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Tiling with Polyominoes*

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INTRODUCTION

An *n-omino* is a set of n rookwise connected unit squares [1]. The simpler figures (monomino, domino, trominoes, tetrominoes, and pentominoes) are shown in Figure 1.

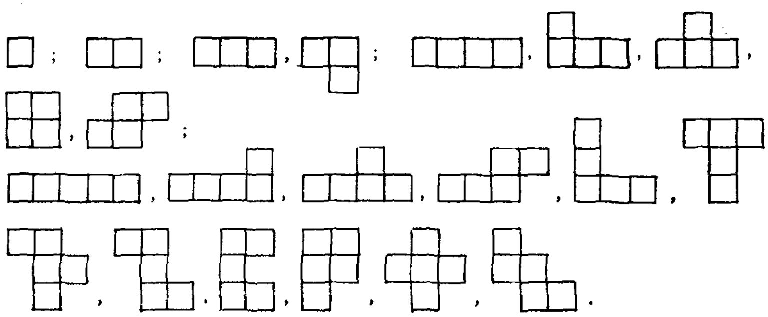

Fig. 1. The simpler polyominoes.

It is interesting to consider the tiling properties of the various polyominoes. Certain polyominoes will tile the infinite plane, as illustrated with the X-pentomino in Figure 2. Some polyominoes can tile an infinite strip, as indicated with the F-pentomino in Figure 3. There are polyominoes

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which are capable of tiling a rectangle, as exemplified by the *Y*-pentomino in Figure 4. Also, certain polyominoes are "rep-tiles" [2], i.e., they can be used to tile enlarged scale models of themselves, as shown with the *P*-pentomino in Figure 5.

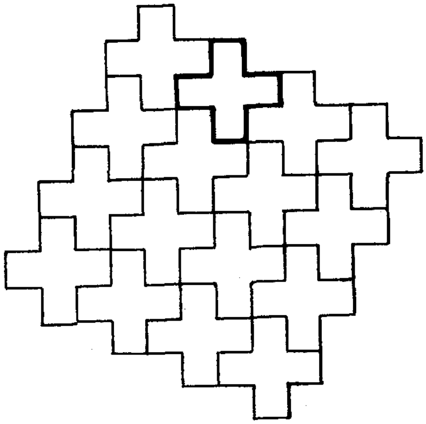


Fig. 2. The *X*-pentomino can be used to tile the plane.

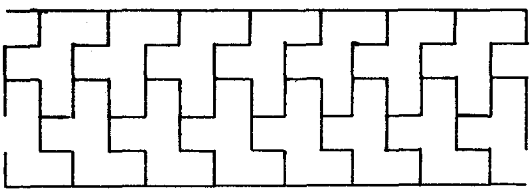


Fig. 3. The *F*-pentomino can be used to tile a strip.

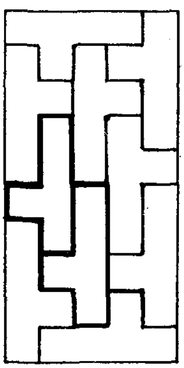


Fig. 4. The *Y*-pentomino can be used to tile a rectangle.

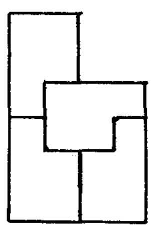


Fig. 5. The *P*-pentomino can be used to tile itself.

The objective of this paper is to establish a hierarchy of tiling capabilities for polyominoes, and to classify all the simpler polyominoes (through hexominoes) as accurately as possible with respect to this hierarchy.

2. The Tiling Hierarchy

In Figure 6, we see the hierarchy of tiling capabilities for polyominoes, in the sense that a polyomino which tiles any of the regions specified in the figure can also tile all the regions listed lower in the hierarchy.

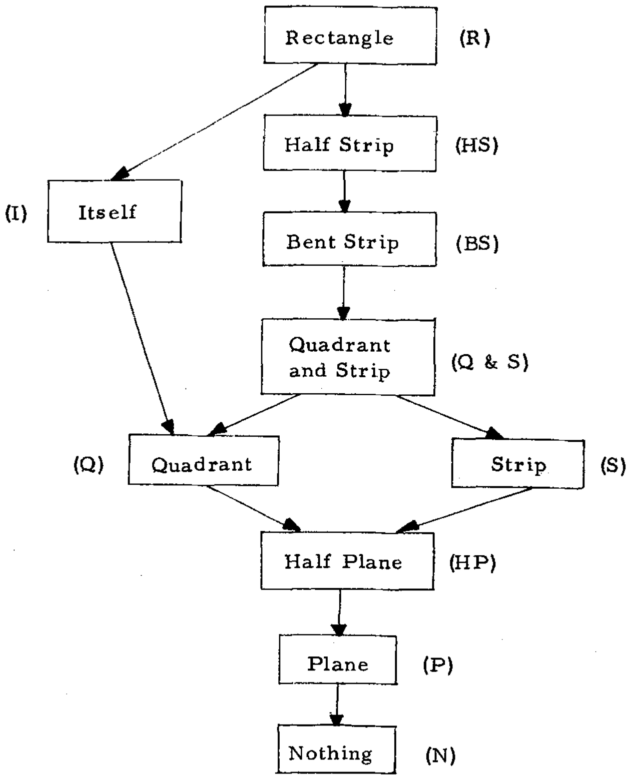


Fig. 6. The hierarchy of tiling capabilities for polyominoes.

To describe the regions more precisely, the plane (P) is the Cartesian xy-plane, extending infinitely in all four directions (right, left, up, and down). The *half-plane* (HP) may be thought of as a right half plane (say to the right of $x = 0$), extending infinitely in three directions (right up, and down). The *quadrant* (Q) can be taken as the "first quadrant," extending infinitely in two directions (right and up). The *strip* (S) may be considered as the vertical strip between $x = 0$ and $x = a$, extending infinitely in two directions (up and down). The *bent strip* (BS) can be taken as that portion of the first quadrant which lies either to the left of $x = a$, or below $y = b$ (or both). The *half-strip* (HS) may be regarded as the upper half-strip bounded by $x = 0$, $x = a$, and $y = 0$, extending infinitely in only one direction (up). Finally, the *rectangle* (R) is finite in all directions, and can be thought of as bounded by $x = 0$, $x = a$, $y = 0$, $y = b$.

The position in the hierarchy labeled *quadrant and strip* (Q & S) refers to the possibility that a polyomino may be capable of tiling either a quadrant or a strip. Finally, the position labeled itself (I) corresponds to the case that a polyomino can tile some enlarged scale model of itself, for some scale factor $c > 1$.

We shall prove that the tiling implications in Figure 6 are valid, and give examples to show that several of them are *true categories* for certain polyominoes, in the sense that the polyomino can tile the region at that point but at no higher point in the hierarchy.

NOTE. The "arms" of a bent strip need not be of equal width. However, when the widths are unequal (say a and b), two such strips can be nested so as to produce a bent strip with each arm having width $a + b$.

3. PROOF OF THE TILING IMPLICATIONS

THEOREM 1.

$$\begin{tikzcd}
(R) \arrow[r] & (HS) \arrow[r] \arrow[d] & (S) \arrow[r] & (HP) \arrow[r] & (P) \arrow[r] & (N) \\
& (BS) \arrow[r] & (Q) \arrow[ur] & & &
\end{tikzcd}$$

PROOF. The tiling of a rectangle (R) can be repeated periodically upward to produce a tiled half-strip (HS). Two half strips can be fitted together at right angles to form a bent strips (BS), or "back-to-back" to form a strip (S). Bent strips can be "nested" to fill a quadrant, as illustrated in Figure 7. Strips can be repeated periodically to the right to fill a half-plane (HP). Also, two quadrants can be put together to form a half-plane. Two half-planes "back-to-back" fill the plane (P).

The implication (P)⟶(N) is vacuous, since N corresponds to no tiling requirement whatever.

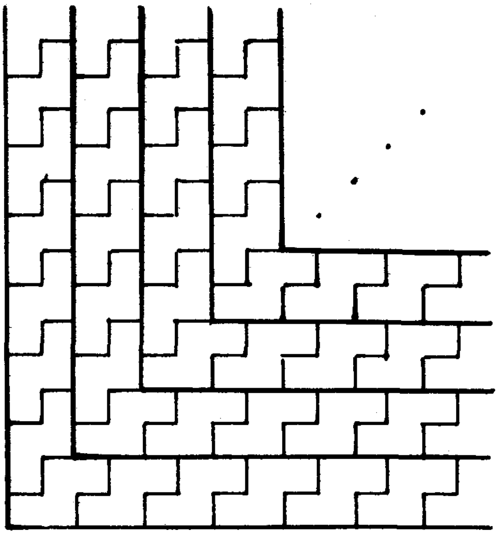


Fig. 7. Bent strips nested to fill a quadrant.

THEOREM 2. (R)⟶(I)⟶(P).

PROOF. If a polyomino tiles an $a \times b$ rectangle, this rectangle can be used to tile an $ab \times ab$ square, and this square can be used n times to form the original n-omino, suitably enlarged.

(If $a = b = 1$, the original polyomino is a monomino, and four of these can be used to form an enlarged scale model. In all other cases, the enlargement factor ab exceeds 1.)

If a polyomino tiles itself, this process of producing an enlarged model can be iterated over and over to cover larger and larger regions of the plane. In the limit, the infinite plane is covered. (A more detailed proof of a stronger result is given for Theorem 5.)

DEFINITION. The smallest rectangle which can contain a given polyomino, and has sides parallel to those of the polyomino, is called the *rectangular hull* of that polyomino.

THEOREM 3. (BS)⟶(S).

PROOF. Suppose a polyomino, A, tiles a bent strip. Let m be the maximum dimension of the rectangular hull of A, and let w be the width of one of the arms of the bent strip. Along that arm of the bent strip, we examine segments of length m. A connected path cutting no polyominoes can always be drawn down any such $m \times w$ segment (cf. Figure 8), since m is the maximum detour which can be caused by the poly-

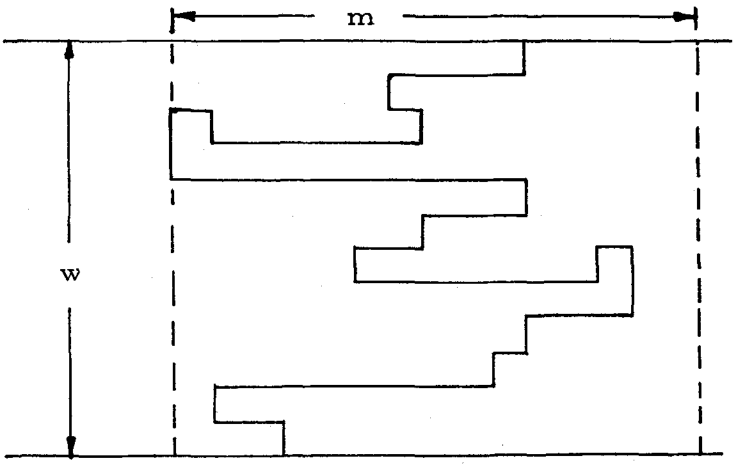


Fig. 8. A path through a segment of length m in an arm of width w of a bent strip.

omino A. There are only a finite number of possible paths through an $m \times w$ pattern of unit tiles; so the semi-infinite arm must somewhere contain two identical top-to-bottom paths. The entire structure between these two identical paths can then be repeated periodically to tile a (doubly infinite) strip.

THEOREM 4. *A polyomino rep-tile always covers at least one corner of its rectangular hull. However, it may fail to cover more than one of the four corners.*

PROOF. Suppose a polyomino occupies none of the four corners of its rectangular hull (cf. Figure 9). If it is a rep-tile, a reduced scale model (with arbitrarily large reduction factor, by iterating the replication process) must be capable of filling every nook and cranny of the original figure. However, it is clear that a *greatly* reduced version will be unable to get into any of the internal corners of the figure (cf. the x-marks in

Figure 9), since the tiling unit has no extremal corners available for the purpose.

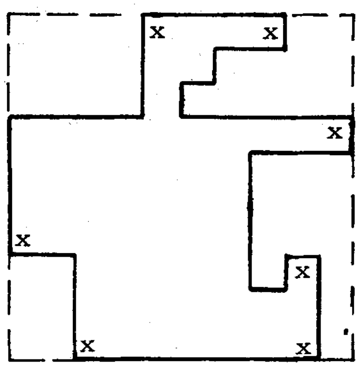


Fig. 9. A polyomino which covers none of the corners of its rectangular hull.

On the other hand, we see in Figure 10 a polyomino which covers only one corner of its rectangular hull, but such that four of them form a square. Hence the figure is a rep-tile, by Theorem 2.

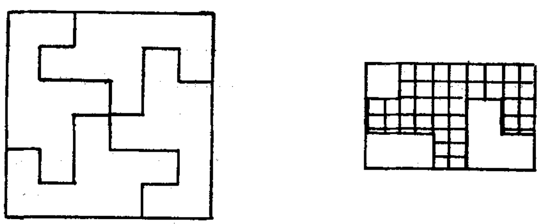


Fig. 10. A rep-tile which covers only one corner of its rectangular hull.

THEOREM 5. *If a polyomino tiles itself, it can tile a quadrant.*

PROOF. By Theorem 4, a polyomino which tiles itself (viz., a "rep-tile") occupies one of the corners of its rectangular hull. Put the figure in the first quadrant, with an occupied corner at the origin.

Divide the polyomino into replicas of itself. (Call this the operation ϱ.) The small replica touching the origin may differ in orientation from the parent by some symmetry operator φ, where φ belongs to the group D_4 of symmetries of the square. If the replication ϱ is iterated eight times, there must exist two origin-touching replicas of different sizes, among the nine sizes present, which have the same orientation, since D_4 contains only eight distinct orientations. Suppose then that ϱ^{α} and ϱ^{β} both result in the same orientation for the figure touching the origin, where $0 \leq \alpha < \beta \leq 8$. Now start with the polyomino in the first quadrant having the orientation resulting from ϱ^{α}, and consider the replication operation $\varrho^{\beta-\alpha}$, followed by an expansion operation ε, which brings the small piece touching the origin back up to the size of the original figure. This process, consisting of $\varrho^{\beta-\alpha}$ followed by ε, may be iterated repeatedly to cover larger and larger portions of the first quadrant, *consistent* with the partial tiling already established. In the limit, the entire first quadrant is covered.

These theorems completely establish the hierarchy shown in Figure 6. In both Theorem 3 and Theorem 5, it is rather easy to produce a non-constructive proof using König's lemma, which in turn requires the Axiom of Choice. For obvious reasons, the constructive proofs given here are preferable.

4. Characteristic Examples

Since Figure 6 goes no higher than rectangles, any polyomino which tiles a rectangle is characteristic for this class. We may use the *Y*-pentomino construction of Figure 4 to illustrate this case.

The skew tetromino tiles a bent strip, as shown in Figure 7. If this were not its characteristic level in the hierarchy, it would be able to tile a half-strip. However, it is trivial to observe that the skew tetromino cannot cover the finite edge of an infinite strip, including both corners. (See Figure 11.)

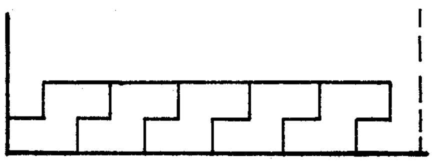


Fig. 11. The skew tetromino cannot cover a finite edge with two corners.

The *F*-pentomino is shown in Figure 3 to tile a strip. If it were possible to do better with this figure, it would have to be capable of filling a corner. However, we see in Figure 12 that if the *F* is placed in the only way which gets it to the corner, a square inaccessible to *F*-pentominoes is produced.

We see in Figure 2 that the X-pentomino tiles the plane. On the other hand, it is clear (cf. Figure 13) that it cannot tile a half-plane, because it cannot even touch an edge without creating two inaccessible squares.

In Figure 14 we see a nonomino which cannot possibly tile the plane because it contains a notch it clearly cannot fill. This can be used as a characteristic example of the class (N), although there are polyominoes of fewer squares belonging to class (N).

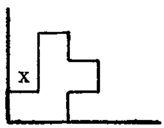


Fig. 12. If the F-pentomino touches a corner, it creates a square inaccessible to F-pentominoes.

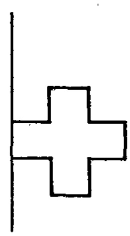


Fig. 13. The X-pentomino cannot fill an edge.

Thus far, no confirmed examples have been found which are characteristic for any of the other classes: (HP), (Q), (Q & S), (HS), and (I). In particular, although it is conceivable that there are rep-tiles at each

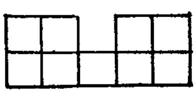


Fig. 14. A polyomino which cannot tile the plane.

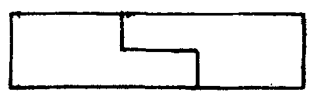
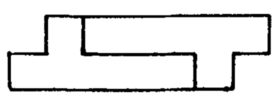


Fig. 15. Two P-polyominoes fit to form a rectangle.

of the levels from (Q) to (R), all known polyomino rep-tiles actually belong to class (R). Of course, we see in Figure 5 that a rep-tile construction need not involve the intermediate formation of a rectangle; but the fact remains that the P-pentomino *also* tiles a rectangle (see Figure 15).

There is a possible suspect for the class (HS). The Y-hexomino can be used to tile a half-strip of width 16, as shown in Figure 16, where the construction continues periodically, using the figure  composed of two Y-hexominoes as the basic unit. However, the possibility that the Y-hexomino can also tile a rectangle has never been ruled out, even though all attempts to *find* a rectangular pattern have been futile to date.

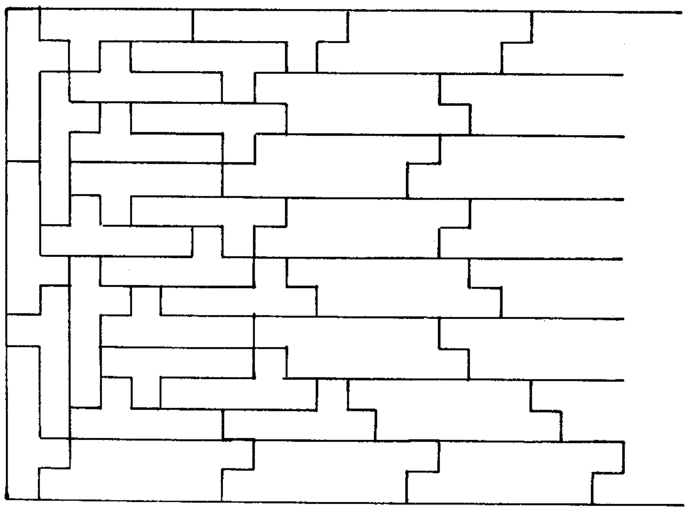


Fig. 16. The Y-hexomino tiles a half-strip.

The principal objective of future investigation in this area should be to determine whether the remaining levels in the hierarchy (Figure 6) possess characteristic examples, or whether the lattice can be collapsed further, eliminating these levels as logically distinct cases.

5. Systematic Classification of Polyomino Tiling Properties

The general problem of classifying an arbitrary polyomino as to its tiling properties is logically undecidable. This means that there is no (finite) computer program capable of making this classification for all polyominoes. This type of undecidability result is obtained by the methods of H. Wang and his students [3, 4]. On the other hand, many individual cases can be decided rather easily, including most of the simpler polyominoes.

In general, the constructions and impossibility proofs corresponding to the classifications in Table I will be left as exercises for the reader.

TABLE I

TILING PROPERTIES OF POLYOMINOES

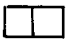
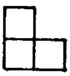
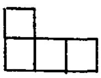
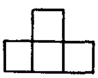
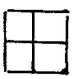
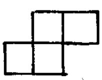
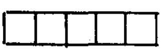
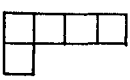
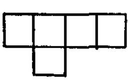
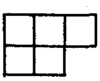
Polyomino	Rec-tangle	Half Strip	Bent Strip	Strip	Plane	Nothing
1. 	√					
2. 	√					
3. 	√					
4. 	√					
5. 	√					
6. 	√					
7. 	√					
8. 	√					
9. 			√			
10. 	√					
11. 	√					
12. 	√					
13. 	√					

TABLE I (*continued*)

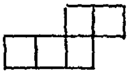
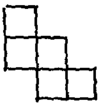
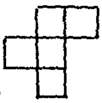
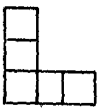
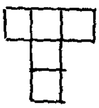
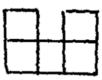
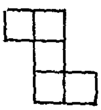
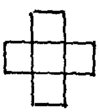
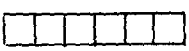
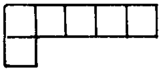
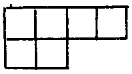
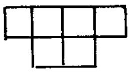
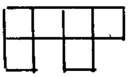
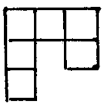
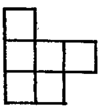
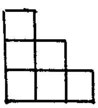
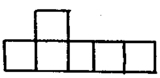
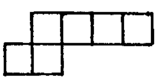
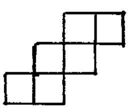
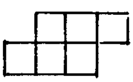
Polyomino	Rec-tangle	Half Strip	Bent Strip	Strip	Plane	Nothing
14. 			√			
15. 			√			
16. 				√		
17. 				√		
18. 					√	
19. 					√	
20. 					√	
21. 					√	
22. 	√					
23. 	√					

TABLE I (*continued*)

Polyomino		Rec-tangle	Half Strip	Bent Strip	Strip	Plane	Nothing
24.		√					
25.		√					
26.		√					
27.		√					
28.		√					
29.		√					
30.		√					
31.			√*				
32.				√			
33.				√			
34.				√			

* In this case, the rectangle has not been ruled out.

Table I (*continued*)

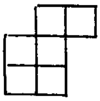
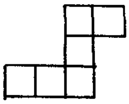
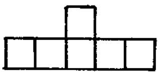
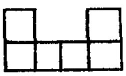
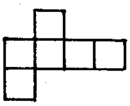
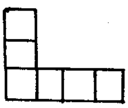
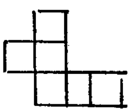
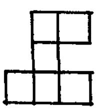
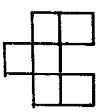
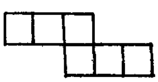
	Polyomino	Rec-tangle	Half Strip	Bent Strip	Strip	Plane	Nothing
35.				√			
36.				√			
37.					√		
38.					√		
39.					√		
40.					√		
41.					√		
42.					√		
43.					√		
44.					√		

TABLE I (*continued*)

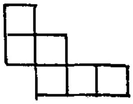
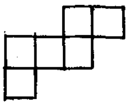
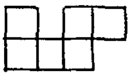
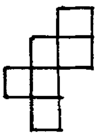
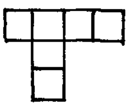
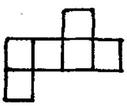
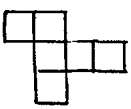
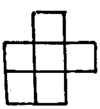
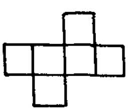
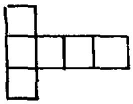
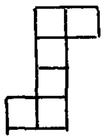
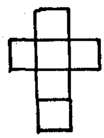
	Polyomino	Rec-tangle	Half Strip	Bent Strip	Strip	Plane	Nothing
45.					√		
46.					√		
47.						√	
48.						√	
49.						√	
50.						√	
51.						√	
52.						√	
53.						√	

TABLE I (*continued*)

Polyomino	Rec-tangle	Half Strip	Bent Strip	Strip	Plane	Nothing
54. 					√	
55. 					√	
56. 					√	

However, two of the pentomino constructions, for strip and bent strip respectively, are shown in Figure 17.

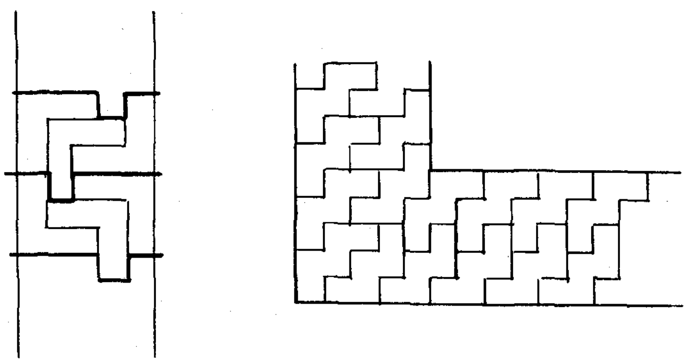


Fig. 17. Constructions for pentomino tilings of a strip and a bent strip.

A particularly intricate construction is the bent strip using the hexomino numbered 36 (the "snake") in Table I (Figure 18). Note that the (N) category is empty, the first examples being certain heptominoes, three of which appear in Figure 19.

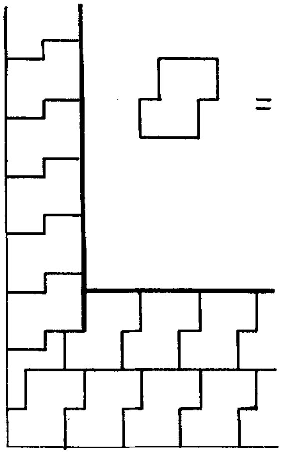


Fig. 18. An elegant bent-strip construction with the snake hexomino.

The hexomino numbered 29 in Table I can be used to tile a 9×12 rectangle. The construction is left as an exercise for the reader.

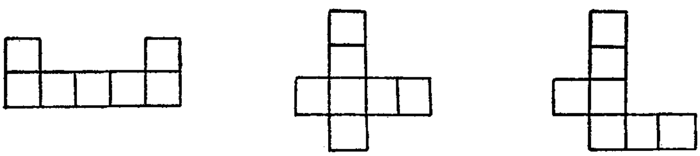


Fig. 19. Three heptominoes that cannot tile the plane.

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2. S. W. Golomb, Replicating Figures in the Plane, *Math. Gaz.* **48**, No. 366 (Dec. 1964), 403–412.
3. H. Wang, Games, Logic, and Computers, *Sci. Amer.* **213**, No. 5 (Nov. 1965), 98–106.
4. R. Berger, *The Undecidability of the Domino Problem*, Harvard Computation Laboratory Rep. BL-37, Oct. 1964.