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Leonhard Euler: The First St. Petersburg Years (1727–1741)

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After reconstructing his tutorial with Johann Bernoulli, this article principally investigates the personality and work of Leonhard Euler during his first St. Petersburg years. It explores the groundwork for his fecund research program in number theory, mechanics, and infinitary analysis as well as his contributions to music theory, cartography, and naval science. This article disputes Condorcet's thesis that Euler virtually ignored practice for theory. It next probes his thorough response to Newtonian mechanics and his preliminary opposition to Newtonian optics and Leibniz–Wolffian philosophy. Its closing section details his negotiations with Frederick II to move to Berlin.

Après avoir reconstruit ses cours individuels avec Johann Bernoulli, cet article traite essentiellement du personnage et de l'oeuvre de Leonhard Euler pendant ses premières années à St. Pétersbourg. Il explore les travaux de base de son programme de recherche sur la théorie des nombres, l'analyse infinie, et la mécanique, ainsi que les résultats de la musique, la cartographie, et la science navale. Cet article attaque la thèse de Condorcet dont Euler ignorait virtuellement la pratique en faveur de la théorie. Cette analyse montre ses recherches approfondies sur la mécanique newtonienne et son opposition préliminaire à la théorie newtonienne de l'optique et a la philosophie Leibniz–Wolffienne. Dans la section finale, l'article traite en détail les rélations d'Euler avec Frédéric II lors de son séjour à Berlin.

Nach der Beschreibung seines Tutorials mit Johann Bernoulli untersucht dieser Artikel hauptsächlich die Persönlichkeit und das Werk Leonhard Eulers während seiner ersten Petersburger Jahre. Diese stellt eine grundlegenden Phase für Eulers Forschungen in Zahlentheorie, Mechanik, und Infinitesimalrechnung dar, sowie auch für seine Beiträge zur Musiktheorie, Kartographie und Nautik. Dieser Artikel kritisiert Condorcets These, wonach Euler die Praxis gegenüber der Theorie weitgehend vernachlässigte. Eulers eigehende Untersuchungen zur Newtonschen Mechanik wie auch seine deutliche Opposition zur Newtonschen Optik und zur Leibniz-Wolffschen Philosophie werden auch thematisiert. Das abschließende Kapitel berichtet über Eulers Verhandlungen mit Friedrich II über seinen Umzug nach Berlin.

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I. INTRODUCTION

During most of the mature European Enlightenment, Leonhard Euler (1707–1783) dominated all branches of the mathematical sciences. This Swiss-born genius especially led in the extraordinary development of calculus, or analysis of the infinite, and its highly successful application to mechanics and astronomy—the chief achievements of 18th-century science. Euler, for example, systematized the

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fundamental concepts for calculus in his *Introductio in analysin infinitorum* (2 vols., 1748), basing it on elementary functions instead of geometric curves, and he largely created its branches of differential equations, calculus of variations, and differential geometry. Beginning with his first landmark work, *Mechanica* (2 vols., 1736), a diffident Euler, not Isaac Newton, principally formulated the differential equations underlying rational mechanics before those of William Rowan Hamilton. His parallel contributions to hydraulics, hydrodynamics, ship theory, elasticity, and especially the mechanics of rigid bodies founded continuum mechanics. Notwithstanding their bulk, Euler's 873 memoirs and books are models of clarity and simplicity that contain, among other things, two lunar theories, built on his improvement of the differential equations for the three bodies problem, and an influential pulse theory of light. Besides this prodigious *oeuvre*, Euler's administrative talents brought to the new Imperial Russian Academy of Sciences in St. Petersburg and the Royal Academy of Sciences in Berlin the status of respected European research centers.

As publication of the 74 volumes in the first three series of Euler's *Opera omnia* nears completion, and the projected 15 volumes of his massive correspondence and notebook in series four are well under way, scholars are fleshing out his life and reevaluating his work.[1] This is providing two indispensable foundations for writing his first full-length biography and thereby filling a serious gap in the history of mathematics and 18th-century European intellectual history. The shorter biographies of Gustav du Pasquier [85], Otto Spiess [139], and Rudolf Fueter [92] are now dated, and Rüdiger Thiele's insights [144] merit fuller treatment. In particular, Euler's correspondence, which included over 2250 letters unpublished until the 1960s [145, 313], is adding fresh information and allowing for greater precision. The virtual impossibility of a single scholar's mastering completely the range, depth, and quantity of Euler's scientific achievements makes the best of Euler scholarship essential to an acute analysis. Among its leading contributors during the past 30 years are Emil Fellmann on biography, early memoirs, and optics, Walter Habicht on astronomy and mechanics, Clifford Truesdell on calculus and rational mechanics, and Adolf Youschkevitch on correspondence and pure mathematics.[2]

A detailed, coherent, rounded biography of Euler also requires knowledge of his cultural setting. Today scholars differ over how far the mathematical sciences are to be perceived as abstract, universal, independent of time and setting, and to what extent they are comprehensible as products of culture and society [125; 140]. These themes and tensions between analyses based on them are crucial to historical research. This paper assumes that a full understanding of Euler's mathematics, including his motivations, his aesthetic sensibility, and the succession of problems he faced, depends partly on knowing his interactions with the St. Petersburg and Berlin Academies and his correspondents, his encounters with reform absolutism, and his participation in Enlightenment literary and religious discourses.[3] Since studies of many of his correspondents are in an early phase and Euler scholarship

[1] On the *Opera omnia,* see the introductory remarks in the References.

[2] See [14–17; 25–27; 54–55; 70; 87; 145; 152].

[3] See [25–27; 73–74; 80; 95; 101; 106; 137; 141].

and research on Enlightenment discourses are continuing, the proposed biography cannot claim completeness.

Drawing on his *Opera omnia* and the new findings,[4] this paper will primarily examine Euler's first St. Petersburg years from 1727 to 1741, when his career began. The second section probes the Bernoulli tutorial, the early academy, and his personality; the third and fourth show him setting the groundwork for his ambitious research program in number theory, infinitary analysis, and rational mechanics. The third section opens with his exact summing in 1735 of the longstanding Basel problem, $\zeta(2)$, and proceeds to his proof that $2^{2^n} + 1$ is composite for $n = 5$, his massive computations of logarithms, e, and π, and his recasting of trigonometric functions as numerical ratios. These are discussed along with his discovery of early beta and gamma functions, his pioneering of a semigeometric calculus of variations, and the introduction into mechanics of a uniform analytic method in his *Mechanica*. The fourth section explores his music theory and state projects in cartography and naval science. After finding missing solutions, often by analogy, Euler returned periodically to perfect methods, make exhaustive computations, and systematize fields. His research signature is a patient, tenacious search in stages for greater precision, completeness, and taxonomic order. The fifth section elucidates his thorough study of the operation of Newton's inverse-square law and preliminary rejection of Newtonian optics and Leibnizian–Wolffian monads. The sixth examines the cause and nature of his eyesight deterioration from 1735 and first details his negotiations to move to Berlin in 1741. This paper disputes the conventional view that Goldbach chiefly instigated Euler's work in number theory and the myth enduring since Condorcet's *Éloge* [78] that he virtually ignored practice for theory.

II. BIOGRAPHICAL INFORMATION: FROM BASEL TO ST. PETERSBURG

In a private tutorial in 1725, the irascible Johann Bernoulli (1667–1748) of the University of Basel had discovered young Euler's mathematical genius. Under Bernoulli's guidance, Euler read classics and leading books and articles of his time in the mathematical sciences. No list of the required readings has survived, but a plausible one can be reconstructed. During the week Euler may have read the *Projet* (1687) and *Nouvelles conjectures sur la pesanteur* (1690) of Pierre Varignon, a recent correspondent of Johann Bernoulli.[5] He also probably reread authors he had consulted while preparing his master's thesis in 1724 that compared the natural philosophy of Descartes with that of Newton. These included Galileo Galilei's *Dialogo ...* (*Dialogue on the Two Chief World Systems, Ptolemaic and Copernican*, 1632), René Descartes's *Principia philosophia* (1644) and *La géométrie* in the fourth of Franz van Schooten's Latin editions (*Geometria*, 1695), and Isaac Newton's *Principia mathematica* (1687) and *Opticks* (1704). The footnotes to his first articles

[4] See, for example, [70; 75; 76; 89; 90; 99; 104; 110; 111; 119; 145; 148–150; 152].

[5] Perhaps Varignon's *Éclaircissemens sur l'analyse des infiniment petits* (publ. posth., 1725) was also available.

and mathematical scholarship in Basel suggest that Euler also examined Jakob Bernoulli's articles on the theory of infinite series (1682–1704, repr. 1713) [50] and *Ars conjectandi* (publ. posth., 1713), Jakob Hermann's *Phoronomia, sive de viribus et motibus corporum solidorum et fluidorum* (1716), Brook Taylor's *Methodus incrementorum directa et inversa* (1715), and John Wallis's *Arithmetica infinitorum* (1656).[6] After Euler completed his weekly readings, Bernoulli for an hour on Saturday afternoons would explain "everything ... [his student] could not understand" [152, 468].

In order not to bother his teacher unnecessarily, Euler managed to reduce to a very few his questions about concepts, methods of solution, and problems. His burgeoning brilliance benefitted immensely from this tutorial. The readings introduced challenging and some invincible problems but did not exhaust him, for they kept him from pursuing unpromising routes in the search for solutions. At the same time, he had to devote great energy attempting to solve the problems in his readings.

The tutorial work and the freedom of the study were exhilarating, and Euler soon decided to become a mathematician and theoretical physicist rather than a rural Evangelical Reformed pastor. With Bernoulli's backing, he obtained his father Paul's assent. Johann Bernoulli and Paul Euler knew each other well, having both boarded at the home of Jakob Bernoulli as students at the University of Basel. In 1726 Leonhard was completing graduate studies there, writing papers on the masting of ships and on algebraic reciprocal trajectories, apparently his first two articles [17, 1–36; 13]. Although he lacked maritime experience, Leonhard won the *accessit* or second place in the Paris Academy prize competition the next year for his paper on ship masting, losing only to Pierre Bouguer, who was becoming France's leading nautical authority. The *accessit* paper reflects a powerful intuition into physics.

In the autumn of 1726, Euler was offered a position in the newly founded Imperial Russian Academy of Sciences in St. Petersburg. The St. Petersburg Academy was the capstone for Peter the Great's program to improve education in Russia and to bring scientific inquiry there from the West. His view of the sciences was utilitarian; they were instruments to advance navigation, cartography, commerce, and manufactures. Working with his aide, the Scot Jacob Bruce, and more closely with the Saxon scholar Gottfried Leibniz, the tsar had broadly conceived of science (*nauka*) as embracing mathematics, natural and social sciences, medicine, history, and philosophy. Disapproving of the independence that the Royal Society of London enjoyed from governmental control, Peter planned an academy based on the Paris and Berlin models with state financing and state projects. It was also to administer a small associated gymnasium and university. Consulting with Bruce, who stressed mathematics, and Leibniz's disciple Christian Wolff, Peter and from 1725 his widow

[6] Besides works by Euclid and Michael Stifel, Euler read books by Claude-Gaspar Bachet de Méziriac, Honoré Fabri, François Viète, and Christian Wolff, along with papers by Joseph Sauveur and by John Flamsteed and William Derham in the *Philosophical Transactions* of the Royal Society of London, perhaps as part of this tutorial.

and successor, Catherine I, had recruited thirteen Germans, two Swiss, and one French scholar to be the members.[7]

Euler's appointment at the St. Petersburg Academy was not unexpected. When Johann Bernoulli's sons, Nicholas II and Daniel, accepted positions there in 1725, they had promised to recommend their close friend Euler for its first vacancy.[8] That vacancy arose when Nicholas II died the next summer from appendicitis. Daniel assumed Nicholas's place in the academy's mathematical–physical division and recommended that Euler fill the resulting vacancy in physiology. Writing on behalf of academy president Lavrentii Blumentrost, Bernoulli offered Euler a modest two hundred ruble pension, free lodging, heat, and light, and 130 rubles for travel expenses [53, 2: 409–410]. Adding 100 rubles from his travel funds was to bring Euler's annual pension to 300 rubles. He responded to the offer with alacrity.

Euler's letter of acceptance in November 1726 stated that he could not travel to Russia until the weather cleared in the spring. Two reasons prompted this delay. As Daniel Bernoulli had recommended in his letter, Euler wanted to study anatomy and physiology based on geometrical principles in Basel in preparation for his new post. More importantly, Euler was a candidate for the vacant chair of physics at his *alma mater*. For this competition, he wrote an essay on acoustics, entitled *Dissertatio physica de sono*, that became a classic [21, 181–197; 15, XXIV–XXIX].[9] He was nevertheless rejected, primarily because at 19 he was too young [76, 84–87]. Benedict Staehelin, an established scholar from a patrician family in Basel and an alumnus of Basel and Paris, was selected instead. This decision ultimately benefitted Euler, because it forced him to move from a small republic into a setting more adequate for his brilliant research and technological work. A few days after Staehelin's appointment on April 5, 1727, he left Basel never to return, although he kept his Swiss citizenship lifelong.

Following a seven-week journey by boat down the Rhine, on foot, and by post wagon across German states, and by ship from Lübeck, Euler arrived in the Russian capital of St. Petersburg on May 17, 1727. Along the way he had stopped in Marburg to meet the heroic rationalist philosopher Christian Wolff, an exile from Halle on charges of unorthodoxy from conservative Pietist theologians. Wolff had recommended most of the academy's original members. In St. Petersburg, Euler was to be at the focal point of Russia's export trade with

[7] In 1726 Lavrentii Blumentrost, the son of a German physician in Russian service, was chosen the first president of this European, not Russian, institution. A member of the influential German colony in Russia, Blumentrost had helped plan the academy and spoke impeccable Russian.

[8] The Bernoulli family lineage in the sciences begins with the brothers Jakob I (1654–1705) and Johann I (1667–1748) and continues mainly with Johann's three sons—Nicholas II (1695–1726), Daniel (1700–1782), and Johann II (1710–1790). As a university student, Euler was a classmate and close friend of Johann II, who likely introduced Euler to his older brothers and supported him for the elder Bernoulli's tutorial. Joachim Otto Fleckenstein has written brief biographies of these Bernoullis [88]. Other biographical sketches appear in the two-volume *Der Briefwechsel von Johann Bernoulli,* Basel: Birkhäuser, 1955, I: 12–15.

[9] *De sono* is often called his doctoral dissertation.

the West and center of a growing navy and Russia's Enlightenment [131]. Its population was to increase during the next three decades to about 150,000, or roughly 10 times the size of Basel.

The initial St. Petersburg Academy was especially attractive to young scholars. It had ample funds and was building strong library holdings, starting with the private library of Peter I, who had many works on geography and military science. It also acquired the library of Peter's son Alexis and private ones from the nobility. Having few students enrolled at the associated gymnasium and university, the academy emphasized and offered considerable freedom for research. Members were, in fact, required to publish so as to enhance the prestige of the academy. In 1728, the academy was to acquire a press from Holland and commence publishing its proceedings, *Commentarii academiae scientiarum imperialis Petropolitanae.* [10] Since half its original members were under 30, the institution was not encumbered by the kind of hierarchies that dominated the Paris Academy and the Royal Society of London.

After its benefactress Catherine I died a week before Euler's arrival,[11] however, the St. Petersburg Academy began to experience financial and political difficulties. Conservative Russian nobles, who controlled the 12-year-old Tsar Peter II and exiled Catherine's chief minister, were hostile to the academy, seeing it as an intrusion into Russian culture by German, Swiss, and French aliens. Unable to grasp the importance of scientific inquiry, those nobles who were governmental ministers delayed payments to the academy, froze funds for unfilled positions, and fomented antagonisms against its foreign members. The nobility also blocked the entrance of students into its associated gymnasium and university, which did not confine themselves to the upper classes [147, 79–80]. Most gymnasium students came from foreign families residing in Russia and only a few from Russian aristocratic families, so the lack of knowledge of Russian among the academicians on the faculty was not initially a problem. Opposition from noble ministers, the emphasis on the sciences over the humanities, and a haphazard curriculum responding to individual faculty interests dampened enrollments at the gymnasium, which fell from 112 at its opening in 1727 to 74 after just two years [147, 78].[12] The Swiss mathematician Jakob Hermann was instructed to write a text on elementary arithmetic for the boy tsar Peter II, and the French astronomer Joseph Delisle had to prepare an introduction to astronomy.

These were the prevailing circumstances when Euler arrived in St. Petersburg as an adjunct (initially called "student"), not in the academy's medical branch but, in accordance with his enthusiasm for mathematical science, in its mathematical–

[10] In 1728 the Academy gained jurisdiction over the *Kunstkamera*, a museum with natural history and ethnographic specimens, and Peter's instruments used in studying physics as a core for its Museum Physicum.

[11] Catherine I should not be confused with Catherine II, the Great, who came to the throne in 1762 and was Euler's later benefactress.

[12] The situation in the academic university was worse: only eight students imported from Vienna and poor prospects that Russian schools could offer instruction sufficient to prepare students for admission.

physical division. Apparently, Daniel Bernoulli and Jakob Hermann helped arrange this. But Euler's annual salary of three hundred rubles was uncertain. On the basis of his Paris Academy paper on the masting of ships and his medical training, the Russian navy offered him a position as a medic, which he accepted. He lodged at the home of Daniel Bernoulli, who had asked him to bring from Switzerland fifteen pounds of coffee, one pound of the best green tea, six bottles of brandy, twelve dozen fine tobacco pipes, and a few dozen packs of playing cards [144, 32]. With his talent for language, Euler quickly mastered Russian and assimilated smoothly into local society.

When the old nobility returned the imperial court to Moscow from St. Petersburg six months later in 1728 in an unsuccessful effort to undo the modernizing reforms of Peter the Great, the academy suffered more tribulations. President Blumentrost accompanied the boy tsar as physician, and conference secretary Christian Goldbach went with him as tutor. Euler had become a close friend of Goldbach, and their correspondence focusing on number theory and calculus soon began. With the departure of Blumentrost, the academy was left under the control of Johann Schumacher, a despotic Alsatian bureaucrat who had purchased scientific equipment and books for Peter the Great [147, 81]. Schumacher adopted the principle of divide and rule.

In the censors of the Russian Orthodox church, the academy had still other antagonists. Opposed to the new sciences, they prohibited publication of articles and books supporting Copernican astronomy. This was not a minor issue at the academy. In 1728 Joseph Delisle and Daniel Bernoulli had taken the lead in attempting to propagate the heliocentric theory. Notably, in 1730 the Russian *philosophe* Antiokh Cantemir (1708–1744), who had studied at the Academic University in 1726/27, completed his Russian translation of Bernard Fontenelle's *Entretiens sur la pluralité des mondes* (*Conversations on the Plurality of Worlds,* 1686). The *Entretiens* in its first evening's presentation discusses Copernican astronomy as being most likely true; in its second and third, it presents findings of Galileo and Kepler; and in its fourth and fifth, it presents Cartesian vortex cosmology [52; 82, 123–129]. In Fontenelle's nongeocentric universe the human race is displaced from its central position. Church censors viewed its Cartesian mechanism as a desacralization of the Earth. Throughout the 1730s, they suppressed publication of Prince Cantemir's translation by the academy press.[13] Euler supported publishing this translation,[14] but to no avail. When it finally appeared in print in 1740, Cantemir insisted that the title page give 1730, the date when his manuscript had actually been completed.

[13] In 1740 Cantemir no longer accepted Fontenelle's Cartesian views. He had become a Newtonian and preferred to get printed his Russian translation of Francesco Algarotti's *Il Neutonianismo per la dame* (1739), but this was never done. See [68, 125–127].

[14] In his *First Satire* (1729), Cantemir had attacked the idea of a hereditary aristocracy and religious opposition to Copernican astronomy. He served as ambassador to England from 1731 to 1738 and to France from 1738 to his death in 1744. In both places, Cantemir acted as a later Mersenne, transmitting scientific knowledge between eastern and western Europe, especially between Euler and his French colleagues.

Angered at the intolerance of censors to the new sciences, two senior academicians left Russia in 1730, when their five-year terms expired. German physicist Georg Bilfinger returned to Tuebingen, while Jakob Hermann, who was a second cousin of Euler's mother, accepted the chair of ethics at the University of Basel, a post he had been offered three years earlier [53, 2: 4].

At the beginning of the 1730s, Euler moved to full-time involvement in academy affairs and rose quickly through its ranks. The academy's relations with the monarchy improved in 1730 with the death of Peter II and the accession of Empress Anna Ivanovna, who surrounded herself with German officials and two years later returned the capital to St. Petersburg. In 1731, when the academy selected him at the age of 23 to succeed Bilfinger as professor of physics and thus become a full member of the institution, Euler left the Russian navy, declining a promotion to lieutenant [47, 1: 17]. Daniel Bernoulli succeeded Hermann in the more prestigious position of premier professor of mathematics. Outside the academy, Euler offered public lectures on logic and mathematics and wrote popular scientific articles for the *St. Petersburg Gazette* supplement [147, 95].

About this time, a combative facet of Euler's personality began to emerge at the academy. In 1731 the genial Euler, known lifelong for a calm disposition and self control, became angry at Blumentrost and Schumacher [145, 304]. The 23-year-old Euler was particularly upset that four others who were receiving lesser salaries—chemist Johann Georg Gmelin, physicist Georg Wolfgang Krafft, historian Gerhard Friedrich Mueller, and physiologist Josias Weitbrecht—had been proposed for professorships with a salary equal to his 400 rubles, while his salary was to be frozen for two years. On January 23, 1731 he wrote sharply to Schumacher:

> It seems to me that it is very disgraceful for me, that I, who up to now have had more salary than the others, shall now be set equal to them. . . . I think that the number of those who have carried [mathematics] as far as I is pretty small in the whole of Europe, and none of them will come for 1000 rubles. [54]

For the author of only seven articles, these were bold words. Schumacher advised Blumentrost not to grant any concession to Euler: otherwise he might grow impudent. By keeping the salary of all five at 400 rubles, instead of giving Euler the 500 he had earlier requested for himself, the bureaucrat Schumacher won this round. But Euler became a tougher salary negotiator, successfully arguing for raises in 1735 and again in 1741, by which time he had acquired a distinguished reputation throughout Europe. Meanwhile, in the autumn of 1732 when Johann II Bernoulli arrived in St. Petersburg to visit his brother and friend, Euler recommended that he be appointed to the academy. Apparently Johann II hoped to gain this position, but Schumacher blocked the appointment, fearing any increase in Swiss representation.

Tired of the censorship, the hostility to Germans, and Schumacher's intrigues and failure to pay his salary in accordance with his contract, Daniel Bernoulli returned to Basel in 1733 as professor of botany and anatomy—a move he had been considering for two years. His brother accompanied him home. When he went back to Basel after a visit to Paris, Bernoulli felt revitalized in the free Swiss air, but his theoretical achievement never regained the level it had reached in his last six years in St. Petersburg closely interacting with Euler. Euler succeeded him as premier mathematician at the academy. His salary was raised to 600 rubles supplemented by a 60 ruble allowance for lodging, wood, and light. From Peter the Great, Jakob Hermann had received a salary of 2000 rubles.

With Daniel Bernoulli's departure, Euler lost more than a friend. Euler had lived at his house for six years. The two dined together often and collaborated on problems in the evening. For the title page of his *Hydrodynamica, sive de viribus et motibus fluidorum commentarii* (1738) [49], Bernoulli wrote that he had completed its first draft before he left St. Petersburg; surely that was with Euler's assistance. After he returned to Basel, the two corresponded about making it ready for publication [53, 2: 411–423, 443–453].

Financial security resulting from his becoming premier mathematician at the academy allowed Euler to contemplate marriage in 1733. He courted Katharina Gsell, a member of the local Swiss colony and the daughter of a painter who taught at the academic gymnasium. The father, Georg Gsell, was the creator of a decorative baroque art. Leonhard and Katharina were married on January 7, 1734 (N.S.). The academy had a poet named Gottlob Juncker write pieces to mark special occasions. At one point, his lengthy poem celebrating the wedding asks in German:

Who would have thought it,
That our Euler should be in love?
Day and night he thought constantly,
How he wanted more to calculate numbers,
His profound learned sense was free. [139, 64–65]

The marriage seems to have been happy. The young couple purchased a comfortable house on the banks of the Neva not far from the academy [148, 164]. They lived quietly there. Their first child, Johann Albrecht, was born on November 27, 1734 (N.S.). His namesake, Johann Albrecht Korff (1697–1766), presided over the academy from 1734 to 1740, and his godfather, Christian Goldbach, was its secretary. As long as these friends were both at the academy, Euler's position was most secure. Johann Albrecht was the first of thirteen children, but only three sons and two daughters survived early childhood.[15]

III. RESEARCH IN PURE MATHEMATICS

From his arrival in St. Petersburg, Euler immersed himself in research with unfailing enthusiasm and, despite hostility to the academy from the Russian nobility and Orthodox clerics, after 1730 he carried out state projects dealing with cartography, science education, magnetism, fire engines, machines, and shipbuilding. He deeply appreciated the freedom to pursue in peace the pleasures of the academy-sheltered study. The core of his research program was now set in place: number

[15] The boys were Johann Albrecht (1734–1800), Karl (1740–1790), and Christoph (1743–1812); and the daughters were Helen (1741–1781) and Charlotte (1746–1781).

theory; infinitary analysis including its emerging branches, differential equations and the calculus of variations; and rational mechanics. He viewed these three fields as intimately interconnected. Studies of number theory were vital to the foundations of calculus, and special functions and differential equations were essential to rational mechanics, which supplied concrete problems. Euler's two-volume *Mechanica, sive motus scientia analytice exposita* (1736) [12] proved to be a milestone in rational mechanics,[16] and his founding of continuum mechanics began with studies in hydraulics, hydrodynamics, elasticity, and ship theory. Euler also now wrote on astronomy, optics, topology, and music. Prior to 1741, he completed nearly 90 memoirs on these topics, 55 of which were in print, and he kept extensive notebooks.

A. Number Theory

By 1729 Euler was drawn deeply into number theory or higher arithmetic, seemingly from its connections with other fields and its aesthetic appeal to the mind. By recovering and proving Pierre Fermat's conjectures, studying Diophantine equations of degree 2, and formulating more major principles for the field than had all his predecessors combined, he was to vitalize and essentially recreate number theory, preparing the way for Carl Friedrich Gauss. By progressing toward the prime number theorem and introducing the law of quadratic reciprocity without a proof—the two fundamental theorems of the field—he was to glimpse what he called its inner *Herrlichkeit* or splendor [148, 285]. Clearly, Fermat and Diophantus were not his only sources in number theory. Among others he was to draw upon were Euclid, François Viète, René Descartes, Christoph Rudolff, Franz van Schooten, Bachet de Méziriac, Frenicle de Bessy, Philippe de la Hire, Joseph Sauveur, and John Wallis from the past and Philippe Naudé and Christian Wolff from his time [2, X].

In a postscript to a letter of December 1, 1729, Goldbach asked Euler whether he knew the conjecture of Fermat, who in responding to a query of Frenicle transmitted through Marin Mersenne in 1640 surmised that all integers $2^{2^n} + 1$, for $n = 0, 1, 2, 3, 4, \ldots$, are prime. This query is generally taken as beginning Euler's work in number theory [55, 1: 10]. Although Goldbach had not read Fermat, he insisted that Euler study the Frenchman's writings [55, 1: 24]. Euler's correspondence on number theory, consisting almost solely of his letters to Goldbach from 1730 until 1756,[17] shows his early progress in the field. His subsequent letters on the subject to Lagrange, which reflect Goldbach's influence, seem to support the thesis that Goldbach had instigated his study of number theory. The letters to Lagrange written after Goldbach's death in 1764, however, are perhaps partly a tribute to his best friend. The conventionality of Goldbach's ideas in the correspondence, together with a review of Euler's articles on number theory in the St. Petersburg *Commentarii* and *Novi commentarii* and in *Nova acta eruditorum*, argues for a lesser though

[16] The full title in English is *Mechanics, or the Science of Motion Set Forth Analytically.* A review of the Staeckel volumes appears in *Centaurus* **26** (1983), 323–335.

[17] The third Silesian or Seven Years War interrupted their correspondence from 1756 to 1762.

important influence. Probably it was not the amateurish Goldbach but the Bernoullis who primarily awakened Euler's deep and subsequently unbroken interest in number theory.[18] What Goldbach did at this point, as André Weil notes [148, 169 ff.], was to keep Euler focused on the subject.

Euler's early correspondence shows that he was already examining number theory in January 1728, when Johann Bernoulli wrote him about the value of the function $y = (-1)^X$ [53, 2: 6–7]. Studying the controversy over logarithms between Leibniz and Bernoulli, which had transpired in a twelve-letter correspondence from 1712 to 1713, and operating with arithmetical and algebraic methods, Euler found that $\log(-1) = \pi i$ [9, XVII].[19] This finding refuted both Leibniz, who had argued in a letter of March 1712 that $\log(-1)$ does not exist, and Johann Bernoulli, who responded in May that $\log(-a) = \log a$ [85].

Euler was to expand his study of logarithms, finding by 1746 that $\log i = (2k + \frac{1}{2})\pi i$ and correctly resolving that the logarithms of all negative real numbers and imaginary numbers are imaginary. Since he did not yet offer proofs of these claims, his findings were not at first accepted. In a letter of January 29, 1747 d'Alembert, for example, expressed doubt that the logarithms of negative real numbers are imaginary. His letter of March 24, 1747 to Euler disagreed with that finding, depicting it as "pretended," and asserting on metaphysical and geometric grounds that $\log(-1) = 0$ [26, 200–202].

The other essential stimulus to Euler's work in number theory was his reading in 1729 of Pierre Fermat's *Varia opera mathematica* (1679) and perhaps John Wallis's *Commercium epistolicum* (1658) that contains a few letters from Fermat but nothing of his number theory. Before 1738, Euler also read Fermat's edition of Diophantus of Alexandria's *Arithmetica* and commentary on it. Fermat's edition, based on the Latin translation of Claude Bachet in 1621, was published posthumously by his eldest son Clément-Samuel in 1670. Prime numbers fascinated Euler. Using refinements of criteria for primes given by Fermat, he began computing them up to 10^7 and a few beyond [48].

In response to Goldbach's letter of December 1729 inquiring about Fermat's conjecture that all numbers $2^{2^n} + 1$ are prime, Euler confirmed that this is correct for $n = 0, 1, 2, 3$, and 4, but by 1732 at the latest discovered that $n = 5$ gives a counterexample [2, XII–XIII]. He factored it, $2^{32} + 1 = 4{,}294{,}967{,}297 = 641 \times 6{,}700{,}417$ [2, 3], and later showed that composite Fermat numbers possess divisors only of the form $2^{m+1} n + 1$. He first studied Christian Wolff's *Elementa matheseos universae* (1730) that discussed when Mersenne numbers, $2^n - 1$, are prime. Wolff had shown that $2^{11} - 1$, $2^{37} - 1$, and $2^{43} - 1$ are composites, while Mersenne had

[18] Euler's request of 1742 to Alexis Clairaut in Paris to begin a search for further manuscripts by Fermat—a search that proved fruitless, as he later reported to Goldbach—suggests his continuing desire to know more about the work of Fermat. See [62, 24; 55, 1: 168]. Euler's research on number theory after 1768 in cooperation with Lagrange regained lost territory covered by Fermat and went beyond it [148, 120].

[19] Euler had his concept of natural logarithms long before he published in 1744 on their base, $e = \lim_{n\to\infty} (1 + 1/n)^n = 2.71828\ldots$.

earlier demonstrated that $n = 2, 3, 5, 13, 17, 19, 31, 67, 127$, and 257 give primes. Study of Euclidean perfect numbers (*Elements* IX.36 states that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect) motivated this work. In a letter of October 15, 1743 to Goldbach, Euler discussed the form divisors must have when Fermat numbers are not prime, and he presented his finding in a paper completed in 1747/1748, entitled "Theoremata circa divisores numerorum," that appeared in the *Novi commentarii* in 1750 [2, 62–85].[20] Only afterward did he formally prove that $2^{2^5} + 1$ is not prime [48].

In 1730 Euler studied what he called Fermat's "not inelegant theorem" that every natural integer is the sum of four integral squares [55, 1: 24]. Bachet, who had stated this theorem in the commentary on his translation of Diophantus, had demonstrated it for integers up to 325 but could not prove it [103, 25]. Probably through some carelessness in his reading, Euler did not seem to realize that Fermat had claimed to have proved this result by using his method of infinite descent. Writing to Goldbach on June 4, 1730, Euler said that his inability to demonstrate that numbers of the form $n^2 + 1$ are sums of four integral squares was blocking his attempts to prove the theorem [25, 135]. A year later he rediscovered Fermat's "outstanding" little theorem: $a^{p-1} \equiv 1 (\mathrm{mod}\ p)$, where p is prime and a and p are relatively prime.[21] He believed that Fermat had stated this theorem "without proof [and obtained it] merely by induction" [148, 174]. Nor did he know that papers in the Hannover Library show that Leibniz had proved it by 1683. Like Fermat, Euler proceeded to his discovery of the theorem experimentally. He began with $2^{p-1} - 1$ and then in 1736 gave a clumsy additive proof based on the binomial expansion of $(1 + 1)^{p-1}$, the divisibility of the binomial coefficients, and an appropriate rearrangement of terms [2, 33–37]. In 1747–1748, he examined the theorem in the form "$a^{p-1} - b^{p-1}$ is divisible by p, when a and b are relatively prime and p prime" [2, 62–86]. Not until 1752 did he realize Fermat's priority for a proof using binomial expansion.

Euler, who in 1741 wrote out for Goldbach from Schooten's *Exercitationes mathematicae* (1657) all primes of the form $4n + 1$ to 3000 [25, 139], also studied Fermat's assertion that for a and b relatively prime, $a^2 + b^2$ does not have a prime divisor of the form $4n + 1$. It took until 1749 to complete his proof with the use of Fermat's little theorem. Here is a significant indication of what André Weil has demonstrated: that Euler's research in number theory was to progress from being largely empirical in his first St. Petersburg years to being increasingly conceptual in Berlin [148, 190].

Beginning with the memoir "De solutione problematum Diophanteorum per numeros integros" (1732/1733, publ. 1738) [2, 6–17], Euler pursued solutions of second degree Diophantine equations. These were important to his work in number theory as well as to his investigations on the rectification of curves. He was apparently not yet aware of the Diophantine equation known as Fermat's last theorem,

[20] See Ferdinand Rudio's comment on p. xiii.

[21] Mathematicians have named this his little theorem to distinguish it clearly from his last theorem.

which he first addressed in a 1738 memoir. It examined Frenicle de Bessy's proof for $n = 4$ and Fermat's method of infinite descent [38]. He read of these in Fermat's notes on Diophantus's *Arithmetica.*

B. The Zeta Function: Origins of Analytic Number Theory

In the book *Novae quadraturae Arithmeticae* (1650), the Bolognese mechanist Pietro Mengoli had posed what is known as the Basel problem, that of summing the reciprocals of square numbers, $\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = 1 + 1/2^2 + 1/3^2 + \ldots$. After John Wallis computed this series to three decimal places, 1.645, in *Arithmetica infinitorum* (1656), and Gottfried Leibniz examined it in the 1670s, Jakob Bernoulli made it widely known among mathematicians in his first dissertation on series (1689) [63, 1071–1072]. The precise summation of the Basel problem was to be as striking and significant as the young Gottfried Leibniz's summation a half century earlier in 1674 of the infinite series $1 - 1/3 + 1/5 - 1/7 + \ldots = \sum_{n=0}^{\infty} (-1)^n(2n + 1)^{-1} = \pi/4$, which is related to the power series expansion of arctangent x. That series now bears his name. By skillfully applying infinitary analysis to solve the Basel problem and others for reciprocals of the next few even number exponents, Euler was to contribute to the origins of analytic number theory [63; 97; 130].

In 1689 Jakob Bernoulli observed that the sum in the Basel problem must lie between 1 and 2 [70, 161]. Obviously it is greater than 1, and Jakob Bernoulli determined that it is less than 2 by comparing it to Leibniz's successful totaling of another series, the reciprocals of triangular numbers—$S = 1 + 1/3 + 1/6 + 1/10 + 1/15 + \ldots$. After Christian Huygens posed the summing of triangular reciprocals as a challenge problem, Leibniz had cleverly divided S by 2 and transformed it into $S/2 = (1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \ldots = 1 - 1/2 + 1/2 - 1/3 + 1/3 - \ldots = 1$, and thus $S = 2$. Jakob Bernoulli recognized that $1/4 < 1/3$, $1/9 < 1/6$, $1/16 < 1/10$ or, more generally, that $1/n^2 < 1/n(n + 1)/2$. This term-by-term comparison of the two series reveals that the sum of the reciprocals of squares must be less than 2.[22] By similarly decomposing or transforming the infinite series in the Basel problem, Jakob Bernoulli sought to improve upon Wallis's close approximation and sum it.

An exact solution of the Basel problem, whose infinite series only slowly converged, still eluded Daniel Bernoulli in 1728, just as it had escaped Leibniz, Jakob, Johann, and Nicholas I. Bernoulli, James Stirling, Abraham de Moivre, and others, but he found the sum to be "very nearly 8/5," as he wrote to Goldbach in August of 1728 [148, 257]. Goldbach responded in January of the next year that $\zeta(2) - 1$ must fall between 16223/25200 and 30197/46800 or 0.6437 and 0.6453 [55, 2: 282]. In simpler fractions, he found that $1\ 16/25 = 1.64 < \zeta(2) < 1\frac{2}{3} = 1.66\ldots$.

Daniel Bernoulli surely discussed the unresolved Basel problem with his boarder and friend Euler, who quickly found an approximation approaching 1.644934, which

[22] Raymond Ayoub conjectures that one reason the Basel problem was tantalizing to mathematicians may have been its superficial resemblance to the series $\sum_{n=1}^{\infty} 1/n(n + 1)$, whose value was known to be $\sum_{n=1}^{\infty} [1/n - 1/(n + 1)] = 1$. See [63, 1072].

he presented in the paper "De summatione innumerabilium progressionum" (1731) [29]. Working with series of the type $\sum_{k=1}^{\infty} x^k/(ak + b)^n$, Euler determined that $\zeta(2) = (\log x)\,[\log(1 - x)] + \sum_{n=1}^{\infty} x^n/n^2 + \sum_{n=1}^{\infty} (1 - x)^n/n^2$. For the case $x = 1/2$, $\zeta(2) = (\log 2)^2 + 2 \sum_{n=1}^{\infty} 1/2^n n^2$. This last series converges far more rapidly than the series for $\zeta(2)$. Euler proceeded to find that $(\log 2)^2 = [\sum_{n=1}^{\infty} 1/n2^n]^2 \sim .480453$ and $2 \sum_{n=1}^{\infty} 1/n^2 2^n \sim 1.164481$, and thus $\zeta(2) \sim 1.644934$ [7, 40–41].[23] He often began mathematical studies with such extended numerical calculations. Imposing computations were a great pleasure, if not an addiction, for Euler, who was probably unsurpassed in history in mental computations.

In his search for the exact sum of the Basel problem, Euler read works of his predecessors and contemporaries on the subject. He likely read Bernard Fontenelle's *Géométrie de l'infini* (1728), which has mistaken statements about the summation problem, but he had not yet read Stirling's *Methodus differentialis* (1730), which on page 28 introduced a method yielding the most precise summation to date, 1.644934066, which is correct to eight decimal places. This work on summing the reciprocals of squares and similar problems prompted Euler's discovery in 1732 of the so-called Euler–Maclaurin summation formula, which Scottish mathematician Colin Maclaurin discovered independently by 1738.[24]

After investigating Newton's algebraic theorems on sums of nth powers of roots of finite polynomials and rashly extending similar rules to transcendental polynomials of the form (1 − the Taylor series of sin x/a), Euler found in 1735 that $\zeta(2)$ depends upon the quadrature of the circle, and thus the value of π [7, 73–86].[25] This discovery was a beautiful triumph, for it allowed him to sum the Basel problem exactly. In his computation, which is given below, $\sum_{n=1}^{\infty} 1/n^{2k} = \zeta(2k) = a_{2k}\,\pi^{2k}$, where a_{2k} is the numerical coefficient of the Euler-Maclaurin summation formula. It was apparently belief in the persistence of patterns that led him to extend rules from finite to infinite polynomials, and his superb intuition kept him from going astray on this dangerous course.

Euler first shared his discovery of the exact summation in a now lost letter to Daniel Bernoulli, who conveyed it to his father Johann. Upon hearing the news Johann remarked: "In this way my brother's [Jakob's] most ardent wish is satisfied ... if only my brother were still alive" [51, 4: 22]. Euler had not yet supplied the method: in a letter of September 12, 1736 Daniel Bernoulli, after listing the sums

[23] His paper "Methodus generalis summandi progressiones" in the *Commentarii*, Volume 6 (1732/33, printed 1738), can yield $1 + 1/4 + 1/9 + \ldots = \int_0^1 ([\ln(1 - x)]/x)\,dx$. See [7, 42–72].

[24] Formally speaking, the Euler–Maclaurin summation formula amounts to

$$s(x) = \sum_{i=1}^{v} f(i) + \int_v^x f(t)\,dt + \sum_{m=1}^{\infty} b_m/m!\,[f^{(m-1)}(x) - f^{(m-1)}(v)],$$

where $z/(1 - e^{-z}) = \sum_{m=0}^{\infty} b_m z^m/m!$ and $f^{(i)}$ give the initial f and successive derivatives for $i > 0$. See [148, 259 − 260]. A century later Carl Jacobi determined the remainder for the Euler–Maclaurin formula.

[25] This paper was not printed until 1740.

of $\zeta(2)$ and $\zeta(4)$, wrote, "The theorem on the sum of [these] series ... is very remarkable. No doubt you have discovered it *a posteriori*. I should very much like to see your solution" [25, 20; 53, 2: 435]. After learning what the sum of $\zeta(2)$ is, Johann Bernoulli was to find a proof in 1742, which turned out to be Euler's [63, 1072].

By December 1735 Euler had presented the paper "De summis serierum reciprocarum" to the St. Petersburg Academy for Volume 7 of the *Commentarii* [7, 73–86]. It gives his method and series computations that include Leibniz's series [7, 79].[26]

In modern notation, Euler starts his proof with the function

$$f(x) = 1 - x^2/3! + x^4/5! - x^6/7! + x^8/9! - \dots. \qquad (1)$$

When $x = 0$, obviously $f(0) = 1$. Euler multiplies $f(x)$ by x/x to obtain

$$\begin{aligned}(x/x)\,f(x) &= x(1 - x^2/3! + x^4/5! - x^6/7! + \dots)/x \\ &= (x - x^3/3! + x^5/5! - x^7/7! + \dots)/x\end{aligned}$$

[7, 74]. The numerator is the Taylor expansion of sin x, so $f(x) = \sin x/x$. The zeroes of $f(x)$ thus occur when $\sin x = 0$, or $x = n\pi$, $n \in \mathbb{Z}$. Euler assumed that $f(x)$ has this representation as an infinite product:

$$\begin{aligned}f(x) &= [(1 - x/\pi)\,(1 + x/\pi)]\,[(1 - x/2\pi)\,(1 + x/2\pi)]\dots \\ &= [1 - x^2/\pi^2]\,[1 - x^2/4\pi^2]\dots.\end{aligned} \qquad (2)$$

This brought him close to summing the Basel problem.

He next multiplied out the infinite product on the right side of (2) and gathered all terms of the same power of x in the equation to obtain

$$f(x) = 1 - (1/\pi^2 + 1/4\pi^2 + 1/9\pi^2 + \dots)x^2 + (\dots)x^4 - \dots.$$

He then equated coefficients of the x^2 terms in both (1) and (2), yielding

$$-1/3! = -(1/\pi^2 + 1/4\pi^2 + 1/9\pi^2 + \dots),$$

or

$$1/3! = 1/\pi^2(1 + 1/4 + 1/9 + 1/16 + \dots),$$

and hence

$$\pi^2/6 = 1 + 1/4 + 1/9 + 1/16 + \dots.$$

The numerical coefficient for the Basel problem is thus $P = 1/3! = 1/6$, and $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

Euler did not stop with $\zeta(2)$. With "multo labore," he extended his procedure, computing the zeta function for the even integers up to 12, but remained silent with regard to the odd integers. The coefficient for $\zeta(4)$, which comes from the x^4

[26] For his use of the Leibniz series, also see his "De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera, ..." in *Miscellanea Berolinensia* (1743), 172–192 in [7, esp. 148–150].

term, is more difficult to obtain. The equation of the coefficients from (1) and (2) is

$$\begin{aligned}1/5! &= 1/2[(1/\pi^2 + 1/4\pi^2 + \ldots)^2 - (1/\pi^4 + 1/16\pi^4 + 1/81\pi^4 \\ &\quad + 1/256\pi^4 + \ldots)] \\ &= 1/2[1/\pi^4(1 + 1/4 + 1/9 + \ldots)^2 - 1/\pi^4(1 + 1/16 + 1/81 \\ &\quad + 1/256 + \ldots)] \\ &= 1/2\pi^4[(\pi^2/6)^2 - (1 + 1/16 + 1/81 + 1/256 + \ldots)] \\ 1/5! &= 1/72 - 1/2\pi^4(1 + 1/16 + 1/81 + 1/256 + \ldots) \\ 1/120 - 1/72 &= -1/2\pi^4(1 + 1/16 + 1/81 + 1/256 + \ldots) \\ -1/180 &= -1/2\pi^4(1 + 1/16 + 1/81 + 1/256 + \ldots) \\ 2\pi^4/180 &= 1 + 1/16 + 1/81 + 1/256 + \ldots + 1/n^4 + \ldots\end{aligned}$$

This connects the reciprocals of the fourth power with π as well:

$$\pi^4/90 = 1 + 1/16 + 1/81 + 1/256 + \ldots + 1/n^4 + \ldots$$

or $\zeta(4) = \pi^4/90$. Euler wrote its coefficient as $Q = P\alpha - 2\beta = 1/90$, and the next coefficient as $R = Q\alpha - P\beta + 3\gamma = 1/945, \ldots$. Euler gives $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450$, $\zeta(10) = \pi^{10}/93555$, and $\zeta(12) = 691\ \pi^{12}/6825 \times 93555$, and could have continued further [7, 80, 85].

Euler declares in "De summis serierum reciprocarum" that "quite unexpectedly, I have found an elegant formula for [what we call $\zeta(2)$]" [7, 70]. This and his rapid sketching of the results of the even integer summations through $\zeta(12)$ suggest a feverish excitement.

By 1736 Euler had found that the rational multiple coefficients of π^2, π^4, π^6, π^8, ... depend upon Bernoulli numbers [7, 436]. Three years later he established the relationship $\zeta(2v) = [2^{2v-1}/(2v)!]B_v\pi^{2v}$, where B_v is a Bernoulli number [7, 434–439]. The generating function for these numbers is the Taylor expansion of $1/e^x - 1) = 1/x - 1/2 + \sum_{v=1}^{\infty} (-1)^{v-1}B_v x^{2v-1}/(2v)!$ [83, 99]. Computing them gives $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$, $B_6 = 691/2730$, and so forth. The appearance of the prime 691 both in the coefficient of $\zeta(12)$ and in B_6 probably suggested the interconnection to Euler. Fermat had partially anticipated the Bernoulli numbers and Jakob Bernoulli had described them in his *Ars conjectandi* (1713) on pp. 96 to 97. On pp. 6 and 19 to 21 of his *Miscellanea analytica* (1730), Abraham de Moivre had called these Bernoulli numbers, and Euler eagerly adopted the name Bernoullian. Not until Part Five of Chapter Two of his *Institutiones calculi differentialis* (1755), however, did he begin to bring out their importance.

In this work, Euler connected the properties of the zeta function with the problem of the distribution of primes. Prime numbers steadily become sparser, seemingly decreasing in number according to a certain regularity. By using the divergence of the harmonic series—an analytic fact—in his study of the distribution of prime numbers, Euler indirectly proved the infinitude of primes. His memoir "Variae observationes circa series infinitas'' (1737) introduced the famous product decomposition formula: let P be the set of primes, then

$$\prod_{p_i \in P} (1 - p_i^{-s})^{-1} = \prod_{p_i \in P} (1 + 1/p_i^s + 1/p_i^{2s} + \ldots). \tag{3}$$

Multiplying out the right side of (3) gives $\sum_{n=1}^{\infty} n^{-s} = \zeta(s)$. For $s = 1$, $\zeta(1)$ is the harmonic series, which diverges. It follows that the corresponding product must have infinitely many factors.

Emil Grosswald and Larry Goldstein demonstrate that these studies of the zeta function were a catalyst for questions leading to the prime number theorem and properly portray Euler's indirect approach as a building stone for much of 19th-century analytic number theory [97, 600–601; 102, 121–129].

Following Euler's difficult and exact summation of the Basel problem his reputation started to grow across Europe. Volume 7 (1734/35) of the St. Petersburg *Commentarii* did not appear in print until 1740, but the results were spread by his friends and correspondents, among them Daniel Bernoulli in Basel, James Stirling in Edinburgh, and Philippe Naudé in Berlin. Both the summations and their lack of rigorous foundations became lively topics for discussion.

In June 1736, Euler sent Stirling his construction of series for logarithms of whole numbers based on the harmonic series as well as his zeta summations [25, 433]. Euler was perplexed when Daniel Bernoulli wrote on March 29, 1738 that Stirling had first solved the Basel problem in his *Methodus differentialis* (1730). Stung by this claim, Euler replied on April 26, that Stirling had introduced methods that nearly computed the Basel problem and had a certain similarity to his method, but asserted that he had not known of this book before solving the problem [25, 22–23].

On June 27, 1738 Stirling finally responded to the logarithmic series and compared Euler's summation formula with that of Maclaurin, which was to appear in Maclaurin's *Treatise of Fluxions* (1742) [146].[27] He asked Euler to publish his independent findings in the *Philosophical Transactions* and to let his name be entered for election to the Royal Society. Partly because he had neglected to answer Euler's first letter for two years, Stirling was afraid that Euler might engage in a priority dispute. Characteristically Euler was gracious and confident, rather than proprietary, about his discoveries. Relieved, Stirling wrote to Maclaurin in October 1738 that Euler

> is under no uneasiness about your having fallen on the same theorem with him, because both his and the demonstration were publicly read in the Academy about four years ago.... [57]

Euler recognized that his initial procedure for the zeta computations was shaky and had to be improved. After responding to criticisms of his first proof of them in ''Démonstration de la somme...'' in 1743 [42], he gave his computational improvement in *Introductio in analysin infinitorum* [5; 45]. After presenting methods in

[27] See footnote 24 for the Euler–Maclaurin summation formula.

Chapters 8–11 for computing trigonometric functions and their logarithms "more easily ... than ... in previous times" [45, 168], in Chapters 10, 11, and 15 he turned to more complex expressions, masterfully manipulating infinite series expansions of trigonometric functions and infinite products. For example, he dealt with Wallis's formula,

$$\frac{\pi}{2} = \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\left(\frac{6}{7}\right)\cdots,$$

and with products of prime numbers used to compute the values of the zeta function for even integers $2v$. These zeta function numbers greatly interested Legendre and Gauss, while Bernhard Riemann referred to the zeta function for reals as Euler's function.

C. Infinitary Analysis

The rapid development of calculus and the creation of its branches was the chief achievement of 18th-century mathematics. And calculus or infinitary analysis was at the center of Euler's research.

Euler's initial studies of calculus in its formative stage rested mostly on the writings of Jakob and Johann Bernoulli, John Wallis, Gregory of St. Vincent,[28] and earlier contributions from mathematicians published in the *Philosophical Transactions* of the Royal Society and Leipzig's *Acta Eruditorum.* From Johann Bernoulli, he knew the exponential function e^x, where the exponent is the variable and the derivative of the function is the function itself. Euler discovered that $e^x = \lim_{n\to\infty} (1 + x/n)^n$ and, employing Taylor's series, computed the decimal expansion of $e = 1/1! + 1/2! + 1/3! + \ldots = 2.718281828\ldots$[29] In 1665 Newton had already discovered that infinite expansion for e [126], but neither he, Taylor, nor Johann Bernoulli, who had independently devised the Taylor power series, had recognized its importance in calculus, which Euler was in the process of establishing.

During the 1730s Euler also proposed independently of Maclaurin the integral test for convergence and advanced the study of the vaguely understood transcendental functions. Following Leibniz and Johann Bernoulli, he divided functions into two classes: algebraic and transcendental. Examples of the latter are logarithmic, exponential, and trigonometric functions—the elementary transcendental functions. Extending results of John Wallis and Jakob Bernoulli on progressions of factorial numbers and attempting to interpolate and determine the general nth term of the sequence $n! = 1, 2, 6, 24, 120, \ldots$ using the infinite product

$$1 \cdot 2^n/(1+n) \cdot 2^{1-n}\, 3^n/(2+n) \cdot 3^{1-n} 4^n/(3+n) \ldots,$$

Euler introduced two new integrals: for positive integers n,

[28] He studied Gregory of St. Vincent's voluminous *Opus geometricum quadraturae circuli et sectionum coni* (1647).

[29] In 1744 Euler made this number the base for natural logarithms.

$$\int_0^1 x^e(1-x)^n\,dx = n!/(e+1)\,(e+2)\ldots(e+n+1), \tag{5}$$

and for nearly all integers n [33],

$$\int_0^1 (-\ln x)^n\,dx = n!. \tag{6}$$

After refining these in *Institutiones calculi integralis* (1768–1770), Euler in 1781 determined the modern form of the second (6). In modern notation, he found that $\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx$.

These two integrals became, along with the zeta function, the most important non-elementary transcendental functions of the eighteenth century. In volume 2 of his *Exercices de calcul intégral* (3 vol., 1811–1817), Adrien-Marie Legendre called these the first and second Eulerian integrals. He and Gauss referred to the second of these transcendental functions as the gamma function, and in 1839 Jacques Binet called the first the beta function [9, LX–LXV].

Guided by his study of harmonic series, his method of computing zeta series by summing terms "until they begin to diverge" [7, 357], his intuition, and his gamma function defined by infinite products, Euler computed in 1735 what is now called the Euler constant $C = \lim_{n\to\infty} (\sum_{k=1}^{n} 1/k - \log n) = 0.577215664901532$. This figure is correct to the first fifteen decimal places [7, 119–122].

By discarding the longstanding representation of trigonometric quantities as Ptolemaic chords and half chords and making them functions that are numerical ratios, Euler recast trigonometry. The discovery by Viète of an infinite product for $2/\pi$,[30] and a similar finding by Wallis in *Arithmetica infinitorum* for $\pi/2$, the infinite series of Gregory and Leibniz for $\pi/4$, and his own summing of zeta numbers had shown that formulas connected with the circle can be expressed solely by integers. Continuing in this vein, Euler now made trigonometry part of analysis and gave it stronger foundations.

Building on his studies and computations of e, the function $\sin x/a$, and related logarithms, Euler discovered by 1743 a form of his identity $e^{ix} = \cos x + i \sin x$, the cardinal formula of analytical trigonometry. His reading of the *Philosophical Transactions* and Johann Bernoulli's correspondence with de Moivre suggest that Euler was generally aware of the pioneering work by Roger Cotes and de Moivre in this direction. The 1714 *Philosophical Transactions* and posthumous *Harmonia mensurarum* (1722) contain Cotes' equivalent of $\log(\cos x + i \sin x) = ix$, though not in this notation, and de Moivre's *Miscellanea analytica* (1730)

[30] In his study of trigonometry, Viète discovered the formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

This finding was one of the first explicit expressions of an infinite process as a mathematical formula.

implied but did not explicitly give the famous theorem $(\cos x + i \sin x)^n = \cos nx + i \sin nx$, known today as de Moivre's formula, from which Euler's identity can be simply derived. Euler, however, was the first to link trigonometric, exponential, and logarithmic functions by his identity, making e the base for natural logarithms. This work, presented in the *Analysin infinitorum* of 1748, led to Euler's derivation of the fundamental equation $e^{\pi i} + 1 = 0$ [45, 1: 75–116].

D. Calculus of Variations

In St. Petersburg Euler also contributed important results and ideas to a semigeometric stage of what he later named the calculus of variations. Variational theory studies the existence of extrema, that is, maximum and minimum values, for integrals, often called functionals, and determines these values. Today, the more appropriate title for this work is calculus of variations in the narrow sense, since the concept of the variation or differential of a functional involves more than extrema [96]. Euler was to apply methods of the calculus of variations widely to physical problems and to envisage variational principles as manifesting general laws of physics. Rüdiger Thiele shows that the field was for him partly a Leibnizian *ars inveniendi* or method of discovery for reaching unknown truths [144, 90–91]. The type of problems Euler considers involves families of curves $C_\alpha: x \to y(x, \alpha)$ defined on an interval $a \leq x \leq b$ and with two fixed end points: that is, $y(a, \alpha)$ and $y(b, \alpha)$ are independent of α and are boundary conditions. In what became his necessary condition, Euler was to seek to determine the values of α that furnish an extremum for the functional of type $\int_a^b f(x, y, y')\, dx$.

Euler gained his first exposure to calculus of variations through Johann Bernoulli, who with Jakob had been a prominent pioneer in the field. In 1696 Johann had posed as a challenge problem to mathematicians the brachistochrone problem: to find the curve of quickest descent in a vacuum.[31] The following year Jakob published in *Acta eruditorum* his first solution, showing that the curve is a cycloid.[32] At the close of that paper he observed that the brachistochrone problem that has no subsidiary conditions or constraints is fairly straightforward, whereas isoperimetric problems that impose restrictions on the admissible curves are often more interesting. He posed two important new problems. Given all cycloids through a specified point, having the same horizontal base, and reaching a lower vertical line, he wanted to find the specific cycloid on which a heavy particle falls in the least time. He also recovered and exceeded the ancient isoperimetric problem of finding among closed curves of a set length the curve that encloses the greatest area, namely a circle. He

[31] Newton rapidly and anonymously solved this in *Philosophical Transactions* (Jan. 1696), and Leibniz did not give his solution since it was similar to that of the Bernoullis.

[32] Solutio problematum fraternorum ... cum propositione reciproca aliorum," *Acta Eruditorum* May 1697: 211–217.

now sought to determine "among all isoperimetric figures on . . . [a given base BN, not the one which] will enclose the greatest area, but is such that a related curve BZN has the property" [98, 47]. In extending his methods and differential equations to attack isoperimetric problems in 1700 and 1701 [90; 98, 35–67], Jakob found that these require a second degree of freedom: two successive ordinates on the extremalizing curve have to vary, that is, be functions of an arc, but each element of the path-length must be unaltered. Jakob traced the curve on an orthogonal coordinate system.

Hermann Goldstine illustrates that after Jakob's death in 1705 Johann accepted and elaborated his isoperimetric condition for the case where the abscissa is the independent variable [98, 35ff.]. In a memoir of 1718 for the Paris Academy, Johann found among curves of the same length the one with the lowest center of gravity and first proved sufficiency for the cycloid to provide the least time of fall in the brachistochrone problem. In his tutorial with Bernoulli, Euler had also studied Brook Taylor's *Methodus incrementorum*, whose Proposition 17 offers a solution to an isoperimetric problem, partly by incorporating results from Jakob Bernoulli [86]. Euler's apparently initial paper, "Constructio linearum isochronarum in medio quocunque resistente" examines rigorously the Frenchman Henry Sully's system of pendulum tautochrone, which belongs to the calculus of variations.

In Russia Euler approached the calculus of variations rather slowly. Although three of his other early articles dealt with algebraic trajectories [11], his only paper in this field up to 1732 is "De linea brevissimi. . . ." Here Euler derived the second-order differential equation that furnishes minimizing lines on conoidal, or cone-shaped surfaces and discusses some special cases. Another hand dates this paper November 1728, but Euler states that Johann Bernoulli had posed this geodesic curve problem to him in a letter of December 10, 1728, on logarithms and second-order differential equations [10, 12]. That Bernoulli brought geodesic curves to his attention at this time suggests that the two men previously had not extensively studied the nascent calculus of variations. The actual date of completion of "De linea brevissimi" appears to have been shortly before Euler's response to his mentor on February 18, 1729, supplying the differential equation solution [11, 1–29]. Euler had still another reason for exploring the field further. In a paper of 1727 for Volume 2 of the *Commentarii* that appeared in 1729, his second cousin Jakob Hermann improved upon the Bernoullis' solutions of the brachistochrone problem. Still, Euler did not display a continuing interest in the early calculus of variations until after he became a full professor at the academy.

From 1732 to 1738 Euler submitted four memoirs to the *Commentarii*[33] that elaborate methods that Hermann had presented in *Phoronomia* (1716), Taylor's strategies in *Methodus incrementorum* (1715), and methods of Johann and especially Jakob Bernoulli [10, 14]. Euler concentrated on finding extremal values for actions of forces, including gravity. The theme of his memoir of 1732, "Problematis isoperi-

[33] See [32; 33; 35; 37].

metrici . . . ,'' that in the broadest sense general solutions are possible indicates his future direction. Although Euler had studied the brachistochrone problem as an 18-year-old and had discussed it with Hermann in 1727, the second of his four memoirs on variational problems in the 1730s, ''De linea celerrimi descensus in medio quocunque resistente'' (1734/1735, publ. 1740), marks the beginning of his sustained, career-long interest in this question. Euler's *Mechanica* also deals extensively with problems involving motion in a resisting medium. His investigation of separate problems and methods up to 1738 comprises preparatory work, whereas his proposal at the start of his 1738 paper classifying variational problems according to how many side conditions they have puts his variational studies on a more systematic course.

By 1740, in the draft of his masterful *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes* . . . (The Method of Finding Curves that Show Some Property of Maximum and Minimum), Euler was transforming the calculus of variations. Constantin Carathéodory, a leading 20th-century contributor to variational theory, has described this book as ''one of the most beautiful mathematical works ever written'' [144, 87]. Euler's correspondence with Clairaut suggests and a letter of March 16, 1746 to Maupertuis states that by the spring of 1741 he had completed an early draft without the appendices [26, 81–102; 27]. Craig Fraser shows that *Methodus inveniendi* founded the initial stage of the calculus of variations by generally formulating variational problems: it offered systematic techniques for deriving standard equations for solutions, replaced previous special cases of problems with general cases by classifying a hundred problems under eleven categories, and made methods of solution more direct [90]. Euler's methods, however, were still semigeometric; it was Lagrange who provided the δ-algorithm of analytic variations in the mid-1750s. *Methodus inveniendi* gives Euler's differential equation, or first necessary condition for extremals, and connects it to problems in mechanics. It also presents at the close of Chapter Three the most elegant solution of the brachistochrone problem up to that time and introduces the principle of least action. This book was not published until 1744.[34]

E. Mechanica (*1736*): *Analytic Format and Response*

Although the half-century-old calculus permitted moderns to solve precisely problems of instantaneous acceleration and motion in physics that the ancient Greeks could not, Newton's *Principia* (1687) and in part Hermann's *Phoronomia* (1716) followed a geometric format with conic sections, curves, and quadratures. While still a student of Johann Bernoulli, Euler had begun planning to write a major treatise on mechanics that resulted in his 135-page *Mechanica*. It appeared as a two-volume supplement to the *Commentarii*. The *Mechanica*'s preface credits Newton and Hermann with developing infinitesimal methods but finds that

[34] For an excellent brief introduction to Euler's contributions to calculus of variations, see [144, 87–95].

neither author made a systematic application [145, 332–335]. Building on the shift among continental masters of calculus from geometric quantities to algebraic formulas, Euler now broke decisively with the older geometric format, systematically introducing differential equations for the mechanics of what we today call mass-points and for their free motion along geodesics on a given surface. This brilliantly reformulated the treatment of motion, based on Newton's second law as well as those in resistant media, topics central to Newton's *Principia*. Euler therein brought into mechanics uniform analytic methods drawn from "both what I have found in the writings of others and what I myself have thought out" [145, 335].

Most of Euler's contemporaries quickly recognized the *Mechanica* as a landmark in the history of physics. In 1737 Johann Bernoulli cited the "genius and acumen" of Euler and the Huguenot pastor and philosopher Jean-Henri-Samuel Formey (1711–1797), the future permanent secretary of the Berlin Academy, praised his use of the analytic method. Formey's review of *Mechanica* in Amsterdam's *Bibliothèque germanique* (tome 39)[35] was the first in western Europe. In 1738 Christian Wolff commended Euler for his use of higher mathematics that made it possible to read the text without difficulty, and Maupertuis wrote of his admiration for the author of this "excellent publication" [27, 38]. In 1740 the Parisian *Mémoires de Trevoux* credited Euler with developing modern mechanics. This review first brought Euler broadly to the attention of the Parisian scientific community. The *Mechanica* clearly catapulted him into the first rank of Europe's mathematical scientists, and his analytical approach to mechanics has been essential to physics ever since.

Not all reviews were without criticism. Cartesian physicists and the English ballistic expert Benjamin Robins interpreted Euler's systematic use of differential equations as a blind submission to calculation. Robins faulted the *Mechanica*'s lack of an experimental base and held that Euler's purely computational technique led to some mistakes, especially in describing a material point moving toward the center of a force. Euler had concluded that the body, on reaching the center, would remain there, whereas Newton's calculations indicated that the mass point would oscillate between the center and a return point [59].

IV. RESEARCH IN APPLIED MATHEMATICS

In St. Petersburg Euler also advanced applied mathematics,[36] beginning with an attempt to establish satisfactory mathematical foundations for music theory. But

[35] See pp. 93–108 and [138, 85 ff.].

[36] Following a catalogue prepared by Paul Staeckel, Gustav Enestrom in 1910 divided Euler's works for the *Opera omnia* into pure and applied mathematics. In "Von Nutzen der Hoeheren Mathematik" (publ. posth. 1847), Euler had spoken of the two purposes of higher mathematics as the utilitarian and plumbing the foundations of truth [22, 408–412]. The pure–applied division cuts across the mathematical fields. Emil Fellmann has precisely shown Euler's works to consist of 58% in mathematics and 41% in mechanics, astronomy, architecture, artillery, machines, and ship propulsion [70, 31]. Clifford Truesdell attests that Series II of Euler's *Opera omnia* on mechanics is not entirely application and that about half of the 29 volumes of Series I on pure mathematics involve some mechanics [145, 323].

his chief contributions to the field arose in connection with major projects on cartography and naval science that were critical to the development of the state.

A. Music Theory

From before his arrival in St. Petersburg, Euler had shown a serious interest in music theory. Notebook sketches that seem to date from 1726, when he was 19, outline chapter by chapter a proposed book on music composition that came to be his *Tentamen novae theoriae musicae* (1739) [41, 353–355; 40]. Early in 1727 he had completed his *De sono*. Having worked on the *Tentamen* in St. Petersburg since 1727 [15, XXIX], Euler wrote on May 25, 1731, to Johann Bernoulli that he had almost completed it [25, 42]. In his response of August 11, Bernoulli criticized existing foundations, observing that precise ideas of harmony had yet to be devised. In what may be taken as a hint of displeasure with Euler for spending so much time on musical research, Bernoulli urged him to complete this treatise and proceed to his planned *Mechanica* [53, 2: 8–11].

In the *Tentamen* Euler sought to make music theory "part of mathematics and deduce in an orderly manner, from correct principles, everything which can make a fitting together and mingling of tones pleasing" [41, 9]. The *Tentamen*, an early case of his pursuit of systemic order, employs computations of a kind to be found in his investigation of acoustics, vibration theory, and optics. His Chapter I, for example, summarizes the principles of acoustics, and its Section 9 gives his formula for a vibrating string's frequency: $355/113 = \sqrt{3166n/a}$, where n is the ratio of a string's stretching weight to its normal weight and a represents the string's length. Among others, Marin Mersenne, Descartes, author of *Compendium musicae* (1650), and Leibniz probably influenced his application of number theory to music. As Nicholas Fuss stated in his *Éloge* [93], this treatise also bespeaks a man who relaxed by playing the clavier.

Euler proposes a dual foundation for music theory: an exact knowledge of sound, which is part of natural science, and auditory perceptions of harmony [41, 27]. Both depend on numerical ratios of frequency. Euler thereby rejects the ancient Aristoxenus, who wanted numerical ratios removed from music theory, but finds Pythagorean principles alone insufficient and, when their limits are unrecognized, a possible source of errors. Pythagorean harmonic principles had periodic vibrations heard as tones and supposed a consonant interval to be made by two tones whose ratio of frequencies is given in small integers: for example, the octave, 2:1, the double octave, 4:1, and the fifth, 3:2. Seventeenth-century music theorists had demonstrated the proportionality of pitch to frequency; that is, musical intervals are frequency ratios, the inverse of length ratios. Sigalia Dostrovsky argues that their measuring of audible vibrational frequencies to their upper and lower limits was critical to evolving vibration theory [84].

Seeking a precise description of the beauty of music, Euler computes tone scales and degrees of agreeableness among tones. His contemporaries calculated these scales only by intervals of the so-called musical integers—2, 3, and 5—as well as their composites; he considers extending these to the number 7 and suggests testing other primes and their powers [24, *il–l*]. Euler's computations, which include logarithms and continued fractions, are not simple. His first approximation in Chapter IV of the ratio of the octave to the fifth, for example, is .30103001/.1760913, which he expresses as

$$1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \dots}}}$$

In Chapter X (Section 20) Euler warns of the great difficulty in applying 7 to the computation of consonance or harmony and rejects its use. It appears that Euler was strongly influenced by Marin Mersenne's *Harmonie universelle* (1636), which had attempted to follow teachings of St. Augustine in explaining harmony by numerical ratios, studied behavior of vibrating strings in seeking a mechanistic base for harmony, appealed to the number 7, and used analogies between music and other elements of the physical world.

In revising drafts of his *Tentamen*, Euler also reviewed Newton's *Principia* and musical experiments conducted from 1700 to 1702 by Joseph Sauveur. Sauveur, a mathematician at Paris's *College Royal* and tutor at Louis XIV's court, had especially studied the frequency and pitch of organ pipes together with the organ builder Deslandes [15, XXX] and had measured the speed of sound using rates of vibration of higher notes. In not confining his study of music to harmonics, Sauveur had developed a new field, which he named *acoustique*. An analogy between the shape of a flute, whose sound does not depend upon its vibrations, and related string theory, Euler believed, would allow progress to be made in the theory of horns. In the *Principia* (Book Two, Section Seven), Newton had analyzed sound as pressure waves through a compressible medium and knew that the velocity of sound is frequency times wavelength and that frequency $v \propto 1/l\sqrt{F/\sigma}$, where l is length, F tension, and σ cross-sectional area. The first edition of the *Principia* (1687) gave the velocity of sound as 968 English ft/sec. But after Newton reviewed frequency measures by Sauveur in 5-foot organ pipes and by Derham [84], the second edition (1713) increased the velocity to 1020 ft/sec. Euler gave a closer approximation of 1100 Rhenish ft/sec [41, 37–38].[37] He measured the speed of sound against that of light (in which he also had a long-standing interest) by the time difference between seeing a bolt of lightning and hearing the thunder [14].

Euler's *Tentamen* had little immediate influence, however, perhaps because it was "for musicians too [advanced in its] mathematical [computations] and for mathematicians too musical" [70, 73]. Today it is known as the third of his three major books from his early St. Petersburg period, of which it is chronologically the first.

[37] The velocity of sound in air is $c = 331.6$ m/sec $+ 0.6(t)$, where t = temperature in centigrade.

B. Russian Cartography

Improvement of Russian mapmaking was the chief state project for Euler in St. Petersburg. This work was desperately needed.

Before 1700 Russian cartography had been crude, little more than rough sketches with no astronomical base and employing no consistent projections, such as Mercator's. A 1697 map of Siberia lacked latitudes and meridians and even reversed north and south. Peter the Great ordered the first Russian geodetic surveys, recruited the French astronomer Joseph Delisle to his academy, and had a three-story observatory built on Vasil'evskii Island. The observatory was among the finest in Europe. Delisle founded the St. Petersburg astronomical school, training young astronomers and geodetic practitioners for cartographic work and geodetic ventures, and he made and used observations of Jupiter's satellites to determine longitude. His many enthusiastic assistants variously included Daniel Bernoulli, Friedrich Christoph Mayer, Georg Wolfgang Krafft, and Georg Wilhelm Richmann.

In the early 1730s Euler assisted Delisle by recording astronomical observations in the journal of the St. Petersburg Observatory. Beginning in late 1734 he independently made observations for constructing meridian tables, and he published those tables the next year. In October affairs turned better for the academy and Euler when his friend Johann Albrecht Korff was named president. In August of 1735 Euler was officially named director of the academy's geography section and his total pension increased to a handsome twelve hundred rubles.[38] Primarily, he was to assist Delisle in tasks related to composing a general map of Russia.

The government's heaviest funding at the academy was for geography projects, and for the next two years Euler attended daily meetings of the government's geography bureau. In addition, he assisted with the government's Second Kamchatka (or Great Northern) Expedition to eastern Siberia, which was to last from 1733 to 1743. This expedition was the most important group research project of the early academy. Led by the Danish explorer Vitus Bering, it exceeded in scope both the Lapland and the Peru geodetic expeditions sponsored by the Paris Academy. Delisle got appointed as chief astronomer of the expedition his ill-prepared brother Louis, who cared little for science, and in the harsh Siberian environment, Louis's work deteriorated. The astronomical observations he and his Russian assistants made turned out to be worthless [123]. Nevertheless, Euler conscientiously evaluated the expedition's astronomical reports and swiftly computed ephemerides. In 1737 he and Delisle prepared a uniform set of instructions for the expedition's geodesists. These instructions were principally for the Russian scholar Vasilii Tatischev's investigation of the economic possibilities of the high Urals [100]. In part to remedy the failure of his brother's observations, Delisle left to join the Kamchatka

[38] In 1735 Euler expanded his list of correspondents to include astronomers Giovanni Poleni in Padua and Giovanni Marinoni in Vienna as well as Danish naval officer Friedrich Weggersloff. They were independent sources of the latest astronomical information from the West.

Expedition in 1740.[39] Afterward, Euler and the German astronomers Gottfried Heinsius and Christian Winsheim did most of the work of the geography section.

The chief goal of these academic and governmental efforts—an atlas with a new "general map of the Russian Empire"—remained incomplete until 1745. When it appeared, the *Russian Atlas*, with its general map and 19 maps of territories, represented the first such effort in Russia to draw upon sophisticated astronomical determination of longitude and accurate geodetic measurements. Euler, by now in Berlin, wrote that this map put the Russians ahead of the Germans in the art of mapmaking [61].

C. Education and Naval Science

From 1735 Euler was increasingly given technical projects on such topics as magnetism, machines, and ship construction, along with educational assignments. In carrying out these state projects he wrote two books. *Scientia navalis* was mainly completed in 1738 [25, letter 210; 46; 16], although Euler corresponded with Johann and Daniel Bernoulli on related hydraulic and hydrodynamic problems up until 1741. But Russian instability and probably his move to Berlin kept it from publication until 1749.[40] The St. Petersburg Academy paid him 1200 rubles for its preparation, thereby doubling his salary one year, besides financing the eventual publication of this book. A two-part introduction to arithmetic, *Einleitung zur Rechen-Kunst* (1738–1740), was intended for use in the academic gymnasium. Vasilii Adodurov, a member of an aristocratic family from Novgorod who was one of Euler's first students at the gymnasium, translated the first part of the *Einleitung* into Russian [22; 147, 94]. Adodurov, the only Russian admitted to membership in the early academy, had been made an adjunct in mathematics in 1733.

Euler's great two-volume *Scientia navalis* on ships follows the *Mechanica* as a second milestone in his program to develop rational mechanics. Outstanding in both theoretical and applied mechanics, it addresses Euler's intense occupation with the problem of ship propulsion. It applies variational principles to determine optimal ship design and first establishes the principles of hydrostatics, which provide a scientific foundation for naval architecture. Euler here also begins developing the kinematics and dynamics of rigid bodies, introducing in part the differential equations for their motion. *Scientia navalis* introduces the concepts of centroid and metacenter separate from center of gravity, and treats such topics as equilibrium of floating bodies, oscillations of ships, and their masting.

The attention in *Scientia navalis* to practical design and ship handling problems essentially refutes the oft expressed opinion that Euler put all his faith in theoretical mathematics and virtually neglected experience. Reading only his published memoirs from the first St. Petersburg period and Benjamin Robins' criticisms of his

[39] Another member of the expedition from 1733 was historian Gerhard Mueller. Following the advice of Euler, he collected archives, from families in Siberia and ethnographic descriptions of its peoples. See [147, 103; 61, 11].

[40] *Scientia navalis* appears in [16].

Mechanica can give rise to this false impression. Numerous authorities have contributed to it, beginning with Condorcet's *Éloge*, which presented Euler as purely an analyst. A similar view dominates biographies of Euler written in the 1920s by Gustav du Pasquier and Otto Spiess [85; 139]. Adolph Youschkevitch's characterization of Euler as seeking above all to express physical problems in mathematical terms reflects a judgment that fails to explore sufficiently the rich interaction between the two [152, 473, 481]. Euler insisted that "experience ... must determine which" theories to accept [47, 1: 223]. Clifford Truesdell and recently Charles Blanc have argued convincingly that experience and observation were of central importance to Euler, especially in formulating and refining differential equations to apply precisely to the physical world at a time when rigorous foundations for calculus were still lacking [145, 97 ff.; 11].

V. RESPONSE TO NEWTONIAN SCIENCE AND WOLFFIAN PHILOSOPHY

The diffusion, criticism, and articulation of Newton's dynamics, based on the inverse-square law of gravitational attraction, and his corpuscular optics, especially the application of higher mathematics to these theories, was the dominant current in the development of the sciences in the early 18th century. On the continent, their spread encountered resistance. In the West, the major center of rivalry between the young Newtonians and the dominant Cartesians was the Paris Academy, where the Newtonians generally triumphed in the 1740s [109]. Valentin Boss has described the transmission of Newtonian ideas to Russia through advisers to Peter the Great by examining their libraries, and he and I have studied the response at the St. Petersburg Academy [68; 74], where the Leibniz–Wolffians—a phrase coined by Bilfinger—challenged them.

Euler was well prepared for the arguments. His master's thesis in Basel had compared Cartesian and Newtonian natural philosophy, and his mentor Johann Bernoulli supported Leibniz and Wolff. In addition, the leading advocate of Newtonian science at the early academy was Euler's friend Daniel Bernoulli.

At the two conference meetings of the academy each week, the volatile Bernoulli had championed ideas of his hero Christian Huygens and Newton against Cartesian and Leibniz-Wolffian critics [20]. When Bilfinger in April 1726 discounted Huygens' proportionality approach to the conservation of *vis viva* (mv^2 or, roughly, kinetic energy), Bernoulli had irately exclaimed "errasti! errasti!" Later he denounced and refused to speak to Bilfinger for criticizing as a "vulgar hypothesis" Newton's inverse-square law of gravitational attraction [68, 110–111]. Dismissing Newton's refutation of Cartesian vortices, Bilfinger had reasoned that the vortex ether had the ability to penetrate other matter without losing motive force. In 1728 Bilfinger won the prize of the Paris Academy for a paper on this topic, entitled "De causa gravitatis ... disquisitio." In the St. Petersburg *Commentarii* Bernoulli rejoined that the Cartesian "hypothesis" was "insufficient" [68, 110–111]. Bernoulli criticized Bilfinger for his lack of mathematical background and the two men argued over the theory of capillarity, which Bernoulli based partly on theorems from Book II of Newton's *Principia* [23, 64–65]. Further controversy arose when Bilfinger and Hermann led a group of Leibniz–Wolffian natural philosophers in defining as primal substance not Cartesian corpuscles or Newton's atoms but animate monads. Added to theoretical differences were the constant attempts of Schumacher to pit people against each other in order to strengthen his control over the academy. The rift between Bernoulli and Bilfinger healed only shortly before Bilfinger left St. Petersburg.

A. Rigorous Confirmation of Newton's Inverse-Square Law

Daniel Bernoulli advanced Newton's inverse-square law of gravitational attraction and Joseph Delisle, the senior astronomer in St. Petersburg, agreed with Newton's proof of the oblate spheroid shape of the Earth [58, 424–428]. Euler was more circumspect about accepting that law's general operation. Like members of the Paris Academy, he insisted on accurate, confirming observational data obtained from studies of the shape of the Earth, the tides, lunar motion, and the paths of comets. In a letter of March 29, 1738, he was to reject Bernoulli's contention that Pierre Maupertuis' measurements from his Lapland expedition of the length of a degree of meridan settled the question of the Earth's shape [25, 22–23].

From its founding the Paris Academy had served as the center for geographical research in France. After Gian Domenico Cassini first thought that the Earth is *aplatie*, an oblate spheroid—a near sphere slightly flattened at the poles and bulging at the equator—Jacques Cassini was concluding between 1713 and 1718 that the Earth is instead *allongie*, an elongated spindle. His position was based on spherical geometry linked to his new measurement of a degree's difference along an arc of the Paris meridian north towards Dunkirk—a measure six times finer than possible with existing instruments. In the 1720s Jacques Cassini blocked a polar geodetic expedition, planned by Delisle while he was still in Paris.

Among the factors prompting Maupertuis' expedition was the spate of papers on geodesy presented to the Paris Academy following the rejuvenation of mapping surveys in France after 1730. David Beeson observes that scathing criticisms shortly after 1730 of the use of cartographic data by Jacques Cassini reopened interest in a definitive determination of the shape of the Earth [64, 100–102]. Particularly devastating was a review in the *Journal historique de la république des lettres* (1732/1733) by J., who was likely editor Elie de Joncourt, of the Venetian Giovanni Poleni's book *Epistolarum mathematicarum fasciculus* (1729). J. pointed out that an error of 20 seconds in measuring celestial arcs could produce the 31 toise difference[41] in separate measurements of arcs of meridian by Jean Picard, Philippe de la Hire, and the Cassinis to the north and south of Paris, that the Cassinis had rejected figures that contradicted their theory, and that within existing margins of error their data could either prove or disprove their position. The 31 toise difference in arcs seemed to indicate a regular decrease occurring to the north. Mary Terrall

[41] 1 toise = 1.95 m = 6.4 English feet.

has found that a dispute among craftsmen over which surveying instruments were superior made Maupertuis' expedition a good test case [142].

In an attempt to confirm their position, Jacques Cassini and his son Cesar-François measured in 1733 and 1734 an arc of the great circle perpendicular to the Paris meridian, but this did not stop the dominance of the Cartesians from eroding [114]. Determining the shape of the Earth required measurements of arcs of meridian in latitudes near the north pole and equator. To obtain these the Paris Academy funded geodetic expeditions to Peru from 1735 to 1744 and to Lapland from 1736 to 1737. Maupertuis and Clairaut arranged the Lapp expedition, which aimed to demonstrate the Newtonian position that the Earth is *aplatie* and not *allongie*.

After gaining the support of Hercule Cardinal Fleury,[42] who financed all the instruments that they needed, Maupertuis and his French–Swedish expedition left Dunkirk for the Gulf of Bothnia in May 1736. Battling swarms of insects in summer that required wreathing the expedition in smoke and enduring bitter cold in winter, they made astronomical observations and surveying triangulations to measure an arc of meridian. To ensure accuracy, Maupertuis had independent observers repeat measurements. In this work, they introduced the new Gordon sector. To corroborate their results, they also gathered pendulum data showing that the force of gravity is weaker at the pole. To his surprise, Maupertuis' results suggested that the Earth is even flatter at the poles than Newton had thought. Upon returning to Paris in August 1737, Maupertuis declared that his expedition's observations and measurements were quite precise. While his claims were dubious and the accuracy of his results was later shown to be poor [114, 335], he gained support from craftsmen for the soundness of his methods and instruments. The Cartesian Cassinis, of course, rejected Maupertuis' results, as did Johann Bernoulli, who cited Clairaut's myopia. But Emilie du Châtelet praised the exactitude of the expedition's measurements and hailed its leader as Sir Isaac Maupertuis [114, 368], marking a major triumph for Newtonian science in Paris.

In a letter of May 20, 1738, Maupertuis initiated correspondence with Euler in St. Petersburg. He expressed his admiration for Euler's *Mechanica* and enclosed a copy of his new book *La Figure de la Terre*, written partially in the form of a popular travelogue describing his visit to Lapland [27, 38]. Euler had closely followed the Lapland expedition and had in April 1738 begun writing observations on the shape of the Earth for the St. Petersburg newspapers. He believed that the use of the expedition for geography and expanding knowledge of the deepest principles of natural science established its importance. His response to Maupertuis' letter was similar in rigor to the position of Clairaut in praising the Lapland data as the most accurate on the subject to date, having been made with the best instruments, but insufficient alone to determine the shape of the Earth [27, 75–77]. Euler withheld judgment, awaiting the results of the Paris Academy's geodetic expedition to Peru, led by Pierre Bouguer and Maupertuis's friend Charles de la Condamine. The Peru

[42] From 1726 to 1743, Cardinal Fleury was first minister to Louis XV.

expedition's measurements and study of pendulum oscillations would complete the basis for a crucial comparison of an arc of meridian at the equator with that in the polar region and confirm whether the Earth has a bulge at the equator [39, 327].[43] Euler now thought that the Earth was shaped like an orange.

As his *Mechanica* and his letter of November 23, 1738 to Maupertuis demonstrate [27, 39–40], Euler was also busy, along with Daniel Bernoulli, Clairaut, and d'Alembert, on another important tool for the confirmation of Newton's inverse-square law: the invention of differential equations aimed at precisely describing the shape of the Earth, the tides, lunar motion, and orbits of comets. This work first unleashed the power of calculus. A key criterion for Euler was that these equations must agree with improved observations. Drawing on his study of the third edition of Newton's *Principia* (1726), especially Book III, Proposition XIX, Problem III [58, 424–428], Euler proposed to calculate the polar flattening, assuming a diminution of centrifugal force at the equator of 1/289. Is that flattening 1/233 to 1/234 or Newton's 1/229 to 1/230? Euler, who rounded Newton's figure to about 1/240, added a new consideration in his letter to Maupertuis and a December article by proposing that the internal density of the Earth is not homogeneous but variable [39, 346]. While in Lapland, Clairaut had similarly decided to determine the Earth's shape by applying hydrodynamic principles and calculus. Proceeding on his and Euler's assumption that the Earth's density increases toward the center, he derived in his *Théorie de la figure de la terre* (1743) a regular rather than a lumpy Earth and obtained more accurate results than those obtained both by the Lapp expedition and by Maclaurin, who had agreed with Newton's proof that the radius of the polar circle was 1/230 less than that of the equator [58]. Clairaut arrived at the figure 1/300, confirming a modified form of the Newtonian shape of the Earth, essentially that of an orange.

In celestial mechanics, the great unanswered question at that time was whether Newton's inverse-square law could alone explain all astronomical motions. In 1747, Clairaut created a stir at the Paris Academy when he announced that the results of a series of differential equations that he had invented utilizing the inverse-square law to describe the motion of the mutually attracting sun, Earth, and moon, the moon acting as the perturbed body, did not agree with observations of the advance of lunar apogee, the farthest point of the moon from the Earth. This finding momentarily heartened the few remaining Cartesians, and even the Newtonians in the Paris Academy, led by Georges-Louis Buffon, believed that a corrective inverse cube or fourth power was needed to account for the progress of lunar apogee. Then Clairaut discovered in December 1748 that his equations, not Newton's inverse-square law, were imperfect. He announced the confirming equations in May 1749 and rapidly informed Euler [26, 6–9].

[43] In a letter of March 20, 1751, Pierre Bouguer claimed that he had sent to Euler his book *La figure de la Terre determinée par les observations de MM. Bouguer, la Condamine, etc.* (1748). When the book did not arrive by 1752, Bouguer stated that Madame Denis had sent Euler a copy bound with other works. As late as 1752, astronomers and physicists in Paris quarreled over the shape of the Earth.

The three-body problem was not new to Euler. Early in his career he had turned to it in studying the third edition of Newton's *Principia*, particularly Book I, Section 11, Theorems 21, 23, and 24. About 1730 he composed a brief tract, *De trium corporum mutua attractione*, that attempted to solve such problems purely by geometric means [119], but the new differential equations he was devising, particularly in his *Mechanica*, appeared more promising. Since his own differential equations for the three-body problem involving Jupiter and Saturn indicated a discrepancy similar to that found initially by Clairaut, Euler remained doubtful that Newton's inverse-square law accounted for lunar motion. For this reason, he had the St. Petersburg Academy select lunar theory as the topic for its 1750 prize competition, which Clairaut won with a new series of difficult differential equations for lunar motion. After making tedious calculations and checking closer approximations to recent observations, Euler was convinced that Newton's inverse-square law of gravitational attraction alone is "entirely sufficient to explain the motion of" lunar apogee, and that this work of Clairaut gave "quite a new lustre to the [gravitational attraction] theory of the great Newton" [20, 1]. These events took place after Euler left St. Petersburg for Berlin in 1741.

B. Euler's Pulse-Theory Optics

The label anti-Newtonian has been applied to Euler's optics [110; 111]; he was the chief Enlightenment opponent of Newton's projectile theory of light, which defined light rays as discontinuous streams of material corpuscles. He rejected this qualitative description. Building on his powerful intuition into physics and a strict analogy between the propagation of sound in air and that of light in the ether, Euler first proposed instead a theory close to the Malebranchean notion that light is propagated as pressure vibrations or waves through the medium of an elastic ether; this accords with the Cartesian view in which an ether pervades the complete universe. But studies of historians Casper Hakfoort and Roderick Home show that in his "Nova theoria lucis et colorum" (1746), Euler modified and strengthened his initial theory through a study of elasticity and by replacing Malebranche's theory with Newton's methods for handling the propagation of a sound wave in Book II (Section VIII) of the *Principia* [104; 111; 99]. This shift in his optical theory occurred between 1744 and 1746 [111, 529–530]. Euler's new theory of light was to prevail east of the Rhine [111, 523, 532].

Historical studies of Eulerian optics thus suggest, in contrast to academic orthodoxy, that the 18th century was not always a time of unfolding triumphs for Newtonian science,[44] that applying an anti-Newtonian label here beclouds his evolving thought and the complex mature Eulerian synthesis in optics, and that he improved his theory not through polemics but generally through disciplined research into concrete problems. As shown below, his strong stance against Newton's corpuscular optics did not prevent Euler from adopting an element

[44] See [77; 91; 111; 132–134; 143].

from Newton's *Principia* to improve his new optics. As in dynamics, he remained open, selective, and nuanced.

Euler's optics in St. Petersburg and the "Nova theoria lucis et colorum" clarify another facet of his work. Both are only formulated mathematically in part. As Euler himself put it, "the method which I have used [in "Nova theoria"], following Newton, is indirect and very far distant from a perfect theory of pulses propagated in an elastic fluid . . ." [15, XXXIII]. In its mathematical description, the longitudinal waves that a vibrator generates produce a linear effect. Light is a succession of pulses and Newton's equation accounts for only the first. In beginning to create the associated wave equations of vibratory motion, Euler restates Newton's geometric equation in algebraic language, thereby removing obscurities from Newton's account of pulses in an elastic medium, while employing approximations for the associated sines and cosines. But in seeking an equation for the velocity of sound he was also to go beyond pure mathematics and draw upon a growing body of experiments, such as William Derham's comparison of the speed of high notes with that of low notes. This work, together with plausible reasoning and an analogy with sound waves, produced the equations for optics. These were not completed until Euler wrote the memoir "Continuation of Research on the Propagation of Sound" in 1759 for Vol. 15 of the Berlin Academy *Mémoires*. For these he thus drew upon empirical mathematics.

C. Emerging Opposition to Wolffian Philosophy

Except in optics, the young Euler tended to avoid scientific disputes. Despite intellectual sympathies, he did not join the argumentative Daniel Bernoulli in attacking Leibniz–Wolffian thought. Nor did he accede to Schumacher's attempts to set him against Bernoulli. But in a letter of August 1736 to Danzig mathematician Karl Ehler, Euler gently began to criticize Christian Wolff's *Philosophia prima sive ontologia* (1729), *Cosmologia generalis* (1731), and the theory of positive and negative infinity given in the latest edition of *Elementa matheseos universae* (1710) [25, 115]. He did not accept the value that Wolff, using l'Hôpital's rule of infinitesimal quantities, assigned to the expression (0/0). While he agreed with Leibniz and Wolff that infinitesimals are absolute zeroes, he was formalistic in arguing that a peculiarity allows their ratio (0/0) to represent a finite number. Michael Segre shows that later in *Institutiones calculi differentialis* (1755), Euler was to reason that $n \cdot 0 = 0$, and so $n \cdot 1 = 0/0$ [6, 69–71, 136]. In a letter of February 1737 to Ehler, he states that no satisfactory book exists on integral calculus and he plans to embark on such a project [25, 116]. Euler wrote that he had not yet read Wolff's forthcoming *Theologia naturalis* (1737), but assumed that it was no better than the *Cosmologia generalis.*

Euler cautiously developed his critique of Wolffian philosophy. At the conference meeting of the academy on May 16, 1738, he submitted a now lost notice on faults occurring in the *Cosmologia generalis*. A letter of July 10, to Bilfinger asked for Bilfinger's comments but suggested that the notice was preliminary and should not be circulated to Wolff [25, 70–71]. A letter of November 3, 1738 to Bilfinger questioned Wolff's reduction of primal substance to animate monads—metaphysical points of energy—a position suggested in his discussion of infinitesimals with Ehler.

Euler was soon to reject as absurd the monadic doctrine, the core of the Wolffian philosophy. In his search for greater accuracy in the sciences, he sought to clarify the properties of the smallest particles of matter—a field in which the microscope had made possible more exact studies. Not until October 1741, however, did he send a letter to Wolff from Berlin denying that properties of corporeal elements could be based on monads, as given in the *Cosmologia*, and citing mathematical errors in the *Elementa* [25, 466].

In north Germany, Wolffian philosophy was exerting its greatest influence in Berlin, where, according to Nicholas Fuss, the scholarly community spoke of nothing but the monadic doctrine and its metaphysical foundations [93, XI]. By 1744 Euler had marshalled his physical arguments against monads. Leibniz had claimed that the Cartesian assignment of extension alone to primal substance did not account for the "two resistances" of impenetrability and Kepler's inertia. His monads were proposed to correct this situation. But Euler argues that they offer a basis neither for impenetrability, since they lack density, nor for Newton's inertia. Moreover, they cannot account for extension or different specific gravities [43]. For Euler, the infinity of variations of physical bodies result from the diversity and arrangement of primal substance, which he calls molecules. Incomparably smaller and rare molecules compose a second species of matter, the subtle fluid that we call the ether. It is the cause of gravity and the medium for the propagation of light.

Euler's opposition to Wolff's monadic doctrine went beyond a scientific basis. In November 1738 he raised a religious issue by suggesting that the monadic doctrine led to atheism [61, 31]. This was not because he held theology to be the queen of the sciences. Euler knew well that pietists and conservative theologians had labelled Wolff's rational, mathematical philosophy as "heathen and atheistic" from the 1720s [91, 41–42]. He also recognized that institutional Christianity, both Catholic and Protestant, was under attack in the 1740s from deists, who branded it a source of fanaticism and brutality, and in Berlin from Wolffian freethinkers and their natural religion of reason. A devout Christian for whom Scripture provided inspiration, Euler felt it his duty to defend religion. In Berlin his tract "Gegen die Einwurfe der Freygeister" (1747) presented his physicotheology. Euler believed that knowledge of the Good is founded partly in knowledge of physical truths, by which he meant precise quantitative laws, and that Wolffian science and monadism cannot provide these [22, 269, 282–283].

VI. POST 1735

A. Eyesight Deterioration

Except for two major health problems, little had disrupted the quiet life of the growing Euler family. Early in 1735, Euler suffered a nearly fatal fever—news which he kept from his parents and the Bernoullis until his recovery [53, 2:

419]. Three years later he became, so it has long been thought, nearly blind in his right eye.

In a letter of August 21, 1740 to Goldbach, the presbyopic Euler lapsed from his typical Latin into German, claiming that his painstaking work on correcting landmaps had overstrained his eyes and by itself caused his eyesight problem, beginning in 1738. "Geography is fatal to me," he wrote. "You know that I have lost an eye and [the other] currently may be in the same danger" [25, 102]. Euler, who seems to have had trouble saying no to new assignments, was asking his friend to intercede to free him from another assignment similar to his cartographic work that he thought threatened his remaining sight. The alarmed Goldbach responded the same day. In a letter of September 12, 1740 to Philippe Naudé, Euler complained that his right eye was almost unusable and that the sight in his left eye was deteriorating. The *Éloge* of his close colleague and grandson-in-law (Paul) Nicholas Fuss (1755–1826) similarly holds that three days of intense astronomical calculations connected with geographical work underlay the loss in the right eye and began a course leading to Euler's total blindness in 1767 [93, lvi–lvii]. Fuss, who was nearly 50 years younger than Euler, did not meet him until seven years later in 1773 and did not become his assistant until 1778. Until recently scholars, among them Gustav Enestrom and Euler's early 20th-century biographer Otto Spiess, would accept his explanation for the fading of his right vision.

Swiss medical historian René Bernoulli has argued persuasively against the notion of a constant deterioration in Euler's sight via a two-stage pathogenesis, in which overstrain produces blindness. He suggests instead that a process in four stages, typical in high stress cases, was more likely with occasional partial remission of the blindness along the way. In this sequence overstrain leads to high fever, followed by an eye abscess, which is a contributing factor in blindness [65]. Euler's problem with fevers dating from 1735 supports this interpretation. His attempt to conceal his fevers from his parents in Switzerland, as revealed in his correspondence with Daniel Bernoulli, may partly account for his simplified and incomplete description. As René Bernoulli also notes, Euler's explanation is not unlike that of younger patients with analogous illnesses today.

Two portraits of Euler provide fundamental information on this medical history. His first known portrait, by a cousin named Johann Georg Brucker, which dates from 1737, shows no detectable eye malady. Brucker paints a bright, confident, and seemingly playful man. A pastel portrait from 1753 by Euler's Basel colleague Emmanuel Handmann gives close attention to the eyes. Handmann details problems with the upper lid and a condition of strabismus in the right eye. The right eye is not yet totally blind, which suggests some partial remission in its condition. The left eye has a strong dark pupil and no special deterioration. A later cataract, not constant deterioration dating from 1738, probably most harmed Euler's left eye. René Bernoulli employs the Handmann portrait to support his account of the course of Euler's problem with his sight.

B. Growing Reputation and the Move to Berlin

In the late 1730s, Euler's reputation was rising in an important forum—the prestigious annual prizes of the Paris Academy of Sciences. Two were offered in alternate years. The greater prize, given in even years, was for the best treatise on astronomy, matter theory, optics, or mechanics. It carried a handsome total award of 2500 livres. The other prize, given in odd years, on longitude and navigation, was for 2000 livres. After taking the *accessit* for his paper on the masting of ships in 1727 at 20,[45] Euler was to win these prizes an astounding 12 times, once under the name of his son (Johann) Albrecht.[46] Five times he won for applied papers on shipbuilding and navigation.

In 1738 Euler shared the Paris Academy prize with two others for papers on the nature and properties of fire. Fire, along with air, heat, electricity, and magnetism, was then a crucial research subject. In the early 18th century, natural philosophers considered fire to be the most volatile of the four Aristotelian elements—earth, air, fire, and water. Lengthy studies of combustion and calcination associated with mining gave rise to another explanation, German chemist and physician Georg Stahl's influential phlogiston theory. According to it, substances rich in phlogiston burn readily. But this qualitative theory lacked adequate phenomenological descriptions and precise quantitative laws. In the late 1730s physicists, like Pieter van Musschenbroek at Leiden, in order to create a richer theoretical framework and better to identify what to measure in the above phenomena, began to posit that they depend upon subtle fluids conveying no mass. Not surprisingly, the prize papers of 1738 were quite diverse in their analysis. Euler described fire as an elastic fluid, while the other winners took different approaches: the Jesuit Louis-Antoine du Fech rejected the view that fire is an element, giving instead a vortex theory explanation, and Jean-Antoine de Crequy, comte de Canaples, asserted that motion in opposing currents of an ethereal fluid causes fire.

Perhaps more interesting than the winners are two who lost—Voltaire and Emilie du Châtelet—who received honorable mention. Upset, Voltaire blamed the outcome on a Cartesian dominance at the Paris Academy. He was Newton's champion. Henry Guerlac, John Heilbron, and Ellen Hine have amply demonstrated that Voltaire's simple division of the academy into Cartesian and Newtonian camps fails to recognize the complexities in arguments and other motivations for research [108; 109]. Voltaire had visited Leiden and conducted experiments at Cirey before writing his essay on the particulate nature of fire. His essay lacks originality and mainly derives from ideas set forth by the Leiden professors Musschenbroek and Willem

[45] The chief work of the French geometer and hydrographer Pierre Bouguer (1698–1758), who won the prize, was to be *Traité du navire, de sa construction et de ses mouvemens* (1746).

[46] Albrecht was an able mathematician, physicist, and astronomer. Born on November 27, 1734 (N.S.), in St. Petersburg, he became an ordinary member of the Berlin Academy in 1754 and director of its astronomical observatory in 1758. He moved to the St. Petersburg Academy in 1766 and served as its secretary from 1769 to his death in 1800.

'sGravesande. They imputed weight to fire particles in heating iron, but held that their upward thrust on absorption results in the heated iron weighing the same as the original. Seemingly, in research C. P. Snow's two cultures were evolving. Even a literary scholar as eminent as Voltaire could not substantially advance the natural sciences without proper instruction in them—a point that Daniel Bernoulli and Euler had been making in general. The 1738 prize competition appears to be the first encounter between Euler and Voltaire.

Again in 1740 Euler shared the Paris Academy prize, this time with three others—Daniel Bernoulli, Colin Maclaurin, and the little-known Jesuit mathematician Antoine Cavalleri, a Toulouse Cartesian—for essays on the subject of the ebb and flow of the tides. Although Euler clearly explained the influence of the sun and the moon on the tides, Daniel Bernoulli chided him in a letter for still questioning the extent to which Newtonian attraction accounts for celestial motions, citing data from the Lapland expedition and the work of Maclaurin supporting the Newtonian idea of the Earth's shape [25, 29–30]. The year 1740 represents a turning point in the polemic between Newtonians and Cartesians: afterward no Cartesian paper won the Paris Academy prize.

As 1740 began, Euler seemed permanently settled in St. Petersburg. His younger brother, Johann Heinrich, a painter who had come from Basel in 1735, lived with him and was an adjunct at the academy, and a second son, Karl, was born. Seeking to make his moribund Brandenburg Society (soon to be the Berlin Academy) a leading European scientific center, the new Prussian monarch Frederick II had his ambassador and close friend Ulrich Friedrich von Suhm attempt to recruit Euler in June and July. Euler declined, finding unenticing the salary offer of from 1000 to 1200 ecus, since the St. Petersburg Academy paid the equivalent of 1600 ecus [147, 95]. In the autumn, when Russia fell into turmoil following the death of Anna, who left only an infant heir, and the overthrow of her German favorite Ernst-Johann Biren, Euler became more receptive to a move to Berlin. His brief autobiography,[47] recorded in 1767, notes that political tensions and heightened hostility toward foreigners during the brief interregnum of the infant Ivan VI made work difficult and life dangerous [145, 95]. This was the time prior to the successful putsch of Elizabeth, a daughter of Peter the Great. Asked to cast a horoscope for the infant tsar, Euler passed that assignment to professor of astronomy Gottfried Heinsius.

Frederick was determined to obtain Euler's services.[48] In November 1740, Suhm died while returning to serve in Berlin, and Frederick appointed the aged Baron Axel von Mardefeld to be his new ambassador to Russia.[49] In February 1741, perhaps partly at the urging of his wife Katharina in the face of the local danger,

[47] The autobiography, which Euler dictated to his eldest son, Johann Albrecht, has not been published. Emil Fellmann has a copy of the manuscript at the Swiss Academy of the Natural Sciences in Basel.

[48] For Frederick's relations with the sciences, see [72; 73; 141].

[49] In Russia von Mardefeld was to discuss the renewal of alliances in general. Recalled from St. Petersburg in 1747 at the age of 83, he died the next year. For more information on von Mardefeld, see [62].

Euler asked von Mardefeld to confirm an offer of a 1600 ecus pension and payment of all travel expenses up to 500 ecus. Suhm had promised to make this proposal to the king. Von Mardefeld responded positively, and Euler accepted the offer.[50] Illness and difficulties in obtaining permission to resign from the St. Petersburg Academy delayed his departure for Berlin until June [27; 61, 14]. In a letter of March 6, 1741, that he delivered to the new Academy president Karl von Brevern,[51] he claimed that he needed to move to a milder climate; otherwise he might completely lose his eyesight and ruin his health [25, 68]. He ultimately left with the goodwill of the St. Petersburg Academy, which later purchased his house, albeit for a paltry 100 rubles, and continued to pay him as an honorary member a 200 ruble annual stipend in Berlin [147, 98; 25, 356]. This was not simply an altruistic gesture. For the academy, Euler purchased books and instruments, wrote extensive reports on the sciences in the West, and tutored Russian students, such as the future mathematician and astronomer Stepan Jakovlevic Rumovskij (1734–1812), who resided with the Euler family. In addition, he sent a stream of papers for the *Commentarii* and was its *de facto* editor.

The Bernoullis had mixed reactions to Euler's going to Berlin. They wanted him to succeed Johann I at the University of Basel. Still, his old master Johann Bernoulli wrote on October 28, 1741, heartily congratulating him for obtaining the Berlin position [87, 207]. He was pleased that Euler would be closer to Basel. Bernoulli urged Euler to visit his home city, expressing a burning wish to see him once again before he died [25, 49–50]. When Frederick invited Johann and his sons Daniel and Johann II to Berlin, they declined. If he were 20 years younger, Johann proclaimed, he would accept, but he was suffering from asthma, gout, and advanced age [87, 210]. In addition, Frederick was involved in the first Silesian War against Austria, a result of his aggression against the young and pregnant Maria Theresa. He could not accept a position "an einer Akademie [wo man] muss einigermassen Subordinaten sein wie in dem Militärstande" [139, 81], Daniel Bernoulli wrote to Euler, adding, "Es scheint dass die Wissenschaften und der Krieg incompatibel seyen" [53, 2: 474].

Euler seems to have disregarded the forebodings of the Bernoullis about Prussia. By 1740 Berlin had grown to a population of 70,000 and had a lively French Huguenot refugee colony that helped make it a social oasis from the military [105]. A comment he is supposed to have made to the Queen Mother Sophia Dorothea of Prussia, just after he arrived in Berlin, suggests his relief at being away from the dangers and anxiety of interregnum Russia. Able to draw out only monosyllables in a conversation with him, the Queen Mother asked why he was so timid and reserved in his speech. Reportedly Euler replied, "Madame, parce que je viens d'un pays ou quand on parle, on est perdu" [78, 290]. Of his early reception in Berlin, Euler wrote ebulliently on January 8, 1746, to the Basel-born chaplain

[50] This compared favorably to the average pension of a junior member of about 300 ecus. See [145, 353].

[51] Von Brevern was named president on May 5 (April 24, O.S.), 1740, after Euler's friend Korff was named ambassador to Denmark on April 19.

Johann Caspar Wettstein: "I can do just what I wish [in my research] .... The king calls me his professor, and I think I am the happiest man in the world." [145, 302]. In this situation, Euler's research was to flourish, and he would rise to the peak of his career.

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In 1975 the Swiss Society and the Soviet Academy of Sciences began the two-part Series Quarta (IV), fulfilling a proposal made by the original publication committee and repeated by Andreas Speiser in 1947. Series IVA will print Euler's correspondence in eight volumes. Series IVA volumes published to date are I (a descriptive inventory), V (correspondence with Clairaut, d'Alembert, and Lagrange), and VI (with Maupertuis and Frederick II). Series IVB is planned to consist of about seven volumes. It will include Euler's notebooks and adversaria, or questions that he covered with his assistants after returning to St. Petersburg in 1766.

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