$ and $\mathscr{O}_{\mathcal{E}}$ by their positive characteristic analogues as introduced in Section 4.1 and the pth power map is replaced by the qth power map in suitable places. See [15, Section 11.7] for the full details.

[24] We remark that in positive characteristic $K_\infty := K(\sqrt[q^\infty]{u})$ is a purely inseparable field extension of K, so the gap between $\mathcal{G}_K$ and $\mathcal{G}_{K_\infty}$ collapses.

[25] Note that not all the π_0-divisible groups come from effective local shtukas – the π_0-divisible groups that come from effective local shtukas are called *divisible Anderson modules* in [12, Section 3].

Remark 4.2.2. Similarly to the p-adic $\mathcal{G}_{K_\infty}$-deformation rings, one can show that $R^{\square,\leqslant h}[\frac{1}{p}]$ (respectively, $R^{\leqslant h}[\frac{1}{p}]$, if it exists) is a formally smooth $\operatorname{Frac}\mathscr{O}$-algebra, and compute the dimension of the equi-dimensional union of connected components defined by fixing a suitable analogue of Hodge type; see [15, Theorem 11.2.9, Corollary 11.3.11] for the precise statements and the proofs.

4.3. Moduli of torsion shtukas of height $\leqslant h$

Let h be a positive integer, and consider a deformation ρ_R of $\bar{\rho}$ over $R \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$ which is of height $\leqslant h$ (i.e. $\rho_R \otimes_R R/\mathfrak{m}_R^n$ is of height $\leqslant h$ for each n). The main examples to keep in mind are universal framed deformation of height $\leqslant h$. Let M_R be the corresponding étale φ-module constructed in the analogous way as in Section 3.1.

With this setting, we have an analogue of Kisin's construction of moduli of finite flat group schemes [21, Proposition 2.1.10].

Proposition 4.3.1. *Under the notation as above, there exists a projective R-scheme $\mathscr{GR}^{\leqslant h}_{\rho_R}$ such that for any R-algebra A with $\mathfrak{m}_R^N \cdot A = 0$ for some N, the set $\operatorname{Hom}_R(\operatorname{Spec} A, \mathscr{GR}^{\mathbf{v}}_{\rho_R})$ is in natural bijection with the set of φ-stable $\mathfrak{S}_A$-lattices $\mathfrak{M}_A$ in $M_R \otimes_R A$. Here, $\mathfrak{S}_A := \mathfrak{S} \otimes_{\mathscr{O}_0} A$ and we view $M_R \otimes_R A$ as a φ-module by A-linearly extending φ_{M_R}.*

Moreover the structure morphism $\mathscr{GR}^{\mathbf{v}}_{\rho_R} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \operatorname{Spec} R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism.

Indeed, the proof of its p-adic analogue (Proposition 3.2) works verbatim in the positive characteristic setting. (We use Proposition 4.1.8(1) to prove the generic isomorphism.) The proof is also worked out in Proposition 11.1.9, Corollary 11.1.11, Proposition 11.2.6 of [15] for the positive characteristic setting.

When $\bar{\rho}$ is 2-dimensional and $h=1$, one can define the $\mathscr{O}$-flat quotient $R^{\square,\mathbf{v}}$ of $R^{\square,\leqslant 1}$ in the similar fashion to Section 3.1[26]; i.e., the universal quotient classifying lifts such that I_K acts via $\chi_{\mathcal{LT}}$ on the determinant. Then the direct analogue of the connected component result (Proposition 3.3) holds for the positive characteristic deformation ring $R^{\square,\mathbf{v}}[\frac{1}{\pi_0}]$. Furthermore, the argument in [19, Section 3] can be adapted to show that $R^{\square,\mathbf{v}}[\frac{1}{\pi_0}]$ is equi-dimensional of dimension $4+[K:\mathbb{F}_q((u_0))]$, which is strongly analogous to the p-adic case. (Compare with [19, Theorem 3.3.8] and [15, Section 11.3.17].) All these results can be generalized to the case with $h>1$ except the connectedness of the "supersingular locus" in $\operatorname{Spec} R^{\square,\mathbf{v}}[\frac{1}{\pi_0}]$ (with the suitable definition of $R^{\square,\mathbf{v}}$).

Acknowledgments

The author deeply thanks his thesis supervisor Brian Conrad for his guidance. The author especially appreciates his careful listening of my results and numerous helpful comments. The author thanks Mark Kisin and Tong Liu for their helpful advices and the anonymous referees for their comments on the presentations and suggesting improvements of the original argument of the previous version.

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[26] In the p-adic case $\operatorname{Spec} R^{\square,\mathbf{v}}_\infty[\frac{1}{p}]$ is a union of connected components of $\operatorname{Spec} R^{\square,\leqslant 1}_\infty[\frac{1}{p}]$. But in the positive characteristic setting, the author could only prove this when K is separable over $k((u_0))$. See [15, Proposition 11.3.7].

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