Topology and its Applications 41 (1991) 235-245
North-Holland

On completeness of spaces of open mappings on continua

Janusz R. Prajs

Institute of Mathematics, Pedagogical University, Ul. Oleska 48, 45-951 Opole, Poland

Received 2 January 1990

Abstract

Prajs, J.R., On completeness of spaces of open mappings on continua, Topology and its Applications 41 (1991) 235-245.

Among other things it is proved that the set of all open mappings between compacta X and Y is topologically complete if X is locally connected and Y is a graph, and this set is not topologically complete if it is nonempty and Y is a manifold of dimension >1, or Y is the Menger universal curve, or Y is a pseudo-arc.

Keywords: Continuous decomposition, metric continuum, open mapping, set in pointlike position, topologically complete space.

AMS (MOS) Subj. Class.: 54F20.

Introduction

When homogeneity with respect to open mappings is studied, the question whether the set of all open mappings between two given compacta is completely metrizable naturally appears. Since the space of all continuous mappings between compacta is completely metrizable, the question is equivalent to the question raised by Charatonik and Maćkowiak [2, Chapter 2], whether all open mappings form a G_δ subset of the set of all continuous mappings. The general negative answer has been presented by Hohti in [3]. He has proved that the set of all open surjections of the Cantor set C onto itself is not a G_δ subset of the space of all continuous autosurjections of C. A modification, due to W.J. Charatonik, is mentioned in [3], to obtain a similar result for open mappings between continua (see Remark 3.2 below). On the other hand McAuley proved in [12] that the space of all open mappings between any locally connected continuum and an arc is topologically complete. In a conversation with the author Hohti asked if McAuley's result can be extended by replacing an arc by any locally connected continuum. Though this extension has occurred to be not true, the question was an inspiration for the results of this paper. The author wishes to express his thanks to Professor Hohti for fruitful discussions on spaces of open mappings.

In the next section we collect a few of general facts concerning spaces of open mappings. In the third section we introduce the notion of a set in pointlike position in the space, and, applying some strong theorems on continuous decompositions of the plane, the Menger universal curve and the pseudo-arc, we use this notion to obtain a number of negative results. In the last section we extend McAuley's result replacing an arc by any graph.

In this paper spaces are assumed to be metric. Spaces of mappings are equipped with the compact open topology and the sup metric. A map $f: X \to Y$ is said to be:
- monotone, if $f^{-1}(y)$ is connected for each $y \in Y$,
- light, if $f^{-1}(y)$ is totally disconnected for each $y \in Y$,
- confluent, if $f(C) = K$ for each continuum $K \subset Y$ and each component C of $f^{-1}(K)$.

The symbols $c(X, Y)$, $o(X, Y)$, $\mathrm{mo}(X, Y)$, $\mathrm{lo}(X, Y)$ denote the sets of all continuous, open, monotone open, light open mappings from X into Y, respectively. Let a space X be compact and $A \subset X$ be a closed set. Then $g_A: X \to X/A$ denotes the natural quotient map obtained by identifying all points of A to a point. Manifold means here connected, compact, finitely or infinitely dimensional manifold. The symbol ab denotes an arc with a and b as the endpoints. An arc ab is called to be free in a space X provided the set $ab - \{a, b\}$ is open in X. A continuum is called a graph if it is the finite union of its free arcs. The unit interval $[0, 1]$ is denoted by I. The symbol $B(A, \varepsilon)$ denotes the open ball around a set A with radius ε.

1. Some general facts

In this section we prove some general facts concerning Borel class of open mappings between compacta. First, recall that given compacta X and Y, the set $o(X, Y)$ is known to be an $F_{\sigma\delta}$ subset of $c(X, Y)$ [2, Proposition 2.1]. The sets of all monotone mappings, light mappings and confluent mappings are known to be G_δ subsets of $c(X, Y)$ (see [5; 4, Theorem 5, p. 109; 13, Theorem (2.10)]).

1.1. Lemma. *Let X, Y, Z be topological spaces and sets $A \subset c(X, Y)$, $B \subset c(Y, Z)$, $C \subset c(X, Z)$ be such that $A \neq \emptyset$ and*
(i) *if $f \in A$ and $g \in B$, then $gf \in C$,*
(ii) *if $f \in c(X, Y)$, $g \in c(Y, Z)$ and $gf \in C$, then $g \in B$.*
If C is a subset of $c(X, Y)$ of Borel class α, then B is a subset of $c(Y, Z)$ of Borel class α.

Proof. Fix $f \in A$ and take the map $F: c(Y, Z) \to c(X, Z)$ defined by $F(g) = gf$. This map is continuous and $F^{-1}(C) = B$ by (i) and (ii). This implies the conclusion. □

1.2. Proposition. *Let X, Y, Z be compacta and $A \subset c(X, Y)$, $B \subset c(Y, Z)$, $C \subset c(X, Z)$, with $A \neq \emptyset$, be the sets of all open (monotone open, light open) mappings in the respective sets of continuous mappings. If C is a G_δ subset of $c(X, Z)$, then B is a G_δ subset of $c(Y, Z)$.*

Proof. For open and monotone open mappings we obtain the conclusion by Lemma 1.1 because these mappings satisfy (i) and (ii). In the case of light open mappings take F defined in the proof of Lemma 1.1. In this case we have only (i) and thus $B \subset F^{-1}(C)$. However, observe that $F^{-1}(C)$ is composed of open mappings only. Intersecting the G_δ set $F^{-1}(C)$ with the G_δ set of all light mappings in $c(Y, Z)$ we obtain B as a G_δ set in $c(Y, Z)$. □

1.3. Corollary. *Let Y be a compactum. If there is a compactum X such that the set $o(X, Y)$ $(\text{mo}(X, Y), \text{lo}(X, Y))$ is a nonempty G_δ subset of $c(X, Y)$, then the set $o(Y, Y)$ $(\text{mo}(Y, Y), \text{lo}(Y, Y)$, respectively) is a G_δ subset of $c(Y, Y)$.*

1.4. Proposition. *Let X and Y be compacta. If the set $o(X \times Y, X \times Y)$ $(\text{mo}(X \times Y, X \times Y), \text{lo}(X \times Y, X \times Y))$ is of Borel class α in $c(X \times Y, X \times Y)$, then the sets $o(X, X)$ and $o(Y, Y)$ $(\text{mo}(X, X)$ and $\text{mo}(Y, Y)$, $\text{lo}(X, X)$ and $\text{lo}(Y, Y))$ are of Borel class α in $c(X, X)$ and $c(Y, Y)$, respectively.*

Proof. Let $A \subset c(X, X)$, $B \subset c(Y, Y)$, $C \subset c(X \times Y, X \times Y)$ denote the set of all open (monotone open, light open) mappings in the respective spaces of continuous mappings. Consider the map $F: c(X, X) \to c(X \times Y, X \times Y)$ defined by $F(f)(x, y) = (f(x), y)$. It is obvious that F is continuous and $F(f) \in C$ if and only if $f \in A$. Thus $A = F^{-1}(C)$ and the conclusion follows for A. The proof for B is similar. □

In the next section we will see that the converse implication to that from Proposition 1.4 does not hold true for $X = Y = I$.

Recall that two spaces X and Y are said to be equivalent with respect to a class $\mathcal{F}$ of mappings provided there are surjections $f: X \to Y$ and $g: Y \to X$ belonging to $\mathcal{F}$.

Question 1. Let compacta X and Y be equivalent with respect to the class $\mathcal{F}$ of all open (monotone open, light open) mappings, and assume the set of all maps $f: X \to X$ in $\mathcal{F}$ is a G_δ set in $c(X, X)$. Does it follow that the set of all maps $f: Y \to Y$ in $\mathcal{F}$ is a G_δ set in $c(Y, Y)$?

2. Sets in pointlike position and some negative answers

Let A be a nonempty closed subset of a compactum X. We say that A is in pointlike position in X provided for each neighborhood U of A in X there is a homeomorphism $h: X/A \to X$ onto X such that $hg_A(x) = x$ for any $x \in X - U$.

The following theorem is crucial in this section.

2.1. Theorem. *Let X and Y be compacta. If there are a map $f \in o(X, Y)$ $(f \in \text{mo}(X, Y)$, $f \in \text{lo}(X, Y))$ and a sequence $\{y_n\} \subset Y$ converging to some $y \in Y$, $y_n \neq y$, such that the sets $f^{-1}(y_n)$ are in pointlike position in X and $f^{-1}(y)$ is nondegenerate, then $o(X, Y)$ $(\text{mo}(X, Y), \text{lo}(X, Y)$, respectively) is not a G_δ set in $c(X, Y)$.*

Proof. Let $F \subset c(X, Y)$ be the set of all open (light open, monotone open) mappings from X into Y and let $f \in F$, y, $\{y_n\}$ satisfy the assumptions of the theorem. Put $A_n = f^{-1}(y_n)$. Suppose $F = \bigcap_{n=1}^{\infty} U_n$, where each U_n is open in $c(X, Y)$. A sequence $\{f_n\} \subset F$ such that $\lim f_n \in \bigcap_{n=1}^{\infty} U_n - F$ will be constructed, obtaining thus a contradiction. We will inductively construct sequences: of positive numbers ε_k, δ_k, of positive integers n_k $(k = 1, 2, \ldots)$, and of mappings $f_k \in F$, $k = 0, 1, \ldots$, which fulfil the following conditions.

(i) If $g \in c(X, Y)$ and $d(f_k, g) \leq \varepsilon_{k+1}$, then $g \in U_{k+1}$,

(ii) $d(f_{k+m}, f_k) < \varepsilon_{k+1}$ for $m = 1, 2, \ldots$,

(iii) $0 < \varepsilon_{k+1} \leq \frac{1}{2}\varepsilon_k$,

(iv) $d(y_{n_k}, y) < \frac{1}{2}\varepsilon_k$,

(v) $B(y_{n_k}, \delta_k) \subset B(y, \frac{1}{2}\varepsilon_k)$,

(vi) $y, y_n \notin B(y_{n_k}, \delta_k)$ for $y_n \neq y_{n_k}$,

(vii) $B(y, \frac{1}{2}\varepsilon_{k+1}) \cap B(y_{n_k}, \delta_k) = \emptyset$.

Namely, put $f_0 = f$. Assume the construction is already done for $k = 0, \ldots, l-1$. Take any $\varepsilon_l > 0$ satisfying (i) for $k = l-1$, and, if $l > 1$, also (iii) and (vii). Find any n_l satisfying (iv) for $k = l$, and find δ_l satisfying (v) and (vi) for $k = l$. Put $W_l = f^{-1}(B(y_{n_l}, \delta_l))$ and, using the pointlike position of A_{n_l}, take a homeomorphism $h_l : X/A_{n_l} \to X$ such that $h_l g_{A_{n_l}}(x) = x$ for $x \notin W_l$. Define $f_l : X \to Y$ by $f_l(x) = f_{l-1}(g_{A_{n_l}}^{-1} h_l^{-1}(x))$. Observe that f_l is well defined and continuous. Moreover note that $f_l \in F$. Since the construction of mappings $f_l, f_{l+1}, \ldots$ is a modification of the map f_{l-1} which changes nothing in $Y - B(y, \frac{1}{2}\varepsilon_l)$ and $X - f_{l-1}^{-1}(B(y, \frac{1}{2}\varepsilon_l))$ condition (ii) is obtained. The construction is complete.

The sequence $\{f_k\}$ converges to some $h \in c(X, Y)$ by (ii) and (iii), and $d(h, f_k) \leq \varepsilon_{k+1}$, $k = 0, 1, \ldots$, by (ii). Applying (i) we infer that $h \in U_k$ for $k = 1, 2, \ldots$. Finally observe that the set $h^{-1}(y) = f^{-1}(y)$ is nondegenerate while $h^{-1}(y_{n_k}) = f_k^{-1}(y_{n_k})$ are degenerate. This implies that the decomposition of X into the fibers $h^{-1}(p)$, $p \in Y$, is not continuous at $h^{-1}(y)$. Hence h is not open. Thus we have $h \in \bigcap_{n=1}^{\infty} U_n - F$. This contradiction completes the proof. □

The remaining part of this section is devoted to applications of Theorem 2.1. First, some results concerning manifolds will be proved. The following proposition is a consequence of the theorem of Moore [4, p. 533] and of [4, Theorem 3, p. 536].

2.2. Proposition. *Let K be a subcontinuum of the interior part of the unit square I^2. Then K is in pointlike position in I^2 if and only if it does not separate I^2.*

Take the decomposition of the plane into pseudo-arcs constructed by Lewis and Walsh [10], consider I^2 as a compactification of the plane, and extend the decomposition to a continuous decomposition of I^2 by taking singletons in the boundary. The quotient space is homeomorphic to I^2 by the theorem of Moore [4, p. 533] again. Since a pseudo-arc does not separate the plane, we obtain the following by Propositions 2.1 and 2.2.

2.3. Proposition. *The sets* $o(I^2, I^2)$ *and* $\mathrm{mo}(I^2, I^2)$ *are not* G_δ *subsets of* $c(I^2, I^2)$.

Now we apply Proposition 1.4 to the above result.

2.4. Proposition. *If* $X = I^n$ *for* $n \geq 2$, *or* X *is the Hilbert cube* I^ω, *then the sets* $o(X, X)$ *and* $\mathrm{mo}(X, X)$ *are not* G_δ *subsets of* $c(X, X)$.

This proposition can also be verified without using Proposition 1.4. In fact, let $X = I^2 \times I^\alpha$, $\alpha = 0, 1, \ldots, \omega$, and take the decomposition $\mathscr{D}$ of I^2 described above. Extend it to $I^2 \times I^\alpha$ by taking the sets $P \times \{q\}$ as elements, where $P \in \mathscr{D}$ and $q \in I^\alpha$. Observe that all elements $P \times \{q\}$ are in pointlike position in $I^2 \times I^\alpha$ and the quotient space is homeomorphic to $I^2 \times I^\alpha$.

If X is a subset of a manifold Y with int $X \neq \emptyset$, we can modify this decomposition of X to obtain all elements in the boundary degenerate, and then extend it to Y by taking singletons in $Y - X$. Then the quotient space is homeomorphic to Y. Hence, in view of Theorem 2.1, Proposition 2.4 remains true for any manifold X with dim $X \geq 2$. Combining this fact with Corollary 1.3 we obtain the general negative answer for manifolds.

2.5. Theorem. *Let* X *be a compactum and* Y *be a manifold of dimension* ≥ 2. *If any of the sets* $o(X, Y)$ *and* $\mathrm{mo}(X, Y)$ *is nonempty, then it is not a* G_δ *subset of* $c(X, Y)$.

Now we are going to prove some results concerning the Menger universal curve M. First recall three strong results which we are going to use.

Characterization Theorem (Anderson [1]). *Every* 1*-dimensional locally connected continuum which has no local separating points and has no open subset imbeddable in the plane, is homeomorphic to* M.

Decomposition Theorem (Wilson [14, Theorem 1]). *For any locally connected continuum* X *there is a monotone open map* $f: M \to X$, *such that each point-inverse set is homeomorphic to* M.

Homeomorphism Extension Theorem (Mayer, Overseegen and Tymchatyn [11, p. 34]). *Let* K *and* L *be closed, nonlocally separating subsets of* M, *and* $h: K \to L$ *be a homeomorphism. Then* h *extends to a homeomorphism of* M *onto itself.*

The following is an observation on the neighborhoods of point $(0, 0, 0)$ in M combined with the homogeneity of M.

2.6. Observation. *For every $x \in M$ there are arbitrarily small open neighborhoods U of x such that* cl U *is homeomorphic to M and* bd U *does not locally separate* cl U.

2.7. Lemma. *For any proper subcontinuum X of M the following conditions are equivalent:*

(a) *X is in pointlike position in M,*

(b) *M/X is homeomorphic to M,*

(c) *$g_X(X)$ is not a local separating point of M/X.*

Proof. (a)→(b) is trivial. Equivalence (b)↔(c) is a consequence of the Characterization Theorem. Assume M/X is homeomorphic to M and let W be any neighborhood of X in M. Find a neighborhood U of $g_X(X)$ as in Observation 2.6 such that $g_X^{-1}(U) \subset W$. Then $K = g_X^{-1}(\text{cl } U)$ is homeomorphic to M by the Characterization Theorem. Moreover, the homeomorphism $h : \text{bd } U \to \text{bd } K$ defined by $h(x) = g_X^{-1}(x)$ can be extended to a homeomorphism $h^* : \text{cl } U \to K$ by the Homeomorphism Extension Theorem, and then to a homeomorphism $h^{**} : M/X \to M$ putting $h^{**}(x) = g_X^{-1}(x)$ for $x \in M/X - \text{cl } U$. We have (a), and thus the proof is complete. □

An easy proof of the following lemma is left to the reader.

2.8. Lemma. *Let $f : X \to Y$ be a monotone surjection between locally connected continua X and Y, $y \in Y$, and put $A = f^{-1}(y)$. If $g_A(A)$ is a local separating point of X/A, then y is a local separating point of Y.*

Now we formulate the main result concerning the space M.

2.9. Theorem. *Let X be a locally connected continuum with infinitely many nonlocally separating points. Then the sets $o(M, X)$ and* $\text{mo}(M, X)$ *are not G_δ subsets of $c(M, X)$.*

Proof. Let $f : M \to X$ be a monotone open map guaranteed by the Decomposition Theorem, and $\{x_n\}$ be a sequence of nonlocally separating points in X converging to some $x_0 \in X$ with $x_n \neq x_0$. The sets $f^{-1}(x_n)$ are in pointlike position in M by Lemmas 2.8 and 2.7. The conclusion follows by Theorem 2.1. □

Note that the sets $o(M, M)$ and $\text{mo}(M, M)$ are not G_δ subsets of $c(M, M)$ by Theorem 2.9. Applying Corollary 1.3 we obtain the following.

2.10. Corollary. *Let X be a compactum. If any of the sets $o(X, M)$ and* $\text{mo}(X, M)$ *is nonempty, then it is not a G_δ subset of $c(X, M)$.*

In view of the Decomposition Theorem and some other results of [14] we have quite good information about open mappings on the Menger universal curve. The situation is completely different when the Sierpiński universal plane curve S is considered. It is not known, for example, if S admits a monotone open map onto itself which is not a homeomorphism. Neither can the author answer the following question.

Question 2. Is the set $o(S, S)$ a G_δ subset of $c(S, S)$?

Now some results concerning pseudo-arc will be presented. The author thanks Dr K. Omiljanowski for a good suggestion in solving the pseudo-arc case.

Let P be a pseudo-arc. First, we will show that each proper subcontinuum of P is in pointlike position in P. We will use the following two results of Lewis.

Stable Homeomorphism Theorem (Lewis [8]). *For any open U in P and any two points $x, y \in U$ such that the irreducible continuum between x and y lies in U, there is a homeomorphism $h: P \to P$ such that $h(x) = y$ and $h(p) = p$ for $p \in P - U$.*

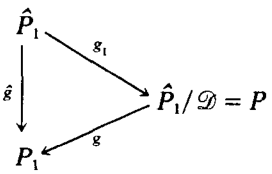
Lifting Homeomorphism Theorem (Lewis [9]). *P admits a continuous decomposition $\mathscr{D}$ into pseudo-arcs such that each homeomorphism $h: P/\mathscr{D} \to P/\mathscr{D}$ can be lifted to a homeomorphism $\hat{h}: P \to P$.*

We will also use the following lemma, perhaps already known and easy to prove with the help of chain coverings. We omit the proof.

2.11. Lemma. *Let U_1 and U_2 be neighborhoods of a subcontinuum P_0 of P such that $\text{int}(P - U_1) \neq \emptyset$. Then there is a homeomorphism $h: P \to P$ such that $h(P_0) = P_0$ and $h(U_1) \subset U_2$.*

2.12. Lemma. *For every $\varepsilon > 0$ and every continuum $P_0 \subsetneq P$ there is a homeomorphism $h: P \to P$ such that $h(x) = x$ for $x \in P - B(P_0, \varepsilon)$ and* diam $h(P_0) < \varepsilon$.

Proof. Consider the following commutative diagram



where $\hat{g}: \hat{P}_1 \to P_1$ is a quotient map guaranteed by the Lifting Homeomorphism Theorem for a pseudo-arc $\hat{P}_1$, and $\mathscr{D}$ is the decomposition of $\hat{P}_1$ into the elements $\hat{g}^{-1}(x)$ for x belonging to a closed set $F \subsetneq P_1$ with int $F \neq \emptyset$, and into singletons in $\hat{P}_1 - \hat{g}^{-1}(F)$. Since any monotone nondegenerate image of a pseudo-arc is a pseudo-arc both P_1 and $\hat{P}_1/\mathscr{D}$ are pseudo-arcs, and we can assume that

(i) $\hat{P}_1/\mathscr{D} = P$,

(ii) $P_0 = g_1\hat{g}^{-1}(x_0) = g^{-1}(x_0)$ for some $x_0 \in P_1 - F$ (all proper nondegenerate subcontinua of P are equivalent—see Lehner's result [6]),

(iii) $g^{-1}(P_1 - F) \subset B(P_0, \varepsilon)$ (by Lemma 2.11).

Fix an element $x_1 \in P_1 - F$ close enough to $\mathrm{bd}(P_1 - F)$, such that diam $g^{-1}(x_1) < \varepsilon$ and the irreducible continuum between x_0 and x_1 is contained in $P_1 - F$. Then there is a homeomorphism $h_1: P_1 \to P_1$ such that $h_1(x_0) = x_1$ and $h(x) = x$ for $x \in F$ by the Stable Homeomorphism Theorem. This homeomorphism can be lifted to a homeomorphism $\hat{h}_1: \hat{P}_1 \to \hat{P}_1$ by the Lifting Homeomorphism Theorem. Observe that the function $h: P \to P$, $h(y) = g_1 \hat{h}_1 g_1^{-1}(y)$, is a well-defined homeomorphism which satisfies our requirements. □

2.13. Theorem. *Each proper subcontinuum of P is in pointlike position in P.*

Proof. Let U be any neighborhood of a proper subcontinuum P_0 of P. Put $\varepsilon_0 = \frac{1}{2}d(P_0, P - U)$ and $\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n$. Take a homeomorphism $h_0: P \to P$ guaranteed by Lemma 2.12 for $\varepsilon = \varepsilon_0$, and put $P_1 = h_1(P_0)$. If $h_0, \ldots, h_n, P_0, \ldots, P_{n+1}$ are already defined, take a homeomorphism $h_{n+1}: P \to P$ guaranteed by Lemma 2.12 for $\varepsilon = \varepsilon_{n+1}$, $P_0 = P_{n+1}$ and put $P_{n+2} = h_{n+1}(P_{n+1})$. Observe that the map $h = \lim_n h_n \ldots h_0$ induces a homeomorphism $h^*: P/P_0 \to P$, defined by $h^*(y) = hg_{P_0}^{-1}(y)$, which can be that required by the definition of the pointlike position of P_0 in P. □

Now, taking any continuous decomposition of P into pseudo-arcs, and applying Theorems 2.1 and 2.13, we see that the sets $o(P, P)$ and $\mathrm{mo}(P, P)$ are not G_δ subsets of $c(P, P)$. Combining this with Corollary 1.3 the main result concerning pseudo-arcs is obtained.

2.14. Theorem. *Let X be a compactum. If any of the sets $o(X, P)$ and $\mathrm{mo}(X, P)$ is nonempty, then it is not a G_δ set in $c(X, P)$.*

We end this section with another proof of the result of Hohti [3] concerning the Cantor set C.

2.15. Proposition (Hohti). *The set $o(C, C)$ is not a G_δ set in $c(C, C)$.*

Indeed, observe that the projection $p: C \times C \to C$ has fibers $\{x\} \times C$ in pointlike position in $C \times C$. Since $C \times C$ is homeomorphic to C, the result is obtained by Theorem 2.1.

3. Open mappings onto locally connected continua and some positive results

We begin this section with the following observation. Let X be any compactum and Y be a locally connected continuum. It is an obvious consequence of a result of Lelek and Read [7, Theorem 5.1] that a map $f: X \to Y$ is light open if and only if it is light confluent. On the other hand, all light confluent maps form a G_δ subset of $c(X, Y)$ (see the beginning of the second section).

3.1. Observation. *For any compactum X and any locally connected continuum Y the set* $\mathrm{lo}(X, Y)$ *is a* G_δ *subset of* $c(X, Y)$.

3.2. Remark. The assumption that Y is locally connected cannot be omitted. In fact, in [3] it is observed (the observation is due to W.J. Charatonik) that for the Cantor set C and the space $\mathrm{cone}(C) = C \times I / C \times \{1\}$ the set $o(\mathrm{cone}(C), \mathrm{cone}(C))$ is not a G_δ set in $c(\mathrm{cone}(C), \mathrm{cone}(C))$. However, the maps used in the argument were light, and thus it actually follows that $\mathrm{lo}(\mathrm{cone}(C), \mathrm{cone}(C))$ is not topologically complete.

In the remaining part of this section we will investigate the class $\mathcal{A}$ of all continua Y satisfying

($\mathcal{A}$) Y is locally connected and for any locally connected continuum X the set $o(X, Y)$ is topologically complete.

One of McAuley's results from [12] may be formulated as follows.

McAuley's theorem. *The unit interval I belongs to $\mathcal{A}$.*

The next proposition is a consequence of the Decomposition Theorem and Corollary 1.3. Similarly as in the previous section, M denotes the Menger universal curve.

3.3. Proposition. *A locally connected continuum Y belongs to $\mathcal{A}$ if and only if the set* $o(M, Y)$ *is topologically complete.*

Indeed, assume there is a locally connected continuum X such that $o(X, Y)$ is not a G_δ set in $c(X, Y)$. The set $o(M, X)$ is nonempty by the Decomposition Theorem, and thus $o(M, Y)$ is not a G_δ set in $c(M, Y)$ by Corollary 1.3. The converse implication is trivial.

Theorem 2.9 implies the following evaluation of the class $\mathcal{A}$.

3.4. Theorem. *If a continuum Y belongs to $\mathcal{A}$, then almost all points of Y locally separate Y.*

Now we modify the argument for McAuley's theorem to prove the following lemma.

3.5. Lemma. *Let X and Y be locally connected continua and an arc $ab \subset Y$ be free in Y. Then the set G of all surjections $f: X \to Y$ such that the map $f|f^{-1}(ab): f^{-1}(ab) \to ab$ is open, is a G_δ set in $c(X, Y)$.*

Proof. Let $\{a_n\}$, $\{b_n\}$ be sequences in $ab-\{a,b\}$ such that $\lim a_n = a$ and $\lim b_n = b$. Denote by $c_0(X,Y)$ the set of all surjections in $c(X,Y)$, and put

$$A_{mn} = \{f \in c_0(X,Y)\text{: there is a point } x \in X \text{ such that } x' = f(x) \in a_n b \text{ and } f(B(x,1/m)) \cap (ax' - \{a, x'\}) = \emptyset\},$$

$$B_{mn} = \{f \in c_0(X,Y)\text{: there is a point } x \in X \text{ such that } x' = f(x) \in ab_n \text{ and } f(B(x,1/m)) \cap (x'b - \{x', b\}) = \emptyset\}.$$

Observe that the sets A_{mn}, B_{mn} are closed in $c_0(X,Y)$. Indeed, let $f_k \in A_{mn}$ with some $x_k \in X$ fulfilling $x'_k = f_k(x_k) \in a_n b$ and $f_k(B(x_k, 1/m)) \cap (ax'_k - \{a, x'_k\}) = \emptyset$, and $f = \lim f_k$. We can assume that $\lim x_k = x_0$, $\lim x'_k = x'_0$ for some $x_0 \in X$ and $x'_0 \in a_n b$. Since the set $ax'_0 - \{a, x'_0\}$ is open in Y, we have $f(B(x_0, 1/m)) \cap (ax'_0 - \{a, x'_0\}) = \emptyset$. This implies that $f \in A_{mn}$. The proof for B_{mn} is similar.

Put $A = \bigcup_{m,n=1}^{\infty} (A_{mn} \cup B_{mn})$. Since $c_0(X,Y)$ is closed in $c(X,Y)$ it suffices to show that $G = c_0(X,Y) - A$. The inclusion $G \subset c_0(X,Y) - A$ is obvious. Let $g \in c_0(X,Y) - A$, U be open in $g^{-1}(ab)$ and $x \in U$. Put $x' = g(x)$. Since $g \notin A_{mn} \cup B_{mn}$ for any m and n, there are sequences $\{p_k\}$, $\{q_k\}$ both converging to x in X such that the arcs $p'_k q'_k$, where $p'_k = g(p_k)$, $q'_k = g(q_k)$, are closed neighborhoods of x' in ab. By the local connectedness of X there are continua L_k in X containing x, p_k and q_k with $\lim \operatorname{diam} L_k = 0$. Take an open set W in X such that $U = g^{-1}(ab) \cap W$ and take L_k contained in W. Then a continuum irreducible either between p_k and $g^{-1}(q'_k) \cap L_k$ or between q_k and $g^{-1}(p'_k) \cap L_k$ contained in L_k, lies in U. Hence $x' \in \operatorname{int} g(U)$. Therefore the map $g|g^{-1}(ab): g^{-1}(ab) \to ab$ is open, and thus $g \in G$. The proof is complete. □

An easy proof of the next lemma we leave to the reader.

3.6. Lemma. *Let a graph Y be the finite union of its free arcs $a_1b_1, \ldots, a_nb_n$. Then for any compactum X and any surjection $f: X \to Y$ the mapping f is open if and only if the maps $f|f^{-1}(a_kb_k): f^{-1}(a_kb_k) \to a_kb_k$ are open for $k = 1, \ldots, n$.*

We obtain the following extension of McAuley's theorem by Lemmas 3.5 and 3.6.

3.7. Theorem. *Any graph belongs to $\mathcal{A}$.*

Theorems 3.4 and 3.7 form a partial solution of the following problem, which ends the paper.

Problem. Find an internal characterization of all continua in $\mathcal{A}$.

References

[1] R.D. Anderson, One-dimensional continuous curves and a homogeneity theorem, Ann. of Math. 68 (1958) 1-16.

[2] J.J. Charatonik and T. Maćkowiak, Around Effros' theorem, Trans. Amer. Math. Soc. 298 (1986) 579-602.
[3] A. Hohti, On the complete metrizability of the spaces of open mappings, Preprint.
[4] K. Kuratowski, Topology II (Academic Press, New York; and PWN, Warsaw, 1968).
[5] K. Kuratowski, On the completeness of the space of monotone mappings and some related problems, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. 16 (1968) 283-285.
[6] G.R. Lehner, Extending homeomorphism of the pseudo-arc, Trans. Amer. Math. Soc. 98 (1961) 369-394.
[7] A. Lelek and D.R. Read, Compositions of confluent mappings and some other classes of functions, Colloq. Math. 29 (1974) 101-112.
[8] W. Lewis, Stable homeomorphisms of the pseudo-arc, Canad. J. Math. 31 (1979) 363-374.
[9] W. Lewis, Continuous curves of pseudo-arcs, Houston J. Math. 11 (1985) 91-99.
[10] W. Lewis and J.J. Walsh, A continuous decomposition of the plane into pseudo-arcs, Houston J. Math. 4 (1978) 209-222.
[11] J.C. Mayer, L.G. Oversteegen and E.D. Tymchatyn, The Menger curve characterization and extension of homeomorphisms of non-locally-separating closed subsets, Dissertationes Math. 252 (1986) 1-45.
[12] L.F. McAuley, Spaces of certain non-alternating mappings, Duke Math. J. 38 (1971) 43-50.
[13] S.B. Nadler Jr, Concerning completeness of the space of confluent mappings, Houston J. Math. 2 (1976) 561-580.
[14] D.C. Wilson, Open mappings of the universal curve onto continuous curves, Trans. Amer. Math. Soc. 168 (1972) 497-515.