ELSEVIER Journal of Pure and Applied Algebra 101 (1995) 335–352
JOURNAL OF PURE AND APPLIED ALGEBRA

Nonassociative algebras related to Hamiltonian operators in the formal calculus of variations

J. Marshall Osborn*, Efim Zelmanov

Department of Mathematics, University of Wisconsin, Van Vleck Hall, 480 Lincoln Drive, Madison, WI 53706, USA

Communicated by K.W. Gruenberg; received 15 January 1993; revised 10 March 1994

Abstract

The main result in this paper is the classification of simple Novikov algebras A with a maximal subalgebra H such that A/H has a finite-dimensional irreducible H-submodule. A second result deals with the extension of Hamiltonian operators.

0. Introduction

In [6] Gelfand and Dorfman formulated conditions for operators in the formal calculus of variations to have the Hamiltonian property. Following [6] consider an algebra of polynomials of symbols $u_\alpha^{(i)}$, where α runs over some set of indices $\Lambda = \{0, 1, 2, \dots\}$. The "differentiation with respect to x" is defined by

$$\mathrm{d}/\mathrm{d}x = \sum_{i,\alpha} u_\alpha^{(i+1)} \partial / \partial u_\alpha^{(i)}.$$

Let H be a matrix differential operator with matrix components H_{ij}, $i, j \in \Lambda$.

Suppose that

$$H_{ij} = \sum_k (c_{ijk} u_k^{(1)} + d_{ijk} u_k^{(0)} \mathrm{d}/\mathrm{d}x). \tag{1}$$

where are the c_{ijk}'s are scalars from the ground field F and $d_{ijk} = c_{ijk} + c_{jik}$ for all $i, j, k \in \Lambda$. Let A be an algebra with basis e_i, $i \in \Lambda$. Define a multiplication on A via $e_i \circ e_j = \sum_k c_{ijk} e_k$. Gelfand and Dorfman [6] proved that H is a Hamiltonian operator if and only if the algebra satisfies the identities

$$(x \circ y) \circ z = (x \circ z) \circ y, \tag{2}$$

$$(x \circ y) \circ z + y \circ (x \circ z) = x \circ (y \circ z) + (y \circ x) \circ z. \tag{3}$$

* Corresponding author.

SSDI 0022-4049(95)00062-N

Now let H be a matrix differential operator of a more general type.

$$H_{ij} = \sum_k (a_{ijk}u_k^{(0)} + c_{ijk}u_k^{(1)} + d_{ijk}u_k^{(0)}\mathrm{d}/\mathrm{d}x), \tag{4}$$

where $a_{ijk} = -a_{jik}, d_{ijk} = c_{ijk} + c_{jik}$. Along with the multiplication $\circ$ on A we define another multiplication $e_i \times e_j = \sum_k a_{ijk}e_k$. Again Gelfand and Dorfman proved that H is a Hamiltonian operator if and only if

(a) the operation $\circ$ satisfies (2) and (3),

(b) the operation $\times$ gives a Lie algebra structure on A, i.e.,

$$u \times v = -v \times u, \quad (w \times u) \times v + (u \times v) \times w + (v \times w) \times u = 0, \tag{5}$$

(c) the operations $\circ$ and $\times$ are related by

$$(w \circ u) \times v - (w \circ v) \times u + (w \times u) \circ v - (w \times v) \circ u - w \circ (u \times v) = 0. \tag{6}$$

Example (S.I. Gelfand, cf. [6]). In any associative algebra with differentiation ∂, the product $a \circ b = a\partial b$ satisfies (2) and (3). In particular, consider the algebra of polynomials in one variable x with the standard differentiation $\partial x = 1$. The basis $e_i = x_i$, $i \geq 0$, gives structural constants

$$c_{ijk} = j\delta_{i+j-1,k}, \qquad e_i \circ e_j = c_{i,j,i+j-1}e_{i+j-1}. \tag{7}$$

Thus, the operator with matrix components

$$H_{ij} = ju_{i+j-1}^{(1)} + (i+j)u_{i+j-1}\mathrm{d}/\mathrm{d}x, \quad i,j \geq 0, \tag{8}$$

is Hamiltonian (this operator was first mentioned in [9]).

In any algebra $(A, \circ)$ which satisfies the identities (2) and (3), the operation $a \times b = ab - ba$ satisfies (5) and (6). Hence, for any scalar $\alpha \in C$, the operator

$$H_{ij} = \sum_k (\alpha(j-i)u_k^{(0)} + ju_{i+j}^{(1)} + (i+j)u_{i+j}^{(0)}\mathrm{d}/\mathrm{d}x), \quad i,j \geq 0 \tag{9}$$

is Hamiltonian.

Balinskii and Novikov [1] (see also [10]) noted that finite-dimensional algebras satisfying (2) and (3) are crucial for the classification of linear Poisson brackets of hydrodynamical type. Papers [1, 10] were followed by a series of papers [2, 3], [11–13, 15] on the classification of finite-dimensional algebras satisfying (2) and (3).

In [11–13] linear algebras satisfying the identities (2) and (3) were called Novikov algebras. These papers study simple finite-dimensional Novikov algebras of characteristic $p > 2$. The first shows that every such algebra is a deformation of certain graded Novikov algebra of dimension p^n, while the second classifies all such algebras which contain an idempotent. The third classifies modules over such an algebra.

In this paper we prove some classification theorems for infinite-dimensional Novikov algebras over an algebraically closed field F of characteristic 0 under assumptions similar to those on Lie algebras in [7]. First we investigate simple Novikov algebras A with maximal subalgebra H such that A/H has a nonzero finite-dimensional submodule. Such an algebra has a Weisfeiler filtration in a natural way, which leads to a graded algebra G associated with A. For convenience, we list the examples of graded simple Novikov algebras that arise in this manner.

Type I: G has a basis $\{x_i\}$ for i an integer with products given by $x_i x_j = x_{i+j}$.

Type II: A has a basis $\{x_i\}$ for i an integer ≥ -1, and products are given by $x_i x_j = (j+1)x_{i+j}$.

Type III: A has a basis $\{x_i\}$ for i an integer ≤ 1, and products are given by $x_i x_j = (j-1)x_{i+j}$.

Type IV: A has a basis $\{x_i\}$ for i an integer, and products are given by $x_i x_j = (j+\beta)x_{i+j}$ for some scalar $\beta \neq 1$.

We remark that the graded algebra of type I is just isomorphic to the algebra of Laurent polynomials, whereas the algebra of type II is just S. Gelfand's example. The algebra III is isomorphic as an algebra to the algebra II. However, the isomorphism reverses instead of preserving the grading.

As for the algebras IV, let Z denote the integers and consider the additive subgroup $Z + Z\beta$ of F and its group algebra $B = F[Z + Z\beta]$. Elements of B are linear combinations $\sum k_i x^{\delta(i)}$ where $k_i \in F$, $\delta(i) \in Z + Z\beta$, and $x^{\delta(i)}$ is identified with the group element $\delta(i)$. The mapping $' : x^{\delta} \to \delta x^{\delta - \beta}$ extends to a differentiation of the algebra B giving rise to the Novikov multiplication, fg' for $f, g \in B$. The subspace $A = F[Z + \beta]$ spanned by elements $x^{i+\beta}$ for $i \in Z$ is closed under the product fg'. Thus A is a Novikov algebra. It is easy to see that $x^{i+\beta}(x^{j+\beta})' = (j+\beta)x^{i+j+\beta}$, so A is the algebra of type IV. The Novikov algebra in this construction with $\beta = 1$ is excluded from type IV because it does not arise from a Novikov algebra A with a maximal subalgebra H with the property that A/H has a finite-dimensional irreducible submodule.

Our first result is the following:

Theorem 1. *Let A be a simple Novikov algebra over an algebraically closed field F of characteristic* 0, *and let A contain a maximal subalgebra H such that A/H has a finite-dimensional irreducible H-submodule. Then the associated graded algebra G is an algebra of type I, II, III, or IV. Conversely, each of these graded algebras satisfies the hypotheses of the theorem. The codimension of H is finite exactly when G is of type II, and in this case the codimension is* 1.

The multiplications for each of the types of graded algebras arising here are special cases of the product $x_i x_j = (\alpha j + \beta)x_{i+j}$. The Hamiltonian operators with correspond to these graded algebras are

$$H_{ij} = \sum_k ((\alpha j + \beta)u^{(1)}_{i+j} + [\alpha(i+j) + 2\beta]u^{(0)}_{i+j}\,\mathrm{d}/\mathrm{d}x), \tag{10}$$

where i, j, k run over all integers.

Let A^* denote the closure of A with respect to the topology induced by the filtration.

Theorem 2. *Let A be and algebra satisfying the hypotheses of Theorem 1, and let the associated graded algebra G be of type II, III, or IV. Further if G is of type IV, suppose that β is not a nonzero integer. Then $A^* = G^*$.*

The proofs of these two theorems are found in Section 2. In Section 3 we prove that the only product $\times$ satisfying (5) and (6) on a graded algebra of type II, or of type IV with β not an integer, is a scalar multiple of the commutator product, $x \times y = x \circ y - y \circ x$. So in these cases the Hamiltonian operator (8) can be extended to the form (4) only as

$$H_{ij} = \sum_k (\alpha(j-i)u_k^{(0)} + (j+1)u_{i+j}^{(1)} + (i+j+2)u_k^{(0)}\mathrm{d}/\mathrm{d}x).$$

When A is of type IV and β a nonzero integer, there exists a Lie product on A which is not a multiple of the commutator product.

1. Modules over finite-dimensional algebras

For our main theorems we need to develop some results about modules over finite-dimensional Novikov algebras over F. Let $R_A = \{R_x \mid x \in A\}$ where R_x is right multiplication by x.

Lemma 1.1. *Let A be a finite-dimensional Novikov algebra, let M b a finite-dimensional module for A, let $0 \neq w \in M$ be such that $Aw = 0$, and let B, C, D be ideals of A. Then*

(a) $wB \cdot C + w \cdot (BC + CB) = C \cdot wB + w \cdot (BC + CB)$. *Further,*

$$\begin{aligned} &D(wB \cdot C + w \cdot (BC + CB)) \\ &\quad \subset wB \cdot DC + w(B \cdot DC + DC \cdot B) + wC \cdot (DB + BD) \\ &\qquad + wD \cdot (BC + CB) + w \cdot D(BC + CB), \end{aligned} \tag{11}$$

$$\begin{aligned} &(wB \cdot C + w \cdot (BC + CB))D \\ &\quad \subset wB \cdot CD + w(B \cdot CD + CD \cdot B) + wD \cdot (BC + CB). \end{aligned} \tag{12}$$

(b) $C(R_A^n w) \subset R_A^{n-1}(wC) + R_A^n(wC)$.

Proof. From $wB \cdot C \subset w \cdot BC + Bw \cdot C + B \cdot wC = w \cdot BC + B \cdot wC$ and $B \cdot wC \subset w \cdot BC + wB \cdot C$ we obtain $wB \cdot C + w \cdot (BC + CB) = C \cdot wB + w \cdot (BC + CB)$. Also,

$$\begin{aligned} &D(wB \cdot C + w \cdot (BC + CB)) \\ &\quad \subset (D \cdot wB)C + (wB \cdot D)C + wB \cdot DC + wD \cdot (BC + CB) + w \cdot D(BC + CB) \\ &\quad \subset DC \cdot wB + (D \cdot wB + w \cdot (DB + BD))C + wB \cdot DC + wD \cdot (BC + CB) \\ &\qquad + w \cdot D(BC + CB) \end{aligned}$$

$$\subset wB\cdot DC + w(B\cdot DC + DC\cdot B) + wC\cdot(DB + BD)$$
$$+ wD\cdot(BC + CB) + w\cdot D(BC + CB),$$
$$(wB\cdot C + w\cdot(BC + CB))D \subset (C\cdot wB + w\cdot(BC + CB))D$$
$$\subset CD\cdot wB + wD\cdot(BC + CB)$$
$$\subset wB\cdot CD + w(B\cdot CD + CD\cdot B) + wD\cdot(BC + CB)$$

to give part (a). The case $n = 1$ of part (b) follows from part (a). For $n \geq 2$ we proceed by induction using the relation $(R_A^k w)c = R_A^k(wc)$ which follows from (2):

$$C(R_A^n w) = C\cdot(R_A^{n-1}w)A \subset C(R_A^{n-1}w)\cdot A + (R_A^{n-1}w)C\cdot A + (R_A^{n-1}w)\cdot CA$$
$$\subset R_A^{n-2}(wC) + R_A^{n-1}(wC) + (R_A^{n-1}(wC))A + (R_A^{n-1}w)C$$
$$\subset R_A^{n-1}(wC) + R_A^n(wC). \quad \square$$

Lemma 1.2. *Let A be a finite-dimensional Novikov algebra, let M be a finite-dimensional faithful irreducible module for A, and let $0 \neq w \in M$ be such that $Aw = 0$. Then $A = Fe$ where e is an idempotent, and $M = Fw$.*

Proof. Let B be a nonzero ideal of A with $B^2 = 0$. Taking $C = B$ and $D = A$ in (11) and (12), we see that $wB\cdot B$ is a submodule of M. On the other hand, taking $D = C = B$ in (11) and (12) shows that B annihilates $wB\cdot B$ on either side. Since A acts faithfully, $wB\cdot B = 0$. On the other hand, the choice $C = D = A$ in (11) and (12) establishes that $wB\cdot A$ is a submodule, and the setting $C = A$, $D = B$ in (11) and (12) show that $wB\cdot A$ is annihilated on both sides by B. Again the fidelity of the action of A implies that $wB\cdot A = 0$.

It follows from Lemma 1.1(b) with $C = A$ that the submodule generated by w is $\sum_n R_A^n w$, and this must be all of M since M is irreducible. But then

$$BM = B\left(\sum_n R_A^n w\right) \subset \sum_n R_A^n(wB) = 0,$$
$$MB = \sum_n (R_A^n w)B \subset \sum_n R_A^n(wB) = 0,$$

which contradicts the fidelity of the action of A. We conclude that A can contain no nonzero ideals which square to zero. Hence, A is a direct sum of ideals generated by orthogonal idempotents (see [15]).

Suppose that A has k orthogonal idempotents, say $e_1, e_2, \ldots, e_k$. Let $M_1 = \{v \in M \mid ve_1 = 0\}$ and observe that M_1 is a submodule since $ve_i\cdot e_1 = ve_1\cdot e_i = 0$ and $e_iv\cdot e_1 = -ve_i\cdot e_1 = 0$ for any $v \in M_1$. If $M_1 = M$, then $e_1M = e_1^2M = e_1M\cdot e_1 \subset Me_1 = 0$, which contradicts the fidelity of the action of A. Then $M_1 = 0$. Using (2) and (3) we see that $ve_1\cdot e_1 = v\cdot e_1^2 + e_1v\cdot e_1 - e_1\cdot ve_1 = ve_1$ for all $v \in M$. Thus, e_1 acts as right identity on M, and by symmetry, each of the e_i's acts like the identity on M. It follows immediately that Fw is a nonzero submodule of M, and hence $M = Fw$. Since the difference between two idempotents annihilates M, there can be only one idempotent in A. □

Proposition 1.3. *Let A be finite demensional, and let M be a finite-dimensional faithful irreducible module for A. Then* $\dim M = 1$ *and* $\dim A = 1$.

Proof. For $x \in A$, let λ_x denote the action of A on M given by $\lambda_x v = xv$ for $v \in M$. Then from the defining relations of a Novikov algebra, $[\lambda_x, \lambda_y] = \lambda_{[x,y]}$ for $x, y \in A$. This says that the operators λ_x for $x \in A$ form a Lie algebra of operators on the vector space M. Since A is a direct sum of 1-dimensional algebras modulo its solvable radical, A^- is solvable as a Lie algebra. So these operators form a solvable Lie algebra of operators. Hence by Lie's theorem, there exists a nonzero element $w \in M$ which is a common eigenvector for all of the λ_x's. If all of the λ_x's annihilate w, then our result follows from Lemma 1.2.

Thus, we may suppose that there exists $x \in A$ with $xw = w$. Then, $wA = xw \cdot A = xA \cdot w \subset Fw$, and so Fw is a submodule of M. By the irreducibility of M, we have $M = Fw$ and $\dim M = 1$. Since the kernel of the action on each side of M is of codimension no more than 1 in A, we see that $\dim A \leq 2$. Suppose first that $\dim A = 2$. Then there exists a nonzero $z \in A$ with $zw = 0$. The fidelity of the action of A implies that $wz \neq 0$, and we can normalize so that $wz = w$. Then there exists $x \in A$ with $wx = 0$ and $xw = w$. We have $zx \cdot w = zw \cdot x = 0$, and $xz \cdot w = xw \cdot z = w = xw$, showing that $zx, xz - x \in Fz$. But then $0 = xz \cdot w + z \cdot xw - zx \cdot w - x \cdot zw = xz \cdot w = xw = w$. This contradiction rules out the case where $\dim A = 2$. So $\dim A = 1$, to complete the proof. □

2. Classification theorems

We consider in this section a Novikov algebra A with a maximal subalgebra H such that A/H has a nonzero finite-dimensional H-submodule.

Lemma 2.1. *Let B be an arbitrary subalgebra of A. Then $N_A(B) = \{a \in A \mid aB + Ba \subseteq B\}$ is a subalgebra of A.*

Proof. Let $a_1, a_2 \in N_A(B)$ and $b \in B$. Then

$$(a_1a_2)b = (a_1b)a_2 \in Ba_2 \subseteq B,$$

$$b(a_1a_2) = a_1(ba_2) + (ba_1 - a_1b)a_2 \in a_1B + Ba_1 \subseteq B. \qquad \square$$

Let V be a minimal nontrivial H-submodule of A/H, and let $\tilde{V}$ be the preimage of V under $A \to A/H$, so $\tilde{V}$ is an H-bimodule, $H \subseteq \tilde{V} \subseteq A$, and $\tilde{V}/H = V$. From Lemma 2.1 it follows that V is an irreducible H-module. Otherwise $\tilde{V} \subseteq N_A(H)$, which implies that $N_A(H) = A$ and H is an ideal of A which contradicts the assumption that A/H is a nonzero H-module.

Consider the Weisfeiler filtration of A induced by H and V [14]. Let K be the kernel of the representation of H on V. Since V is finite dimensional, it follows that H/K is finite dimensional as well. Now let $K_1 = \{a \in K \mid Va + aV \in K\}$, and in general let $K_{i+1} = \{a \in K_i \mid Va + aV \in K_i\}$. We define $A_0 = H$, $A_1 = K$, and $A_i = K_{i-1}$ for $i \geq 2$. Let $\tilde{V}_i$ denote the linear span of all products of elements from $\tilde{V}$ having i factors and an arbitrary arrangement of brackets. Define $A_{-i} = \sum_{j=1}^{i} \tilde{V}_j$ for $i \geq 1$. It is easy to see that

$$\cdots \supset A_{-1} \supset A_0 \supset A_1 \supset \cdots$$

is a filtration, that is $A_iA_j \subseteq A_{i+j}$ for all i, j. Since $\sum_{-\infty}^{\infty} A_i$ is a subalgebra of A properly containing H, it follows that $A = \sum_{-\infty}^{\infty} A_i$. Let G be the graded algebra associated with the filtration above. Thus, $G = \bigoplus_{i \in Z} G_i$, where $G_i = A_i/A_{i+1}$ and the multiplication is defined by

$$(a + A_{i+1})(b + A_{j+1}) = ab + A_{i+j+1}, \quad \text{for } a \in A_i, b \in A_j.$$

Remark 2.2. Since $G_{-1} = V$ is an irreducible faithful module over $G_0 = H/K$, it follows from the results of the last section that $\dim G_{-1} = \dim G_0 = 1$. Let elements x_{-1} and x_0 span G_{-1} and G_0, respectively. There exist scalars $\alpha, \beta \in F$ such that

$$x_0^2 = \beta x_0, \quad x_0x_{-1} = (\beta - \alpha)x_{-1}.$$

Note that by the very definition of the Weisfeiler filtration the graded algebra G is transitive, that is if $x \in G_i$ for $i \geq 0$ and if $G_{-1}x = xG_{-1} = (0)$, then $x = 0$.

Our objective in the next series of lemmas is to establish that $\dim G_i = 1$ for $i \geq 1$, and that a basis $\{x_i\}$ with $x_i \in G_i$ can be chosen so that $x_ix_j = (j\alpha + \beta)x_{i+j}$ for all i and j.

Lemma 2.3. $x_{-1}x_0 = \beta x_{-1}$.

Proof. If $\alpha \neq \beta$, then

$$(\beta - \alpha)x_{-1}x_0 = (x_0x_{-1})x_0 = (x_0x_0)x_{-1} = \beta x_0x_{-1} = \beta(\beta - \alpha)x_{-1},$$

which implies $x_{-1}x_0 = \beta x_{-1}$. If $\alpha = \beta$, then $x_0x_{-1} = 0$. By the transitivity of G we may assume that $x_{-1}x_0 = \gamma x_{-1}$ where $\gamma \neq 0$. But then

$$\gamma x_{-1}x_0 = (x_{-1}x_0)x_0 = (x_0x_{-1})x_0 + x_{-1}(x_0x_0) - x_0(x_{-1}x_0)$$
$$= x_{-1}(x_0x_0) = \beta x_{-1}x_0,$$

which implies $\beta = \gamma$. $\square$

Lemma 2.4. *For any* $x_i \in G_i$,

(a) $x_i x_0 = \beta x_i$ *and*

(b) $x_0 x_i = (i\alpha + \beta)x_i$.

Proof. (a) The assertion is valid for $i = 0$, and we proceed by induction on i. For $i \geq 1$, $(x_i x_0)x_{-1} = (x_i x_{-1})x_0 = \beta x_i x_{-1}$ by the inductive assumption. Hence,

$$(x_i x_0 - \beta x_i)x_{-1} = 0.$$

We also have

$$x_{-1}(x_i x_0) = x_i(x_{-1}x_0) + (x_{-1}x_i - x_i x_{-1})x_0$$
$$= \beta x_i x_{-1} + \beta(x_{-1}x_i - x_i x_{-1}) = \beta x_{-1}x_i$$

by Lemma 2.3 and the inductive assumption. Thus,

$$x_{-1}(x_i x_0 - \beta x_i) = 0.$$

By the transitivity of G we conclude that $x_i x_0 = \beta x_i$.

(b) Again the assertion is valid for $i = 0$, and we proceed by induction on i. We have

$$x_{-1}(x_0 x_i) = x_0(x_{-1}x_i) + (x_{-1}x_0 - x_0 x_{-1})x_i$$
$$= ((i-1)\alpha + \beta)x_{-1}x_i + \alpha x_{-1}x_i = (i\alpha + \beta)x_{-1}x_i,$$

so

$$x_{-1}(x_0 x_i - (i\alpha + \beta)x_i) = 0.$$

On the other hand,

$$(x_0 x_i)x_{-1} = (x_i x_0)x_{-1} + x_0(x_i x_{-1}) - x_i(x_0 x_{-1})$$
$$= \beta x_i x_{-1} + ((i-1)\alpha + \beta)x_i x_{-1} - (-\alpha + \beta)x_i x_{-1} = (i\alpha + \beta)x_i x_{-1},$$

so

$$(x_0 x_i - (i\alpha + \beta)x_i)_{x-1} = 0.$$

Again by the transitivity of G we conclude that $x_0 x_i = (i\alpha + \beta)x_i$. □

Lemma 2.5. $\dim G_i \leq 1$ *for any* $i \geq 1$.

The proof of this lemma will be given in 4 steps. For any $a \in A$, let $\rho(a)$ denote the right ideal of A generated by a.

Step 1: $\dim G_i \leq 1$ *for* $i \geq 1$ *if* $x_0 \in \rho(x_{-1})$ *and* $x_{-1} \in \rho(x_0)$.

Proof. It follows from the hypotheses of this step that if $a \in G$ satisfies $x_{-1}a = 0$ then $\rho(x_{-1})a = 0$ and hence $x_0 a = 0$. Similarly, $x_0 a = 0$ implies $x_{-1}a = 0$. Now let i be the first natural number such that $\dim G_i \geq 2$. In particular, $i \geq 1$ and $\dim G_{i-1} \leq 1$. Then there exists a nonzero element $a \in G_i$ such that $x_{-1}a = 0$. By the remark above, this implies $x_0 a = 0$, so $i\alpha + \beta = 0$. Then $x_0 G_i = (0)$ and hence $x_{-1} G_i = (0)$. Again since $\dim G_i > \dim G_{i-1}$, there exists a nonzero element $b \in G_i$ such that $bx_{-1} = 0$. This contradicts the transitivity of G. □

Step 2: $\dim G_1 \leq 1$.

Proof. If $\dim G_1 \geq 2$, there exists a nonzero element $a \in G_1$ such that $ax_{-1} = 0$. By the transitivity of G we have $x_{-1}a \neq 0$, and moreover we can assume that $x_{-1}a = x_0$. Hence, $x_0 \in \rho(x_{-1})$. Similarly, there exists $0 \neq b \in G_1$ such that $x_{-1}b = 0$. In this case we may assume that $bx_{-1} \neq 0$, and moreover that $bx_{-1} = x_0$. We have

$$x_0 b = (x_{-1}a)b = (x_{-1}b)a = 0\,.$$

On the other hand, $x_0 b = (\alpha + \beta)b$. Hence $\alpha + \beta = 0$, which implies $\alpha - \beta \neq 0$. Now, $x_0 x_{-1} = (-\alpha + \beta)x_{-1}$, and thus $x_{-1} \in \rho(x_0)$. By Step 1, $\dim G_1 \leq 1$. □

Step 3: $\dim G_i \leq 1$ *for* $i \geq 1$ *if* $\alpha \neq \beta$.

Proof. For any $a_i \in G_i$ with $i \geq 1$ we have

$$(-\alpha + \beta)x_{-1}a_i = (x_0 x_{-1})a_i = (x_0 a_i)x_{-1} = (i\alpha + \beta)a_i x_{-1}\,. \tag{13}$$

This implies that if $a_i \neq 0$ then $a_i x_{-1} \neq 0$. Indeed, if $a_i x_{-1} = 0$ then $x_{-1}a_i = 0$, which contradicts the transitivity of G.

Now let $i \geq 1$ be the first integer such that $\dim G_i \geq 2$. Then there exists $0 \neq a_i \in G_i$ with $a_i x_{-1} = 0$. Thus, $x_{-1}a_i \neq 0$. But from what we have proved above it follows that

$$R(x_{-1})^{i-1}(x_{-1}a_i) = (\dots(x_{-1}a_i)x_{-1})\dots)x_{-1} \neq 0\,.$$

Hence, $x_0 \in \rho(x_{-1})$. On the other hand, $x_0 x_{-1} = (-\alpha + \beta)x_{-1}$, and hence $x_{-1} \in \rho(x_0)$. Then $\dim G_i \leq 1$ by Step 1. □

Step 4: $\dim G_i \leq 1$ *for* $i \geq 1$ *if* $\alpha = \beta$.

Proof. Choose an arbitrary nonzero element $x_1 \in G_1$. From (13) it follows that $x_1 x_{-1} = 0$, and hence $x_{-1}x_1 = \gamma x_0$ for some $\gamma \neq 0$. Suppose that $\dim G_i \geq 2$ but $\dim G_{i-1} \leq 1$ for some $i \geq 2$. For an arbitrary element $x_i \in G_i$ we have

$$(x_{-1}x_i)x_1 = (x_{-1}x_1)x_i = \gamma x_0 x_i = \gamma(i\alpha + \beta)x_i\,,$$

where $\gamma(i\alpha + \beta) \neq 0$. Since $\dim G_i > \dim G_{i-1}$, there exists a nonzero $x_i \in G_i$ such that $x_{-1}x_i = 0$. But then $0 = (x_{-1}x_i)x_1 = \gamma(i\alpha + \beta)x_i$, a contradiction. □

This completes the proof of Lemma 2.5. □

Lemma 2.6. *There exists a nonzero element* $x_1 \in G_1$ *such that* $x_i x_j = (j\alpha + \beta)x_{i+j}$ *whenever* $-1 \leq i, j, i + j \leq 1$.

Proof. From the choice of α, β and from Lemmas 2.3 and 2.4, it follows that for an arbitrary element $x_1 \in G_1$ the assertion of the lemma is true for all products except possibly $x_{-1}x_1$ and x_1x_{-1}. By (13) we have

$$(-\alpha+\beta)x_{-1}x_1 = (\alpha+\beta)x_1x_{-1}. \tag{14}$$

Suppose that $\alpha-\beta \neq 0$ and $\alpha+\beta \neq 0$. At least one of the products $x_1x_{-1}, x_{-1}x_1$ is nonzero by transitivity, and hence both are nonzero by (14). We can divide x_1 by a scalar so that $x_{-1}x_1 = (\alpha+\beta)x_0$. Then $x_1x_{-1} = (-\alpha+\beta)x_0$ by (14).

Now let $\alpha=\beta$. Then $\alpha+\beta \neq 0$ and thus $x_1x_{-1}=0$. Then $x_{-1}x_1 \neq 0$ and we can normalize x_1 so that $x_{-1}x_1 = (\alpha+\beta)x_0$. Similarly, if $\alpha+\beta=0$ then $\alpha-\beta\neq 0$ and therefore $x_{-1}x_1=0$. We can then normalize x_1 so that $x_1x_{-1} = (-\alpha+\beta)x_0$. □

Lemma 2.7. *Let $\alpha+\beta \neq 0$, and let $x_0 \in G_0$ and $x_1 \in G_1$ be given. Define the elements $x_k \in G_k$ for $k \geq 2$ inductively by $x_k = (\alpha+\beta)^{-1}x_{k-1}x_1$. Then*

(a) *$x_{-1}x_k = (k\alpha+\beta)x_{k-1}$ and $x_kx_{-1} = (-\alpha+\beta)x_{k-1}$ for $k \geq 0$.*

(b) *$x_ix_j = (j\alpha+\beta)x_{i+j}$ for $i,j \geq 0$.*

Proof. (a) The assertion follows from Lemma 2.6 for $k=0,1$, and we proceed by induction on k. If $k \geq 2$,

$$\begin{aligned} x_{-1}(x_{k-1}x_{-1}) &= x_{k-1}(x_{-1}x_1) + (x_{-1}x_{k-1} - x_{k-1}x_{-1})x_1 \\ &= (\alpha+\beta)x_{k-1}x_0 + k\alpha x_{k-2}x_1 = \beta(\alpha+\beta)x_{k-1} + k\alpha(\alpha+\beta)x_{k-1} \\ &= (k\alpha+\beta)(\alpha+\beta)x_{k-1} \end{aligned}$$

by the inductive hypothesis. Since $x_k = (\alpha+\beta)^{-1}x_{k-1}x_1$, it follows that $x_{-1}x_k = (k\alpha+\beta)x_{k-1}$. Now

$$(x_{k-1}x_1)x_{-1} = (x_{k-1}x_{-1})x_1 = (-\alpha+\beta)x_{k-2}x_1 = (-\alpha+\beta)(\alpha+\beta)x_{k-1}.$$

Hence, $x_kx_{-1} = (-\alpha+\beta)x_{k-1}$.

(b) If $i=0$ or $j=0$ the assertion follows from Lemma 2.4. Thus, we may assume that $i \geq 1$ and thus $x_i = (\alpha+\beta)^{-1}x_{i-1}x_1$. We have

$$(x_{i-1}x_1)x_j = (x_{i-1}x_j)x_1 = (j\alpha+\beta)x_{i+j-1}x_1$$

by the inductive assumption. □

Corollary. *If $\alpha+\beta \neq 0$, then x_k spans G_k for $k \geq 1$.*

Now let us examine the positive part of G under the assumption that $\alpha+\beta=0$. Since $x_1^2=0$ in this case, one possibility is that $G_2=0$, which implies that $G_i=0$ for $i \geq 2$ by transitivity. Suppose that $G_2 \neq 0$. We observe for nonzero $x_i \in G_i$ where $i \geq 2$ that (13) implies that $x_{-1}x_i = 0$ if and only if $x_ix_{-1}=0$ since $-\alpha+\beta \neq 0 \neq i\alpha+\beta$.

By the transitivity of G, $x_{-1}x_i$ and x_ix_{-1} are both nonzero. Since $G_2 \neq 0$, our observation implies that $x_{-1}G_2 \neq (0)$. Choose $x_2 \in G_2$ such that

$$x_{-1}x_2 = (2\alpha + \beta)x_1 . \tag{15}$$

Lemma 2.8. *Let $\alpha + \beta = 0$ and let $x_2 \in G_2$ satisfy* (15). *Define $x_k \in G_k$ for $k \geq 3$ by $x_k = (k\alpha + \beta)^{-1}x_1x_{k-1}$. Then*

(a) *$x_{-1}x_k = (k\alpha + \beta)x_{k-1}$ and $x_kx_{-1} = (-\alpha + \beta)x_{k-1}$ for $k \geq 0$;*

(b) *$x_ix_j = (j\alpha + \beta)x_{i+j}$ for $i, j \geq 0$.*

Proof. (a) Again we may assume that $k \geq 2$. By (13) the assertions $x_{-1}x_k = (k\alpha + \beta)x_{k-1}$ and $x_kx_{-1} = (-\alpha + \beta)x_{k-1}$ are equivalent. Hence by the choice of x_2, both assertions are valid for $k = 2$. For $k \geq 3$, we have

$$\begin{aligned}(x_1x_{k-1})x_{-1} &= (x_1x_{-1})x_{k-1} = (-\alpha + \beta)x_0x_{k-1} \\ &= (-\alpha + \beta)((k-1)\alpha + \beta)x_{k-1} .\end{aligned}$$

It remains to note that $((k-1)\alpha + \beta)x_1x_{k-1} = x_k$.

(b) Again if $i = 0$ or $j = 0$, the assertion is valid. Thus, we assume that $i \geq 1$ and $j \geq 1$. By our remarks just before Lemma 2.7, it is sufficient to prove that $(x_ix_j)x_{-1} = (j\alpha + \beta)x_{i+j}x_{-1}$. By part (a) and by the inductive hypothesis,

$$(x_ix_j)x_{-1} = (x_ix_{-1})x_j = (-\alpha + \beta)x_{i-1}x_j = (-\alpha + \beta)(j\alpha + \beta)x_{i+j-1}$$

and $(j\alpha + \beta)x_{i+j}x_{-1} = (j\alpha + \beta)(-\alpha + \beta)x_{i+j-1}$ by part (a). This finishes the proof of the lemma. □

Corollary. *For any $k \geq 0$, $x_k \neq 0$, and thus x_k spans G_k.*

Now let us see what happens with the negative part of G. By the very definition of the Weisfeiler filtration, $\sum_{i \leq -1} G_i$ is generated by G_{-1}, which is the linear span of x_{-1}. If $\alpha = \beta$, then $x_{-1}^2 = 0$, and so $G_i = (0)$ for $i < -1$. Hence, A_{-1} is a subalgebra, and so $A = A_{-1}$ by the maximality of H. This shows that H has codimension 1 in A when $\alpha = \beta$. We turn to the case $\alpha \neq \beta$.

Lemma 2.9. *If $\alpha \neq \beta$, define $x_i = (-\alpha + \beta)^{-1}x_{i+1}x_{-1}$ for $i \leq -2$. Then, for $i < 0$,* (12) *holds and*

$$\begin{aligned}x_0x_i &= (i\alpha + \beta)x_i, & x_ix_0 &= \beta x_i, \\ x_1x_i &= (i\alpha + \beta)x_{i+1}, & x_ix_1 &= (\alpha + \beta)x_{i+1} .\end{aligned}$$

Proof. The proof of the first two relations proceeds by induction on $|i|$ in a manner identical to the proof of Lemma 2.4. From the first relation it follows that (13) holds for $i < 0$. The fourth relation follows by definition, and the third one from this using ((13). □

Lemma 2.10. *For arbitrary integers i, j, we have*

$$x_i x_j = (j\alpha + \beta)x_{i+j}.$$

Proof. If $i \geq -1$ and $j \geq -1$, the result follows from Lemmas 2.7 and 2.8. Thus, we may assume that at least one of i, j is less than -1, and hence that $\alpha \neq \beta$. We will use induction on $|i| + |j|$.

Let $i \leq -2$. Then $x_i = (-\alpha + \beta)^{-1} x_{i+1} x_{-1}$ and we have

$$(x_{i+1}x_{-1})x_j = (x_{i+1}x_j)x_{-1} = (j\alpha + \beta)x_{i+j+1}x_{-1} = (j\alpha + \beta)(-\alpha + \beta)x_{i+j}$$

by the inductive assumption. Hence, we can assume that $i \geq -1$ and $j \leq -2$.

If $i = -1$, then by (13),

$$x_{-1}x_j = (-\alpha + \beta)^{-1}(j\alpha + \beta)x_j x_{-1} = (j\alpha + \beta)x_{j-1}.$$

The assertion is true for $i = 0$. Let $i \geq 1$. Using $x_j = (-\alpha + \beta)x_{j+1}x_{-1}$, we have

$$x_i(x_{j+1}x_{-1}) = x_{j+1}(x_i x_{-1}) + (x_i x_{j+1} - x_{j+1}x_i)x_{-1}.$$

Since $|j + 1| < |j|$ and $|i - 1| < |i|$, the induction assumption implies

$$x_{j+1}(x_i x_{-1}) = (-\alpha + \beta)((i - 1)\alpha + \beta)x_{i+j},$$

$$x_i x_{j+1} - x_{j+1}x_i = (j + 1 - i)\alpha x_{i+j+1}.$$

Hence $x_i(x_{j+1}x_{-1}) = (-\alpha + \beta)(j\alpha + \beta)x_{i+j}$, which proves the lemma. □

The definition of the Weisfeiler filtration implies that the negative part $G_{-1} + G_{-2} + \cdots$ of G is generated by x_{-1}, giving the following:

Corollary. *G_i is spanned by x_i even for $i < 0$.*

Proof of Theorem 1. We have shown that A has a basis $\{x_i\}$ satisfying $x_i x_j = (j\alpha + \beta)x_{i+j}$. If $\alpha = 0 = \beta$, A_{-1} is a subalgebra properly larger than $H = A_0$, so that $A = A_{-1}$. But because the products in G are all zero, H is an ideal of A, to contradict simplicity. Thus, the case $\alpha = 0 = \beta$ does not arise here. If $\alpha = 0$ and $\beta \neq 0$, we can replace each x_i by β^{-1} times itself to obtain type I.

We may suppose then for the remainder of the proof that $\alpha \neq 0$. Replacing each x_i by α^{-1} times itself, and replacing β by $\beta\alpha$, our product reduces to $x_i x_j = (j + \beta)x_{i+j}$. If $\beta = 1$, then A_{-1} is a subalgebra properly containing H, and so $A = A_{-1}$. This gives an algebra of type II. If $\beta = -1$, then $x_1^2 = 0$. In this case it is possible that $x_i = 0$ for $i \geq 2$, which leads to type III. If any $x_i \neq 0$ for $i \geq 2$, it is easy to see that all of them are nonzero and that we have an algebra of type IV. When $\beta \neq 1, -1$, neither x_1 nor x_{-1} is nilpotent, so that all x_i's are nonzero for $i \in Z$. So, this case leads only to type IV. □

Proof of Theorem 2. Let the graded algebra G have the basis $\{x_i\}$ with products given by $x_i x_j = (j+\beta)x_{i+j}$ (i runs over all integers Z when A is of type IV, $i \geq -1$ for type II, and $i \leq 1$ for type III). We consider first types II and IV. We will show that A^* has a dense subalgebra which is (topologically) isomorphic to G. This will imply the assertion of the theorem. Choose a set of elements $\{\tilde{a}_i\}$ in A which are preimages of the elements $\{x_i\}$ in G, so that $\tilde{a}_i \in A_i$ but $\tilde{a}_i \notin A_{i+1}$, and $\tilde{a}_i\tilde{a}_j \equiv (j+\beta)\tilde{a}_{i+j} (\operatorname{mod} A_{i+j+1})$.

Our first aim is to find an element $a_0 \in A_0^*$ which is a preimage of x_0 in G and which has the property that $a_0^2 = \beta a_0$. We will use induction on n to construct a sequence of elements $a_{0,n}$ in A such that $a_{0,1} = \tilde{a}_0$, $a_{0,n}^2 - \beta a_{0,n} \in A_n$, and $a_{0,n+1} \equiv a_{0,n} (\operatorname{mod} A_n)$. Supposee that elements $a_{0,1}, \ldots, a_{0,n}$ have been constructed. For any $\gamma \in F$ we have

$$\begin{aligned}
&(a_{0,n} + \gamma\tilde{a}_n)^2 - \beta(a_{0,n} + \gamma\tilde{a}_n) \\
&\quad = (a_{0,n}^2 - \beta a_{0,n}) + \gamma(a_{0,n}\tilde{a}_n + \tilde{a}_n a_{0,n} - \beta a_n) + \gamma^2\tilde{a}_n^2 \\
&\quad \equiv (a_{0,n}^2 - \beta a_{0,n}) + \gamma(n+\beta)\tilde{a}_n (\operatorname{mod} A_{n+1}).
\end{aligned}$$

We can choose γ so that the right side lies in A_{n+1} since $n + \beta \neq 0$ by the hypotheses of Theorem 2. Let $a_{0,n+1} = a_{0,n} + \gamma\tilde{a}_n$. The sequence $\{a_{0,n}\}$ has been defined, and the element $a_0 = \lim_{n\to\infty} a_{0,n}$ has the required property.

Next we construct a preimage $a_{-1} \in A_{-1}^*$ of x_{-1} such that $a_0 a_{-1} = (-1+\beta)a_{-1}$. To do this we shall again construct a sequence of elements $a_{-1,n}$ in A such that $a_{-1,1} = \tilde{a}_{-1}$, $a_0 a_{-1,n} - (-1+\beta)a_{-1,n} \in A_n^*$ and $a_{-1,n+1} \equiv a_{-1,n} (\operatorname{mod} A_n)$. Suppose that the elements $a_{-1,1}, \ldots, a_{-1,n}$ have been constructed. For any scalar $\gamma \in F$ we have

$$\begin{aligned}
&a_0(a_{-1,n} + \gamma\tilde{a}_n) - (-1+\beta)(a_{-1,n} + \gamma\tilde{a}_n) \\
&\quad \equiv (a_0 a_{-1,n} - (-1+\beta)a_{-1,n}) + \gamma(n+1)\tilde{a}_n (\operatorname{mod} A_{n+1}^*).
\end{aligned}$$

Find γ so that the right side lies in A_{n+1}^*, and let $a_{-1,n+1} = a_{-1,n} + \gamma\tilde{a}_n$. Then define $a_{-1} = \lim_{n\to\infty} a_{-1,n}$.

In the same way we can find a preimage a_1 of x_1 with the property that $a_0 a_1 = (1+\beta)a_1$.

Let $L(a_0), R(a_0)$ denote, respectively, the operators of left and right multiplication by the element a_0. It is not difficult to prove that the operator $L(a_0): A_0^* \to A_0^*$ is a bijection. Hence, an arbitrary element $a \in A_0^*$ can be represented as $a = a_0 b$ for some $b \in A_0^*$. Then $aa_0 = (a_0 b)a_0 = (a_0 a_0)b = \beta a_0 b = \beta a$. We have shown that $(R(a_0) - \beta)A_0^* = (0)$, and in particular, $a_1 a_0 = \beta a_1$. Since $(R(a_0) - \beta)A_{-1}^* \subseteq A_0$, we have $(R(a_0) - \beta)^2 A_{-1}^* = (0)$, giving $(a_{-1}a_0)a_0 = 2\beta a_{-1}a_0 - \beta^2 a_{-1}$. The calculation

$$\begin{aligned}
a_0(a_{-1}a_0) &= a_{-1}(a_0 a_0) + (a_0 a_{-1} - a_{-1}a_0)a_0 \\
&= \beta a_{-1}a_0 + (-1+\beta)a_{-1}a_0 - (a_{-1}a_0)a_0 \\
&= \beta a_{-1}a_0 + (-1+\beta)a_{-1}a_0 - 2\beta a_{-1}a_0 + \beta^2 a_{-1} = -a_{-1}a_0 + \beta^2 a_{-1}
\end{aligned}$$

shows that $(L(a_0) + 1)a_{-1}a_0 = \beta^2 a_{-1}$. We have $a_{-1}a_0 = \beta a_{-1} + a^*$ where $a^* \in A_0^*$. Since $(L(a_0) + 1)a_{-1} = \beta a_{-1}$, it follows that $(L(a_0) + 1)a^* = 0$. In view of the restrictions we have imposed on β, the operator $L(a_0) + 1$ is invertible on A_0^*. Thus $a^* = 0$ and $a_{-1}a_0 = \beta a_{-1}$.

Similarly we have $a_{-1}a_1 = (1 + \beta)a_0 + a^*$, for some $a^* \in A_1^*$. Now

$$a_0(a_{-1}a_1) = a_{-1}(a_0a_1) + (a_0a_{-1} - a_{-1}a_0)a_1 = \beta a_{-1}a_1,$$

or in operator form, $(L(a_0) - \beta)(a_{-1}a_1) = 0$, which implies $(L(a_0) - \beta)a^* = 0$. Since the operator $L(a_0) - \beta$ is bijective on A_1^*, it follows that $a^* = 0$ and $a_{-1}a_1 = (1 + \beta)a_0$. Then the formula (13) implies that

$$(1 + \beta)a_1a_{-1} = (-1 + \beta)a_{-1}a_1 = (1 + \beta)(-1 + \beta)a_0.$$

Since $\beta \neq -1$ by assumption, we conclude that $a_1a_{-1} = (-1 + \beta)a_0$. It is easy to see that the elements a_{-1}, a_0, a_1 generate a subalgebra which is dense in A^* and topologically isomorphic to G.

Now let G be of type III. We shall prove that in this case $A \cong G$. Let elements $a_i \in A$ for $i \leq 1$ be preimages of the elements x_i. Since $A_2 = (0)$, it follows that $a_1a_0 = -a_1$ and $a_0a_1 = 0$. Suppose that $a_0^2 = -a_0 + \gamma a_1$ and $a_1a_{-1} = -2a_0 + \xi a_1$ for $\gamma, \xi \in F$. Then

$$(a_1a_{-1})a_0 = (-2a_0 + \xi a_1)a_0 = -2(-a_0 + \gamma a_1) - \xi a_1.$$

On the other hand,

$$(a_1a_{-1})a_0 = (a_1a_0)a_{-1} = -a_1a_{-1} = 2a_0 - \xi a_1.$$

We conclude that $\gamma = 0$ and $a_0^2 = -a_0$.

Let $a_0a_{-1} = -2a_{-1} + \eta a_0 + \varepsilon a_1$ for $\eta, \varepsilon \in F$. The element $a_{-1} - \eta a_0 - \frac{1}{2}\varepsilon a_1$ is a preimage of x_{-1} as well. Taking $a_{-1} - \eta a_0 - (\frac{1}{2})\varepsilon a_1$ instead of a_{-1}, we get $a_0a_1 = -2a_{-1}$. Now

$$-2a_1a_{-1} = a_1(a_0a_{-1}) = a_0(a_1a_{-1}) + (a_1a_0 - a_0a_1)a_{-1}.$$

Since $a_1a_{-1} = -2a_0 + \xi a_1$ and $a_0a_1 = 0$, the right side is equal to $-2a_0^2 - a_1a_{-1} = 2a_0 - a_1a_{-1}$. Therefore, $a_1a_{-1} = -2a_0$. Also

$$-2a_{-1}a_1 = (a_0a_{-1})a_1 = (a_0a_1)a_{-1} = 0.$$

$$-2a_{-1}a_0 = (a_0a_{-1})a_0 = a_0^2a_{-1} = -a_0a_{-1} = 2a_{-1},$$

so $a_{-1}a_1 = 0$ and $a_{-1}a_0 = -a_{-1}$. It is not difficult to see that the elements a_{-1}, a_0, a_1 generate A and that the mapping $a_i \to x_i$ for $i = -1, 0, 1$ is extendable to an isomorphism. □

For algebras of type I, or type IV with $\beta = 0, -1, -2, \ldots$, the assertion of the theorem is not valid. To see this for type I, consider the algebra with basis $\{x_i\}_{i \in Z}$ and the multiplication

$$x_ix_j = x_{i+j} + (j\alpha + \beta)x_{i+j+1},$$

where $\alpha, \beta \in F$ and $\alpha \neq 0$. Then the graded algebra G associated with A is of type I. But A is not commutative, so is not embeddable in G^*.

Let B be the algebra with the basis $\{x_i\}_{i \in Z}$ and the multiplication

$$x_i x_j = (j - k)x_{i+j} + x_{i+j+2k+1},$$

where k is a fixed nonnegative integer. Then B is a Novikov algebra with associated graded algebra G of type IV (with $\beta = -k$). It is not difficult to check that G^* contains a nonzero element x_0 which squares to a scalar multiple of itself, whereas B^* contains no such element. Thus, B^* and G^* are not isomorphic.

3. Lie products on Novikov algebras

In this section we find all products $\times$ defined on a Novikov algebra A of type II, and for certain algebras of type IV, with the property that

$$(wu) \times v - (wv) \times u + (w \times u)v - (w \times v)u - w(u \times v) = 0, \tag{16}$$

and with the property that A is a Lie algebra under $\times$. Such a product $\times$ on A will be called a Lie algebra structure defined on A. We also exhibit a counterexample for the remaining algebras of type IV. Since type III algebras are isomorphic to type II algebras (they differ abstractly only in having different filtrations), the answer for type III is the same as for type II.

Theorem 3.1. *The only Lie algebra structure possible on a Novikov algebra A of type II, or of type IV when β is not a nonzero integer, is a scalar multiple of the structure A^-.*

Proof. We work with the basis $\{x_i\}$ of A where i ranges over the integers ≥ -1 for type II and over all integers for type IV. Let $\times$ be a Lie structure on A, and suppose that $x_i \times x_j = \sum_k a_{ijk} x_k$. Since $\times$ is anticommutative, $a_{jik} = a_{ijk}$. If A is of type II, we take $a_{ijk} = 0$ if either $i < -1$ or $j < -1$. Substituting $u = x_i$, $v = x_j$, $w = x_k$ in (16) gives

$$\begin{aligned} 0 &= (i + \beta)x_{k+i} \times x_j - (j + \beta)x_{k+j} \times x_i + \sum_l a_{kil} x_l x_j \\ &\quad - \sum_l a_{kjl} x_l x_i - x_k \sum_l a_{ijl} x_l \\ &= (i + \beta)\sum_l a_{k+i,j,l} x_l - (j + \beta)\sum_l a_{k+j,i,l} x_l + (j + \beta)\sum_l a_{kil} x_{l+j} \\ &\quad - (i + \beta)\sum_l a_{kjl} x_{l+i} - (l + \beta)\sum_l a_{ijl} x_{l+k}. \end{aligned}$$

The coefficient of x_l is

$$\begin{aligned} 0 &= (i + \beta)a_{k+i,j,l} - (j + \beta)a_{k+j,i,l} + (j + \beta)a_{k,i,l-j} \\ &\quad - (i + \beta)a_{k,j,l-i} - (l - k + \beta)a_{i,j,l-k}. \end{aligned} \tag{17}$$

Setting $j = 0$ gives

$$0 = (i + \beta)a_{k+i,0,l} - (i + \beta)a_{k,0,l-i} - (l - k + \beta)a_{i,0,l-k}, \tag{18}$$

and setting $k = 0$ in this gives

$$0 = (i + \beta)a_{i,0,l} - (i + \beta)a_{0,0,l-i} - (l + \beta)a_{i,0,l} = (i - l)a_{i0l},$$

using the anticommutativity of $\times$. Thus, $a_{i0l} = 0$ except when $l = i$. Hence, $x_i \times x_0 = a_{i0i}x_i$. Setting $l = i + k$ in (18) and dividing by $i + \beta$, we obtain

$$a_{k+i,0,k+i} = a_{k,0,k} + a_{i,0,i} \tag{19}$$

for $i \neq -\beta$. If $i = -\beta$ and $k \neq -\beta$, we can reverse the roles of i and k to see that (19) still holds. Thus, (19) holds when A is of type II (and hence $\beta = 1$). When A is of type IV, the validity of (19) when $i = -\beta = k \neq 0$ follows from the cases of (16) that we have already established, since

$$\begin{aligned} a_{-\beta,0,-\beta} + a_{-\beta,0,-\beta} &= a_{-\beta,0,-\beta} + a_{\beta,0,\beta} + a_{-2\beta,0,-2\beta} \\ &= a_{0,0,0} + a_{-2\beta,0,-2\beta} = a_{-2\beta,0,-2\beta}. \end{aligned}$$

Thus $a_{i,0,i} = ia_{1,0,1}$ for all $i \in Z$.

Now taking $k = 0$ in (17) gives

$$\begin{aligned} 0 &= (i + \beta)a_{i,j,l} - (j + \beta)a_{j,i,l} + (j + \beta)a_{0,i,l-j} - (i + \beta)a_{0,j,l-i} - (l + \beta)a_{i,j,l} \\ &= (i + j - l + \beta)a_{i,j,l} + (j + \beta)a_{0,i,l-j} - (i + \beta)_{0,j,l-i}. \end{aligned} \tag{20}$$

The case when $l = i + j$ is

$$\beta a_{i,j,i+j} = (j + \beta)a_{i,0,i} - (i + \beta)a_{j,0,j}, \tag{21}$$

and for $l \neq i + j$, Eq. (20) becomes $0 = (i + j - l + \beta)a_{i,j,l}$. Thus, $a_{ijl} = 0$ except when $l = i + j$ or when $l = i + j + \beta$. When $\beta \neq 0$, we deduce from (21) that $a_{i,j,i+j} = (i - j)a_{1,0,1}$ for all i, j. If β is not an integer, then l is never $i + j + \beta$, and so the theorem holds. We are left with the case when A is of type II and hence $\beta = 1$.

Writing $c_{ij} = a_{i,j,i+j+1}$, Eq. (17) with $l = i + j + k + 1$ becomes

$$0 = (i + 1)c_{k+i,j} - (j + 1)c_{k+j,i} + (j + 1)c_{ki} - (i + 1)c_{kj} - (i + j + 2)c_{ij}. \tag{22}$$

Setting $j = -1$, we obtain $0 = (i + 1)c_{k+i,-1} - (i + 1)c_{k,-1} - (i + 1)c_{i,-1}$, or

$$c_{k+i,-1} = c_{k,-1} + c_{i,-1}$$

for $i \neq -1$. When $k = -1$ we obtain $c_{i-1,-1} = c_{i,-1}$. Since $c_{0,-1} = 0$ by the last paragraph, we have $c_{n,-1} = 0$ for any positive integer n by induction. For $n < 0$, we also have $c_{n,-1} = 0$ since $c_{n,-1} = c_{-n,-1} - c_{-2n,-1} = 0$. Then $k = -1$ in (22) yields

$$0 = (i+1)c_{i-1,j} - (j+1)c_{j-1,i} + (j+1)c_{-1,i} - (i+1)c_{-1,j} - (i+j+2)c_{ij}$$

or

$$\begin{aligned}(i+j+2)c_{ij} &= (i+1)c_{i-1,j} - (j+1)c_{j-1,i} + (j+1)c_{-1,i} - (i+1)c_{-1,j} \\ &= (i+1)c_{i-1,j} - (j+1)c_{j-1,i}.\end{aligned}$$

It is easy to deduce from this using induction that $c_{nm} = 0$ for any positive integers n, m. This completes the case when A has type II. □

If A is an algebra of type IV with β a nonzero integer, then A has Lie products $\times$ which satisfy (16) and which are not scalar multiples of the commutator. For example, let $x_i \times x_{-2\beta} = -x_{-2\beta} \times x_i = (i+\beta)x_{i-\beta}$ where $i \neq 0, \ -2\beta$, let $x_i \times x_{-i-2\beta} = (-i-\beta)x_{-\beta}$ where $i \neq 0, \ -2\beta$, and let all other products $x_i \times x_j$ be zero. It can be verified that the product $\times$ defined in this way satisfies the Jacobi identity and the identity (16).

Acknowledgements

The authors thank the referee for many useful comments.

References

[1] A.A. Balinskii and S.P. Novikov, Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras, Dokl. Akad Nauk SSSR 283 (1985), 1036–1039; English transl. Soviet Math. Dokl. 32 (1985) 228–231.

[2] V.P. Cherkashin, Left-symmetric algebras with commuting right multiplications, Vestnik Moscov. Univ. Ser. I Mat. Mekh. 1988, (5), 47–50 (in Russian).

[3] V.T. Filippov, A class of simple nonassociative algebras, Mat. Zametiki 45 (1989) 101–105.

[4] I.M. Gel'fand and L.A. Dikii, Asymptotics of the resolvent of Sturm-Liouville equations and the algebra of Korteweg-de Vries equations, Uspekhi Mat. Nauk 30 (5), (1975) 67–100.

[5] I.M. Gel'fand and L.A. Dikii, The structure of a Lie algebra on the formal calculus of variations, Funktsional Anal. i. Prilozhen, 10 (4) (1976) 13–29.

[6] I.M. Gelfand and I.Ya. Dorfman, Hamiltonian operators and related algebraic structures, Funktsional Anal. i. Prilozhen, 13 (1979), 13–30; English Transl. Funct. Anal. Appl. 13 (1979) 248–262.

[7] V. Guillemin, Infinite dimensional primitive Lie algebras, J. Differential Geom. 4 (1970) 257–282.

[8] V.G. Kac, Simple graded Lie algebras of finite growth, Math. USSR Izv. 2 (1968) 1271–1311.

[9] B.A. Kuperschmidt and Yu.I. Manin, Equations of long waves with a free surface, I, Conservation Laws and solutions, Funkktisional Anal. i. Prilozhen, 11, (1977) 31–42; II Hamiltonian structure and higher equations, Funktsional Anal. Prilozhen 12 (1978) 25–32.

[10] S.P. Novikov, The geometry of conservative systems of hydrodynamical type. The method of averaging for field theoretical systems, Uspekhi Mat. Nauk. 40 (1985) 79–89; English Transl. Russian Math. Surveys 40 (1985) 85–98.

[11] J.M. Osborn, Modules for Novikov algebras, Proceedings of The II International Conference on Algebra, Barnaul, 1991.
[12] J.M. Osborn, Novikov algebras, Nova J. Algebra Geom. 1 (1992) 1–14.
[13] J.M. Osborn, Simple Novikov algebras with an idempotent, Comm. Algebra 20 (1992) 2729–2753.
[14] B.Yu. Weisfeiler, On filtered Lie algebras and their associated graded algebras, Funktsional Anal. i. Prilozhen 2 (1968) 94–95; English transl. Funct. Anal. Appl. 2 (1968) 88–89.
[15] E.I. Zelmanov, On a class of local translation invariant Lie algebras, Soviet Math. Dokl. 35 (1987) 216–218. English transl. Soviet Math. Dokl. 35 (1987) 216–218.