Journal of Algebra 337 (2011) 126–140

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The isomorphism problem for universal enveloping algebras of nilpotent Lie algebras ☆

Csaba Schneider [a,b], Hamid Usefi [c,*]

[a] Centro de Álgebra da Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal
[b] Informatics Research Laboratory, Computer and Automation Research Institute, 1518 Budapest Pf. 63, Hungary
[c] Department of Mathematics, University of Toronto, Bahen Centre, 40 St. George St., Toronto, ON, Canada, M5S 2E4

ARTICLE INFO

Article history:
Received 14 September 2010
Available online 29 April 2011
Communicated by Michel Van den Bergh

Keywords:
Isomorphism problem
Nilpotent Lie algebras
Enveloping algebras

ABSTRACT

In this paper we study the isomorphism problem for the universal enveloping algebras of nilpotent Lie algebras. We prove that if the characteristic of the underlying field is not 2 or 3, then the isomorphism type of a nilpotent Lie algebra of dimension at most 6 is determined by the isomorphism type of its universal enveloping algebra. Examples show that the restriction on the characteristic is necessary.

1. Introduction

In this paper we examine the isomorphism problem for universal enveloping algebras of Lie algebras. It is known that two non-isomorphic Lie algebras may have isomorphic universal enveloping algebras; see for instance [RU, Example A]. All such known examples require that the characteristic of the underlying field is a prime. In this paper we focus on nilpotent Lie algebras and prove the following main result.

Theorem 1.1. *The isomorphism type of a nilpotent Lie algebra of dimension at most* 6 *is determined by the isomorphism type of its universal enveloping algebra over any field of characteristic not* 2 *nor* 3.

☆ The first author was supported by the FCT project PTDC/MAT/101993/2008 and by the Hungarian Scientific Research Fund (OTKA) grant 72845. The second author was supported by an NSERC PDF.

* Corresponding author.
E-mail addresses: csaba.schneider@gmail.com (C. Schneider), usefi@math.toronto.edu (H. Usefi).
URL: http://www.sztaki.hu/~schneider (C. Schneider).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.03.026

Theorem 1.1 is a consequence of Theorems 3.1 and 4.1. As shown by examples in Sections 3 and 4, the requirement about the characteristic of the underlying field is necessary. In addition to proving Theorem 1.1, we classify the possible isomorphisms between the universal enveloping algebras of nilpotent Lie algebras of dimension at most 5 over an arbitrary field, and those of dimension 6 over fields of characteristic different from 2 (see Theorems 3.1 and 4.1).

Little progress has ever been made on the isomorphism problem for universal enveloping algebras. For a Lie algebra L, let $U(L)$ denote its universal enveloping algebra (see Section 2 for the definitions). Several invariants of L are known to be determined by $U(L)$. For instance, if L is finite-dimensional, then the (linear) dimension of L coincides with the Gelfand–Kirillov dimension of $U(L)$ (see [KL]). More recently Riley and Usefi [RU] proved that the nilpotence of L is determined by $U(L)$ and, for a nilpotent L, the nilpotency class of L can be determined using $U(L)$. Moreover the isomorphism type of $U(L)$ determines the isomorphism type of the graded algebra $\mathrm{Gr}(L)$ associated with the lower central series of L (see Section 2 for the definitions). Malcolmson [M] showed that if L is a 3-dimensional simple Lie algebra over a field of characteristic not 2, then L is determined by $U(L)$ up to isomorphism. Later Chun, Kajiwara, and Lee [CKL] generalized Malcolmson's result to the class of all Lie algebras with dimension 3 over fields of characteristic not 2.

By proving Theorem 1.1, we verify that the isomorphism problem for universal enveloping algebras has a positive solution in the class of nilpotent Lie algebras with dimension at most 6 over fields of characteristic different from 2 and 3. The proof of this result relies on the classification of nilpotent Lie algebras with dimension at most 6. The classification of such Lie algebras of dimension at most 5 has been known for a long time over an arbitrary field. In dimension 6, several classifications have been published, but they were often incorrect, and they usually only treated fields of characteristic 0. Recently de Graaf [dG] published a classification of 6-dimensional nilpotent Lie algebras over an arbitrary field of characteristic not 2. As the classification by de Graaf has been obtained making heavy use of computer calculations, and was checked by computer for small fields [Sch], we consider this classification as the most reliable in the literature. The reason we do not treat 6-dimensional nilpotent Lie algebras over fields of characteristic 2 is that, in this case, we do not know of a similarly reliable classification.

Our strategy in proving Theorem 1.1 is to determine all pairs of nilpotent Lie algebras L_1, L_2 with dimension at most 6, such that the graded algebras $\mathrm{Gr}(L_1)$ and $\mathrm{Gr}(L_2)$ associated with the lower central series are isomorphic. We know from [RU] that this is a necessary condition for the isomorphism $U(L_1) \cong U(L_2)$. Such pairs can be read off from the list of nilpotent Lie algebras with dimension at most 6 in [dG]. Next, for all such pairs, we either argue that $U(L_1)$ cannot be isomorphic to $U(L_2)$, or we exhibit an explicit isomorphism between $U(L_1)$ and $U(L_2)$. Initially, computer experiments played a role in determining the isomorphisms between universal enveloping algebras of nilpotent Lie algebras. Recent work by Eick [E] describes a practical algorithm to decide isomorphism between finite-dimensional nilpotent associative algebras. Her algorithm was implemented in the ModIsom package [MI] of the GAP computational algebra system [GAP] and the implementations work for algebras of dimensions up to about 100 over small finite fields. We used this implementation to decide isomorphisms between the finite-dimensional, nilpotent quotients $\Omega(L)/\Omega^k(L)$ of the augmentation ideals $\Omega(L)$ of $U(L)$ (see Section 2 for notation). We remark here an interesting observation. If L is nilpotent of class c then based on our calculations the isomorphism type of the quotient $\Omega(L)/\Omega^{c+1}(L)$ determines the isomorphism type of L. So, the question remains whether $\Omega(L)/\Omega^{c+1}(L)$ determines the isomorphism type of L in all dimensions.

2. Preliminaries

In this section we summarize some important facts about universal enveloping algebras of Lie algebras; see [D] for a more detailed background. We assume from now on that Lie algebras are finite-dimensional, even though most of the results referred to in this section hold for a larger class of Lie algebras.

Let L be a Lie algebra over a field $\mathbb{F}$. The universal enveloping algebra $U(L)$ of L is defined as follows. For $i \geqslant 0$, let $L^{\otimes i}$ denote the i-fold tensor power of L and set

$$\mathcal{T} = \bigoplus_{i=0}^{\infty} L^{\otimes i}.$$

The space $L^{\otimes 0}$ is one-dimensional, generated by the unit of $\mathcal{T}$, and is usually identified with $\mathbb{F}$. The sum $\mathcal{T}$ can be considered as an algebra over $\mathbb{F}$ with respect to the obvious multiplication defined on the generators of $\mathcal{T}$ as

$$(v_1 \otimes \cdots \otimes v_k) \cdot (u_1 \otimes \cdots \otimes u_l) = v_1 \otimes \cdots \otimes v_k \otimes u_1 \otimes \cdots \otimes u_l,$$

for all $v_1, \ldots, v_k, u_1, \ldots, u_l \in L$. The algebra $\mathcal{T}$ is usually referred to as the *tensor algebra* of L. Let $\mathcal{I}$ denote the two-sided ideal of $\mathcal{T}$ generated by elements of the form $[u, v] - u \otimes v + v \otimes u$. Then the *universal enveloping algebra* $U(L)$ of L is defined as the quotient $\mathcal{T}/\mathcal{I}$. We view $L = L^{\otimes 1}$ as a Lie subalgebra of $U(L)$. Universal enveloping algebras have the following universal property.

Lemma 2.1. *Suppose that L is a Lie algebra, A is an associative algebra, and let $\varphi : L \to A$ be a Lie algebra homomorphism. Then there is a unique associative algebra homomorphism $\overline{\varphi} : U(L) \to A$ such that $\overline{\varphi}|_L = \varphi$.*

The linear subspace spanned by a subset X of a vector space is denoted by $\langle X\rangle_{\mathbb{F}}$. Recall that the *center* $Z(L)$ of a Lie algebra L is the ideal

$$\big\langle x \in L \;\big|\; [x, a] = 0, \text{ for all } a \in L\big\rangle_{\mathbb{F}}.$$

The center of the universal enveloping algebra plays an important role in our arguments. The *center* $Z(A)$ of an associative algebra A is defined as

$$Z(A) = \langle x \in A \mid ax = xa, \text{ for all } a \in A\rangle_{\mathbb{F}}.$$

It is clear that the subalgebra of $U(L)$ generated by the center $Z(L)$ of L lies in $Z(U(L))$.

Note that $\mathcal{T}_+ = L^{\otimes 1} \oplus L^{\otimes 2} \oplus \cdots$ is a two-sided ideal in $\mathcal{T}$, and the image of $\mathcal{T}_+$ in $U(L)$ is referred to as the *augmentation ideal* of $U(L)$ and is denoted by $\Omega(L)$. The following is proved in [RU, Lemma 2.1].

Lemma 2.2. *For Lie algebras L and K, $U(L) \cong U(K)$ if and only if $\Omega(L) \cong \Omega(K)$.*

Investigating the isomorphism between $U(L)$ and $U(K)$ will often be carried out, using Lemma 2.2, through studying the isomorphism between $\Omega(K)$ and $\Omega(L)$. For $i \geqslant 1$, let $\Omega^i(L)$ denote the ideal of $\Omega(L)$ generated by the products of elements of $\Omega(L)$ with length at least i. This way we obtain a descending series in $\Omega(L)$:

$$\Omega(L) \geqslant \Omega^2(L) \geqslant \cdots.$$

The sequence $\Omega^i(L)$ is a filtration on $\Omega(L)$: for $x \in \Omega^i(L)$ and $y \in \Omega^j(L)$ we have that $xy \in \Omega^{i+j}(L)$.

A basis for $\Omega^i(L)$ can usually be constructed as follows. Let L^i denote the i-th term of the *lower central series* of L; that is $L^1 = L$, and, for $i \geqslant 1$, $L^{i+1} = [L^i, L]$. For an element $v \in L$, we define the *weight* $w(v)$ of v as the largest integer i such that $v \in L^i$. A basis $\mathcal{B} = \{v_1, \ldots, v_d\}$ of a Lie algebra L is said to be *homogeneous* if the basis elements with weight at least i form a basis for L^i. An element of $U(L)$ of the form $m = v_{i_1} v_{i_2} \ldots v_{i_k}$ with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$ is said to be a *Poincaré–Birkhoff–Witt monomial*, or more briefly, a *PBW monomial*, in $\mathcal{B}$. The *weight* $w(m)$ of such a monomial is defined as $w(v_{i_1}) + w(v_{i_2}) + \cdots + w(v_{i_k})$.

Theorem 2.3. *(See Proposition 3.1(1) in [R].) Let L be a Lie algebra with a homogeneous basis $\mathcal{B}$ and let $t \geqslant 1$. Then the set of all PBW monomials in $\mathcal{B}$ with weight at least t forms an $\mathbb{F}$-basis for $\Omega^t(L)$, for every $t \geqslant 1$.*

In this paper we are interested to discover, for a pair of nilpotent non-isomorphic Lie algebras L_1 and L_2, if $U(L_1)$ can be isomorphic to $U(L_2)$. Often we use the graded algebras $\mathsf{Gr}(L_i)$ associated with the lower central series of the L_i to rule out the isomorphism between $U(L_1)$ and $U(L_2)$. For a Lie algebra L, $\mathsf{Gr}(L)$ is defined as the algebra on the linear space

$$\mathsf{Gr}(L) = L/L^2 \oplus L^2/L^3 \oplus \cdots$$

with respect to the multiplication given by the rule

$$\left[x + L^{i+1}, y + L^{j+1}\right] = [x, y] + L^{i+j+1}, \quad \text{for all } x \in L^i \text{ and } y \in L^j.$$

The following is proved in [RU, Proposition 4.1].

Theorem 2.4. *For any Lie algebra L, the isomorphism type of $U(L)$ determines the isomorphism type of* $\mathsf{Gr}(L)$. *Consequently, if L is nilpotent of class* 2, *then the isomorphism type of $U(L)$ determines the isomorphism type of L.*

Suppose that L_1 and L_2 are nilpotent Lie algebras such that $U(L_1) \cong U(L_2)$. Then Theorem 2.4 implies, for all i, that $\dim L_1^i = \dim L_2^i$, and that the nilpotency classes of L_1 and L_2 coincide. The second statement of Theorem 2.4 is an easy consequence of the first statement if L is finite-dimensional. Indeed, if L is a finite-dimensional Lie algebra of nilpotency class 2, then it is always isomorphic to $\mathsf{Gr}(L)$. The same assertion without a restriction on the dimension is proved in [RU, Section 5].

The following lemma can be found in [RU, Lemma 5.1].

Lemma 2.5. *Let L and K be Lie algebras and let $\varphi : \Omega(L) \to \Omega(K)$ be an isomorphism. Then $\varphi(L^i + \Omega^{i+1}(L)) = K^i + \Omega^{i+1}(K)$, for every positive integer i.*

3. Nilpotent Lie algebras with dimension at most 5

In this section we determine all isomorphisms between the universal enveloping algebras of nilpotent Lie algebras with dimension at most 5. A classification of such Lie algebras is well known and can be found, for instance, in [dG]. The main result of this section is the following.

Theorem 3.1. *Let L and K be nilpotent Lie algebras of dimension at most* 5 *over a field $\mathbb{F}$ such that $U(L) \cong U(K)$. Then one of the following must hold*:

(i) $L \cong K$;
(ii) $\operatorname{char} \mathbb{F} = 2$; *further L and K are isomorphic to the Lie algebras $L_{5,3}$ and $L_{5,5}$ or they are isomorphic to the Lie algebras $L_{5,6}$ and $L_{5,7}$ in Section* 5 *of* [dG].

Since there is a unique isomorphism class of nilpotent Lie algebras with dimension 1, and there is a unique such class with dimension 2, the isomorphism problem of universal enveloping algebras is trivial in these cases. Up to isomorphism, there are two nilpotent Lie algebras with dimension 3, an abelian, and a non-abelian. By Theorem 2.4 their universal enveloping algebras must be non-isomorphic. The number of isomorphism classes of 4-dimensional nilpotent Lie algebras is 3. One of these algebras is abelian, the second has nilpotency class 2, and the third has nilpotency class 3. Again, by Theorem 2.4, their universal enveloping algebras are pairwise non-isomorphic. This proves Theorem 3.1 for Lie algebras of dimension at most 4.

There are 9 isomorphism classes of nilpotent Lie algebras with dimension 5 and they are listed at the beginning of Section 4 in [dG]. To simplify notation, we denote the Lie algebra $L_{5,i}$ in de Graaf's list by L_i. Inspecting this list, we find that the sequence $(\dim L^1, \dim L^2, \ldots)$ for these Lie algebras, after omitting the trailing zeros, are: (5), (5, 1), (5, 2, 1), (5, 1), (5, 2, 1), (5, 3, 2, 1), (5, 3, 2, 1), (5, 2), (5, 3, 2). Therefore Theorem 2.4 implies that if $U(L_i) \cong U(L_j)$ then either $i = j$, or $\{i, j\} = \{3, 5\}$, or $\{i, j\} = \{6, 7\}$. The Lie algebras that are involved in these possible isomorphisms are as follows:

$$L_3 = \langle x_1, x_2, x_3; x_4; x_5 \mid [x_1, x_2] = x_4,\ [x_1, x_4] = x_5 \rangle;$$
$$L_5 = \langle x_1, x_2, x_3; x_4; x_5 \mid [x_1, x_2] = x_4,\ [x_1, x_4] = x_5,\ [x_2, x_3] = x_5 \rangle;$$
$$L_6 = \langle x_1, x_2; x_3; x_4; x_5 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5,\ [x_2, x_3] = x_5 \rangle;$$
$$L_7 = \langle x_1, x_2; x_3; x_4; x_5 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5 \rangle.$$

Products that are zero are omitted from the multiplication tables above. For instance $[x_1, x_3] = 0$ in L_3 and in L_5. Note that the bases of the Lie algebras given above are homogeneous. Further, basis elements of different weights are separated by a semicolon. The multiplication tables given in [dG] are somewhat different from the ones above, as we changed the orders of certain basis elements in order to work with homogeneous basis. A possible source of confusion is that the symbols x_i are used to denote elements of different Lie algebras, but we believe that using different letters or introducing a subscript or superscript would unnecessarily complicate the notation.

Lemma 3.2. *If $\mathbb{F}$ is a filed of characteristic not 2, then $\Omega(L_3)/\Omega^4(L_3) \ncong \Omega(L_5)/\Omega^4(L_5)$ and $\Omega(L_6)/\Omega^5(L_6) \ncong \Omega(L_7)/\Omega^5(L_7)$; consequently $\Omega(L_3) \ncong \Omega(L_5)$ and $\Omega(L_6) \ncong \Omega(L_7)$. Otherwise, $\Omega(L_3) \cong \Omega(L_5)$ and $\Omega(L_6) \cong \Omega(L_7)$.*

Proof. Suppose that $\operatorname{char} \mathbb{F} \neq 2$ and set $B_i = \Omega(L_i)/\Omega^4(L_i)$, for $i = 3, 5$. First we show that $B_3 \ncong B_5$. We claim that $Z(B_5) \leqslant (B_5)^2$ while $Z(B_3) \nleqslant (B_3)^2$, which will imply that $B_3 \ncong B_5$. As $x_3 \in Z(B_3) \setminus (B_3)^2$, the second assertion of the claim is valid. In order to prove the first assertion, let $w \in Z(B_5)$. We write w as a linear combination of PBW monomials:

$$w = \sum_{i_1 \leqslant \cdots \leqslant i_n} \alpha_{i_1,\ldots,i_n} x_{i_1} \cdots x_{i_n}.$$

Then $w \equiv \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \pmod{(B_5)^2}$. Since w is a central element in B_5, we have $[x_1, w] = [x_2, w] = 0$, and so

$$0 = [x_1, w] \equiv \alpha_2 [x_1, x_2] = \alpha_2 x_4 \quad \bigl(\mathrm{mod}\ (B_5)^3\bigr),$$
$$0 = [x_2, w] \equiv \alpha_1 [x_2, x_1] = -\alpha_1 x_4 \quad \bigl(\mathrm{mod}\ (B_5)^3\bigr).$$

Hence, $\alpha_1 = \alpha_2 = 0$. Since $x_5 \in L_5^3$, we have

$$\begin{aligned} 0 = [x_2, w] &= \alpha_3 [x_2, x_3] + \alpha_{1,1} \bigl[x_2, x_1^2\bigr] + \alpha_{1,2} [x_2, x_1 x_2] + \alpha_{1,3} [x_2, x_1 x_3] \\ &= \alpha_3 x_5 + \alpha_{1,1}(-2 x_1 x_4 + x_5) - \alpha_{1,2} x_2 x_4 - \alpha_{1,3} x_3 x_4. \end{aligned}$$

Since $\operatorname{char} \mathbb{F} \neq 2$, we deduce that $\alpha_3 = 0$. Thus, $w \in (B_5)^2$ as claimed.

Now set $B_i = \Omega(L_i)/\Omega^5(L_i)$, for $i = 6, 7$. Suppose, to the contrary, that $f : B_7 \to B_6$ is an isomorphism. Since B_7 is generated by x_1 and x_2, the map f is determined by the images $f(x_1)$ and $f(x_2)$. As above, let us write $f(x_1)$ and $f(x_2)$ as linear combinations of PBW monomials:

$$f(x_1) = \sum_{i_1 \leqslant \cdots \leqslant i_n} \alpha_{i_1,\ldots,i_n} x_{i_1} \cdots x_{i_n},$$

$$f(x_2) = \sum_{i_1 \leqslant \cdots \leqslant i_n} \beta_{i_1,\ldots,i_n} x_{i_1} \cdots x_{i_n}.$$

As

$$f\big([x_1, x_2]\big) \equiv (\alpha_1\beta_2 - \alpha_2\beta_1)[x_1, x_2] = (\alpha_1\beta_2 - \alpha_2\beta_1)x_3 \quad \big(\mathrm{mod}\ (B_6)^3\big), \tag{1}$$

we obtain that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. Eq. (1) also gives

$$0 = f\big([x_1, x_2, x_2]\big) \equiv \beta_1(\alpha_1\beta_2 - \alpha_2\beta_1)[x_1, x_2, x_1] = -\beta_1(\alpha_1\beta_2 - \alpha_2\beta_1)x_4 \quad \big(\mathrm{mod}\ (B_6)^4\big).$$

Since $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, we must have $\beta_1 = 0$. This implies that $\alpha_1 \neq 0$ and $\beta_2 \neq 0$. Furthermore, modulo $(B_6)^4$, we have

$$\begin{aligned} f\big([x_1, x_2]\big) \equiv{}& \alpha_1\beta_2 x_3 + (\alpha_1\beta_3 + \alpha_2\beta_{1,1} - \alpha_{1,1}\beta_2)x_4 \\ &+ (\alpha_1\beta_{1,2} - 2\alpha_2\beta_{1,1} + 2\alpha_{1,1}\beta_2)x_1x_3 + (2\alpha_1\beta_{2,2} - \alpha_2\beta_{1,2} + \alpha_{1,2}\beta_2)x_2x_3. \end{aligned}$$

Hence $f([x_1, x_2, x_2])$ is equal to the following:

$$\begin{aligned} &\big(-\alpha_1\beta_2^2 + \alpha_1\beta_2\beta_{1,1}\big)x_5 - 2\alpha_1\beta_2\beta_{1,1}x_1x_4 \\ &\quad - \alpha_1\beta_2\beta_{1,2}x_2x_4 + \beta_2(\alpha_1\beta_{1,2} - 2\alpha_2\beta_{1,1} + 2\alpha_{1,1}\beta_2)x_3x_3. \end{aligned}$$

As $[x_1, x_2, x_2] = 0$ in $\Omega(L_7)$, we must have $f([x_1, x_2, x_2]) = 0$. Since $\mathrm{char}(\mathbb{F}) \neq 2$ and $\alpha_1\beta_2 \neq 0$, we get $\beta_{1,1} = 0$. Thus the coefficient of x_5 in $f([x_1, x_2, x_2])$ is $-\alpha_1\beta_2^2$. This, implies that $f([x_1, x_2, x_2]) \neq 0$, which is a contradiction. Thus $B_6 \ncong B_7$, as claimed.

Let us now assume that $\mathrm{char}\,\mathbb{F} = 2$, and prove $\Omega(L_3) \cong \Omega(L_5)$ and $\Omega(L_6) \cong \Omega(L_7)$. Note that the map from L_3 to $\Omega(L_5)$ induced by $x_1 \mapsto x_1$, $x_2 \mapsto x_2$, $x_3 \mapsto x_3 + x_1^2$ is a Lie homomorphism. Hence, by Lemma 2.1, it can be extended to a homomorphism φ from $\Omega(L_3)$ to $\Omega(L_5)$. Since x_1, x_2, and $x_3 + x_1^2$ form a generating set for $\Omega(L_5)$, φ is onto. We need to show that φ is injective. Since $\dim \Omega(L_3)/\Omega^i(L_3) = \dim \Omega(L_5)/\Omega^i(L_5)$ (see Theorem 2.3), the homomorphism φ induces an isomorphism

$$\varphi_i : \Omega(L_3)/\Omega^i(L_3) \to \Omega(L_5)/\Omega^i(L_5),$$

for every $i \geqslant 1$. Let x be a non-zero element in $\Omega(L_3)$ such that $\varphi(x) = 0$. By Theorem 2.3, there exists a positive integer i such that $x \in \Omega^{i-1}(L_3) \setminus \Omega^i(L_3)$. As φ_i is an isomorphism, we get $\varphi_i(x) \neq 0$, which is a contradiction. Hence φ is injective, and so $\Omega(L_3) \cong \Omega(L_5)$.

One can show precisely the same way that $x_1 \mapsto x_1$, $x_2 \mapsto x_2 + x_1^2$ extends to an isomorphism from $\Omega(L_7)$ to $\Omega(L_6)$. $\square$

Lemma 3.2 in combination with Lemma 2.2, and the argument preceding it proves Theorem 3.1.

4. 6-dimensional nilpotent Lie algebras

The isomorphisms between universal enveloping algebras of nilpotent Lie algebras of dimension 6 in characteristic different from 2 are described by the following theorem.

Theorem 4.1. *Let L and K be nilpotent Lie algebras of dimension 6 over a field $\mathbb{F}$ of characteristic not 2. If $U(L) \cong U(K)$, then one of the following must hold*:

(i) $L \cong K$.
(ii) $\operatorname{char} \mathbb{F} = 3$; *further L and K are isomorphic to one of the following pairs of Lie algebras in* [dG, *Section* 5]: $L_{6,6}$ *and* $L_{6,11}$; $L_{6,7}$ *and* $L_{6,12}$; $L_{6,17}$ *and* $L_{6,18}$; $L_{6,23}$ *and* $L_{6,25}$.

Throughout this section $\mathbb{F}$ denotes a field with characteristic different from 2. The proof of Theorem 4.1 is presented in this section. De Graaf [dG] lists the isomorphism types of 6-dimensional nilpotent Lie algebras over an arbitrary field $\mathbb{F}$ of characteristic different from 2. De Graaf denotes these Lie algebras by $L_{6,i}$ or $L_{6,i}(\varepsilon)$ where $1 \leqslant i \leqslant 26$ and ε is a field element. To simplify notation and to distinguish between the Lie algebras of Sections 3 and 4 we will denote $L_{6,i}$ with K_i and $L_{6,i}(\varepsilon)$ with $K_i(\varepsilon)$. Inspecting the list of Lie algebras, we obtain that the isomorphisms among the graded Lie algebras associated with the lower central series of these Lie algebras are as follows:

(1) $\mathsf{Gr}(K_3) \cong \mathsf{Gr}(K_5) \cong \mathsf{Gr}(K_{10})$;
(2) $\mathsf{Gr}(K_6) \cong \mathsf{Gr}(K_7) \cong \mathsf{Gr}(K_{11}) \cong \mathsf{Gr}(K_{12}) \cong \mathsf{Gr}(K_{13})$;
(3) $\mathsf{Gr}(K_{14}) \cong \mathsf{Gr}(K_{16})$;
(4) $\mathsf{Gr}(K_{15}) \cong \mathsf{Gr}(K_{17}) \cong \mathsf{Gr}(K_{18})$;
(5) $\mathsf{Gr}(K_{23}) \cong \mathsf{Gr}(K_{25})$;
(6) $\mathsf{Gr}(K_{24}(\varepsilon)) \cong \mathsf{Gr}(K_9)$, for all $\varepsilon \in \mathbb{F}$.

Let L and K be 6-dimensional non-isomorphic nilpotent Lie algebras over $\mathbb{F}$. If $U(L) \cong U(K)$ then, by Theorem 2.4, L and K must both occur in one of these families. In Lemmas 4.2–4.7 we examine the possible isomorphisms between the universal enveloping algebras of the Lie algebras that occur in one of the families above. Using Lemma 2.2 we only examine the possible isomorphisms between the augmentation ideals.

As with the 5-dimensional Lie algebras, we change the multiplication tables of the algebras presented in [dG] in order to work with homogeneous basis. The bases elements of different weights are separated by a semicolon. Further, as in Section 3, we omit products of the form $[x_i, x_j] = 0$ from the multiplication tables of the Lie algebras. For $i = 3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 23, 25$ we set $U_i = U(K_i)$ and $\Omega_i = \Omega(K_i)$. Further, let $U_{24}(\varepsilon)$ and $\Omega_{24}(\varepsilon)$ denote $U(K_{24}(\varepsilon))$ and $\Omega(K_{24}(\varepsilon))$, respectively.

4.1. Family (1)

First we deal with the isomorphism $\mathsf{Gr}(K_3) \cong \mathsf{Gr}(K_5) \cong \mathsf{Gr}(K_{10})$ where

$$K_3 = \langle x_1, x_2, x_3, x_4; x_5; x_6 \mid [x_1, x_2] = x_5,\ [x_1, x_5] = x_6 \rangle;$$
$$K_5 = \langle x_1, x_2, x_3, x_4; x_5; x_6 \mid [x_1, x_2] = x_5,\ [x_1, x_5] = x_6,\ [x_2, x_3] = x_6 \rangle;$$
$$K_{10} = \langle x_1, x_2, x_3, x_4; x_5; x_6 \mid [x_1, x_2] = x_5,\ [x_1, x_5] = x_6,\ [x_3, x_4] = x_6 \rangle.$$

Lemma 4.2. *The algebras Ω_3, Ω_5, and Ω_{10} are pairwise non-isomorphic.*

Proof. For $i = 3, 5, 10$, let $B_i = \Omega_i / (\Omega_i)^4$. It is enough to prove that the centers $Z_i = Z(B_i)$ have different dimensions. The quotient B_i is spanned by the images of the PBW monomials with weight at most 3 and we will identify such a monomial with its image. We claim that $Z_3 = \langle x_3, x_4, (B_3)^3\rangle_{\mathbb{F}}$. Set $C = \langle x_3, x_4, (B_3)^3\rangle_{\mathbb{F}}$. Clearly, $C \leqslant Z_3$. Let $z \in Z_3$. Then z is a linear combination of PBW monomials with weight at most 3. As usual, $\alpha_{i_1,\dots,i_n}$ is the coefficient of $x_{i_1}\cdots x_{i_n}$ in the PBW representation of z. We may assume without loss of generality that all monomials in C occur with coefficient zero. First we compute that

$$[x_1, z] = \alpha_2 x_5 + \alpha_{1,2}x_1x_5 + 2\alpha_{2,2}x_2x_5 + \alpha_{2,3}x_3x_5 + \alpha_{2,4}x_4x_5 + \alpha_5 x_6.$$

We deduce that $\alpha_2 = \alpha_{1,2} = \alpha_{2,2} = \alpha_{2,3} = \alpha_{2,4} = \alpha_5 = 0$. Now

$$[z, x_2] = \alpha_1 x_5 + 2\alpha_{1,1}x_1x_5 - \alpha_{1,1}x_6 + \alpha_{1,3}x_3x_5 + \alpha_{1,4}x_4x_5.$$

This implies that $\alpha_1 = \alpha_{1,1} = \alpha_{1,3} = \alpha_{1.4} = 0$. Therefore $Z_3 = C$ as claimed. Since x_3, x_4, x_3x_3, x_3x_4, x_4x_4 together with the PBW monomials with weight 3 form a basis for C, we obtain that $\dim C = \dim Z_3 = 30$. Similar argument shows that $\dim Z_5 = 29$, and $\dim Z_{10} = 28$. Thus the Z_i have different dimensions, as required. $\square$

4.2. Family (2)

We examine the following family of Lie algebras:

$$\begin{aligned}
K_6 &= \big\langle x_1, x_2, x_3; x_4; x_5; x_6 \bigm| [x_1,x_2]=x_4,\ [x_1,x_4]=x_5,\ [x_1,x_5]=x_6,\ [x_2,x_4]=x_6\big\rangle;\\
K_7 &= \big\langle x_1, x_2, x_3; x_4; x_5; x_6 \bigm| [x_1,x_2]=x_4,\ [x_1,x_4]=x_5,\ [x_1,x_5]=x_6\big\rangle;\\
K_{11} &= \big\langle x_1, x_2, x_3; x_4; x_5; x_6 \bigm| [x_1,x_2]=x_4,\ [x_1,x_4]=x_5,\ [x_1,x_5]=x_6,\ [x_2,x_4]=x_6,\ [x_2,x_3]=x_6\big\rangle;\\
K_{12} &= \big\langle x_1, x_2, x_3; x_4; x_5; x_6 \bigm| [x_1,x_2]=x_4,\ [x_1,x_4]=x_5,\ [x_1,x_5]=x_6,\ [x_2,x_3]=x_6\big\rangle;\\
K_{13} &= \big\langle x_1, x_2, x_3; x_4; x_5; x_6 \bigm| [x_1,x_2]=x_4,\ [x_1,x_4]=x_5,\ [x_1,x_5]=x_6,\ [x_2,x_3]=x_5,\ [x_4,x_3]=x_6\big\rangle.
\end{aligned}$$

Lemma 4.3. *The enveloping algebras Ω_6, Ω_7, Ω_{11}, Ω_{12}, and Ω_{13} are pairwise non-isomorphic provided that* $\operatorname{char}\mathbb{F} \neq 3$. *If* $\operatorname{char}\mathbb{F} = 3$ *then $\Omega_6 \cong \Omega_{11}$ and $\Omega_7 \cong \Omega_{12}$ and there is no more isomorphism among the algebras in this family.*

Proof. We claim, for $i \in \{6, 7, 11, 12\}$, that $\Omega_{13} \ncong \Omega_i$. Note that $L_3 \cong K_i/K_i^4$, while $L_5 \cong K_{13}/K_{13}^4$, where L_3 and L_5 are 5-dimensional Lie algebras defined in Section 3. Thus, $\Omega(L_3)/\Omega^5(L_3) \cong \Omega_i/(K_i^4 + \Omega_i^5)$ and $\Omega(L_5)/\Omega^5(L_5) \cong \Omega_{13}/(K_{13}^4 + \Omega_{13}^5)$. Suppose that $f : \Omega_i \to \Omega_{13}$ is an isomorphism. By Lemma 2.5, $f(K_i^4 + \Omega_i^5) = K_{13}^4 + \Omega_{13}^5$. Hence, f induces an isomorphism between $\Omega_i/(K_i^4 + \Omega_i^5)$ and $\Omega_{13}/(K_{13}^4 + \Omega_{13}^5)$. Thus,

$$\Omega(L_3)/\Omega^5(L_3) \cong \Omega(L_5)/\Omega^5(L_5).$$

However Lemma 3.2 shows that $\Omega(L_3)/\Omega^4(L_3) \ncong \Omega(L_5)/\Omega^4(L_5)$. This contradiction implies that $\Omega_{13} \ncong \Omega_i$, as claimed.

Next we show that $\Omega_6 \ncong \Omega_7$ and that $\Omega_{11} \ncong \Omega_{12}$. First we argue that $\Omega_6 \ncong \Omega_7$. Suppose on the contrary that $f : \Omega_7 \to \Omega_6$ is an isomorphism. Then f is determined by the images $f(x_1)$, $f(x_2)$, $f(x_3)$. Write $f(x_i)$ as a linear combination of PBW monomials and let $\alpha_{i_1,\dots,i_n}$, $\beta_{i_1,\dots,i_n}$ and $\gamma_{i_1,\dots,i_n}$ denote the coefficients of $x_{i_1}\cdots x_{i_n}$ in $f(x_1)$, $f(x_2)$, $f(x_3)$, respectively. Since $x_3 \in Z(\Omega_7)$, we have

$$0 = \big[x_1, f(x_3)\big] \equiv \gamma_2 x_4 \mod (\Omega_6)^3 \quad\text{and}\quad 0 = \big[x_2, f(x_3)\big] \equiv -\gamma_1 x_4 \mod (\Omega_6)^3.$$

Therefore $\gamma_1 = \gamma_2 = 0$. Let us compute modulo $(\Omega_6)^4$ that

$$0 = f\big([x_2, x_1, x_2]\big) \equiv \beta_1(\alpha_1\beta_2 - \beta_1\alpha_2)x_5.$$

If $\alpha_1\beta_2 - \beta_1\alpha_2 = 0$, then, as $\gamma_1 = \gamma_2 = 0$, it follows that $f(x_i)$ are linearly dependent modulo $(\Omega_6)^2$, which is impossible. Thus $\beta_1 = 0$ and $\alpha_1\beta_2 \neq 0$. Now computation shows, modulo $(\Omega_6)^5$, that

$$\begin{aligned} 0 &= f\big([x_1, x_2, x_2]\big) \\ &\equiv \big(-\alpha_1\beta_2^2 + \alpha_1\beta_2\beta_{11}\big)x_6 - 2\alpha_1\beta_2\beta_{1,1}x_1x_5 - \alpha_1\beta_2\beta_{1,2}x_2x_5 - \alpha_1\beta_2\beta_{1,3}x_3x_5 + \delta\beta_2x_4x_4 \end{aligned}$$

where $\delta = \alpha_1\beta_{1,2} - 2\alpha_2\beta_{1,1} + 2\alpha_{1,1}\beta_2$. Considering the coefficients of x_6 and x_1x_5, we deduce that either $\alpha_1 = 0$ or $\beta_2 = 0$, which is a contradiction.

Let us now show that $\Omega_{11} \not\cong \Omega_{12}$. Assume by contradiction that $f : \Omega_{12} \to \Omega_{11}$ is an isomorphism. Then f is determined by the images $f(x_1)$, $f(x_2)$, and $f(x_3)$. As above, we write $f(x_i)$ as a linear combination of PBW monomials and we let $\alpha_{i_1,\dots,i_n}$, $\beta_{i_1,\dots,i_n}$ and $\gamma_{i_1,\dots,i_n}$ denote the coefficients of $x_{i_1}\cdots x_{i_n}$ in $f(x_1)$, $f(x_2)$, $f(x_3)$, respectively. Let us compute modulo $(\Omega_{11})^3$ that

$$0 = f\big([x_1, x_3]\big) \equiv (\alpha_1\gamma_2 - \gamma_1\alpha_2)x_4$$

and

$$0 \equiv f\big([x_2, x_3]\big) \equiv (\beta_1\gamma_2 - \beta_2\gamma_1)x_4.$$

If the vector (γ_1, γ_2) is not zero, then the vectors (α_1, α_2) and (β_1, β_2) must be its scalar multiples and so the $f(x_i)$ are linearly dependent modulo $(\Omega_{11})^2$. Hence $\gamma_1 = \gamma_2 = 0$. Now we get a contradiction using the same argument as in the previous paragraph.

Now assume that $\operatorname{char}\mathbb{F} \neq 3$. Set $B_i = \Omega_i/(\Omega_i)^5$, for $i = 6, 7, 11, 12$. Let Z_i denote the center of the B_i. We claim that $\dim Z_6 = \dim Z_7 = 29$ and $\dim Z_{11} = \dim Z_{12} = 28$. We only compute $\dim Z_7$, as the computation of the other Z_i is very similar. We claim that

$$Z_7 = \big\langle x_3, x_3x_3, x_3x_3x_3, (B_7)^4\big\rangle_{\mathbb{F}}.$$

Set $C = \langle x_3, x_3x_3, x_3x_3x_3, (B_7)^4\rangle_{\mathbb{F}}$. Clearly, $C \leqslant Z_7$. Let $z \in Z_7$ and write z as a linear combination of PBW monomials with weight at most 4. We may assume that all PBW monomials in C occur with coefficient 0 in z. Let $\beta_{i_1,\dots,i_k}$ be the coefficient of $x_{i_1}\cdots x_{i_k}$. Then $\beta_3 = \beta_{3,3} = \beta_{3,3,3} = 0$. First we obtain that

$$0 = [z, x_5] = \beta_1[x_1, x_5] = \beta_1x_6,$$

and so $\beta_1 = 0$. Also, $[z, x_1^3] = -3\beta_2x_1^2x_4 + 3\beta_2x_1x_5 - \beta_2x_6$. So, $\beta_2 = 0$. Next compute that

$$\begin{aligned} \big[z, x_1^2\big] &= -2\beta_{1,2}x_1^2x_4 - 4\beta_{2,2}x_1x_2x_4 - 2\beta_{2,3}x_1x_3x_4 \\ &\quad + (\beta_{1,2} - 2\beta_4)x_1x_5 + 2\beta_{2,2}x_2x_5 + \beta_{2,3}x_3x_5 + 2\beta_{2,2}x_4x_4 + \beta_4x_6. \end{aligned}$$

This implies that $\beta_{1,2} = \beta_{2,2} = \beta_{2,3} = \beta_4 = 0$. Further,

$$\big[z, x_2^2\big] = 4\beta_{1,1}x_1x_2x_4 - 2\beta_{1,1}x_2x_5 - 2\beta_{1,1}x_4x_4 + 2\beta_{1,3}x_2x_3x_4.$$

Thus $\beta_{1,1} = \beta_{1,3} = 0$. Then

$$0 = [z, x_1] = -\beta_{1,1,2}x_1x_1x_4 - 2\beta_{1,2,2}x_1x_2x_4 - \beta_{1,2,3}x_1x_3x_4 - 3\beta_{2,2,2}x_2x_2x_4 - 2\beta_{2,2,3}x_2x_3x_4$$
$$- \beta_{2,3,3}x_3x_3x_4 - \beta_{1,4}x_1x_5 - \beta_{2,4}x_4^2 - \beta_{2,4}x_2x_5 - \beta_{3,4}x_3x_5 - \beta_5x_6,$$

and so $\beta_{1,1,2} = \beta_{1,2,2} = \beta_{1,2,3} = \beta_{2,2,2} = \beta_{2,2,3} = \beta_{2,3,3} = \beta_{1,4} = \beta_{2,4} = \beta_{3,4} = \beta_5 = 0$. Further

$$0 = [z, x_2] = 3\beta_{1,1,1}x_1x_1x_4 - 3\beta_{1,1,1}x_1x_5 + \beta_{1,1,1}x_6$$
$$+ 2\beta_{1,1,3}x_1x_3x_4 - \beta_{1,1,3}x_3x_5 + \beta_{1,3,3}x_3x_3x_4.$$

Hence $\beta_{1,1,1} = \beta_{1,1,3} = \beta_{1,3,3} = 0$. Thus $z \in C$, and so $C = Z_7$ as required. Similar calculations show that

$$Z_6 = \langle x_3, x_3x_3, x_3x_3x_3, (B_6)^4\rangle_{\mathbb{F}};$$
$$Z_{11} = \langle x_3x_3, x_3x_3x_3, (B_6)^4\rangle_{\mathbb{F}};$$
$$Z_{12} = \langle x_3x_3, x_3x_3x_3, (B_6)^4\rangle_{\mathbb{F}}.$$

Thus the dimensions of Z_i are as claimed. Hence, if $\operatorname{char}\mathbb{F} \neq 3$ then, $\Omega_6 \ncong \Omega_{11}, \Omega_{12}$ and $\Omega_7 \ncong \Omega_{11}, \Omega_{12}$.

Combining the results of the last two paragraph, we obtain that the algebras Ω_6, Ω_7, Ω_{11}, Ω_{12}, Ω_{13} are pairwise non-isomorphic if $\operatorname{char}\mathbb{F} \neq 3$.

Now suppose that $\operatorname{char}\mathbb{F} = 3$. We are required to show that $\Omega_6 \ncong \Omega_{12}$, $\Omega_7 \ncong \Omega_{11}$, $\Omega_6 \cong \Omega_{11}$ and that $\Omega_7 \cong \Omega_{12}$. Suppose that $f : \Omega_{12} \to \Omega_6$ is an isomorphism. Write $f(x_2)$ and $f(x_4)$ as linear combinations of PBW monomials and let $\beta_{i_1,\dots,i_n}$ and $\gamma_{i_1,\dots,i_n}$ denote the coefficient of $x_{i_1} \cdots x_{i_n}$ in $f(x_2)$ and $f(x_4)$ respectively. By Lemma 2.5, $f(x_4) \equiv \gamma_4x_4 \pmod{(\Omega_6)^3}$. Then, modulo $(\Omega_6)^4$, $0 = f([x_2, x_4]) \equiv \beta_1\gamma_4x_5$, which gives that $\beta_1 = 0$. As $x_4 = [x_1, x_2]$, we find that

$$\gamma_{1,1,1} = \gamma_{1,1,2} = \gamma_{1,1,3} = \gamma_{1,2,2} = \gamma_{1,2,3} = \gamma_{1,3,3} = \gamma_{2,2,2} = \gamma_{2,2,3} = \gamma_{2,3,3} = \gamma_{3,3,3} = 0.$$

Then

$$0 = f\big([x_2, x_4]\big) = (\beta_2\gamma_4 - \beta_{1,1}\gamma_4)x_6 - \beta_2\gamma_{1,4}x_4x_4 + \beta_{1,2}\gamma_4x_2x_5 + \beta_{1,3}\gamma_4x_3x_5 - \beta_{1,1}\gamma_4x_1x_5.$$

This implies that $\beta_2\gamma_4 = 0$, and in turn that $\beta_2 = 0$. However, this gives that $f(x_1)$, $f(x_2)$, $f(x_3)$ are linearly dependent modulo $(\Omega_6)^2$, which is impossible. Hence $\Omega_6 \ncong \Omega_{12}$, and very similar argument shows that $\Omega_7 \ncong \Omega_{11}$.

Finally we need to show that if $\operatorname{char}\mathbb{F} = 3$, then $\Omega_6 \cong \Omega_{11}$ and $\Omega_7 \cong \Omega_{12}$. The argument presented in the proof of Lemma 3.2 shows that the map $x_1 \mapsto x_1$, $x_2 \mapsto x_2$, and $x_3 \mapsto x_3 + x_1^3$ can be extended to isomorphisms between Ω_6 and Ω_{11} and between Ω_7 and Ω_{12}. □

4.3. Family (3)

$$K_{14} = \big\langle x_1, x_2; x_3; x_4; x_5; x_6 \;\big|\; [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5, [x_2, x_5] = x_6,$$
$$[x_3, x_4] = -x_6,\ [x_2, x_3] = x_5\big\rangle;$$
$$K_{16} = \big\langle x_1, x_2; x_3; x_4; x_5; x_6 \;\big|\; [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5,$$
$$[x_2, x_5] = x_6,\ [x_3, x_4] = -x_6\big\rangle.$$

Lemma 4.4. *The algebras Ω_{14} and Ω_{16} are not isomorphic.*

Proof. Note that K_{14} and K_{16} are algebras with maximal class. Further, $K_{14}/(K_{14})^5 \cong L_6$ and $K_{16}/(K_{16})^5 \cong L_7$. Now the argument presented in the first paragraph of the proof of Lemma 4.3 shows that $\Omega_{14} \ncong \Omega_{16}$. $\square$

4.4. Family (4)

$$\begin{aligned}
K_{15} = \langle & x_1, x_2; x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5,\ [x_2, x_3] = x_5, \\
& [x_1, x_5] = x_6,\ [x_2, x_4] = x_6\rangle; \\
K_{17} = \langle & x_1, x_2; x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5,\ [x_1, x_5] = x_6, \\
& [x_2, x_3] = x_6\rangle; \\
K_{18} = \langle & x_1, x_2; x_3; x_4; x_5; x_6 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5,\ [x_1, x_5] = x_6\rangle.
\end{aligned}$$

Lemma 4.5. *If* $\operatorname{char}\mathbb{F} \neq 3$ *then Ω_{15}, Ω_{17} and Ω_{18} are pairwise non-isomorphic*; *otherwise the only isomorphism among these algebras is $\Omega_{17} \cong \Omega_{18}$.*

Proof. Note that $K_{15}/(K_{15})^5 \cong L_6$ and $K_{17}/(K_{17})^5 \cong L_7 \cong K_{18}/(K_{18})^5$. Applying the same method as in the proof of Lemma 4.3 yields that $\Omega_{15} \ncong \Omega_{17}$ and $\Omega_{15} \ncong \Omega_{18}$.

Suppose that $\operatorname{char}\mathbb{F} \neq 3$, and let us show that $\Omega_{17} \ncong \Omega_{18}$. Assume, by contradiction, that $f : \Omega_{17} \to \Omega_{18}$ is an isomorphism. Then f is determined by the images $f(x_1)$ and $f(x_2)$. Let $f(x_1) \equiv \alpha_1 x_1 + \alpha_2 x_2$ and $f(x_2) \equiv \beta_1 x_1 + \beta_2 x_2$ modulo Ω_{18}^2. Let $\delta = \alpha_1\beta_2 - \alpha_2\beta_1$. Clearly, $\delta \neq 0$. Note that $f(x_3) \equiv \delta x_3 \pmod{\Omega_{18}^3}$. Thus, $f(x_4) \equiv \alpha_1 \delta x_4 \pmod{\Omega_{18}^4}$ and $f(x_6) \equiv \alpha_1^3 \delta x_6 \pmod{\Omega_{18}^6}$. Also, $f([x_2, x_3]) \equiv \beta_1 \delta x_4 \pmod{\Omega_{18}^4}$. We deduce that $\beta_1 = 0$. Now we write

$$\begin{aligned}
f(x_2) &\equiv \beta_2 x_2 + u_0 + x_1 u_1 + x_1^2 u_2 + x_1^3 u_3 \quad \left(\operatorname{mod} \Omega_{18}^4\right), \\
f(x_3) &\equiv \delta x_3 + v_0 + x_1 v_1 + x_1^2 v_2 + x_1^3 v_3 \quad \left(\operatorname{mod} \Omega_{18}^4\right),
\end{aligned}$$

where each u_i and v_i is a linear combination of (possibly trivial) PBW monomials that do not involve x_1. Since $f(x_3) \equiv \delta x_3 \pmod{\Omega_{18}^3}$, we deduce that weight of $v_0 + x_1 v_1 + x_1^2 v_2$ is at least 3. Similarly, weight of $u_0 + x_1 u_1$ is at least 2. Note that $[u_0, v_i] = [u_i, v_0] = 0$. So,

$$0 \equiv f\big([x_3, x_2]\big) \equiv \beta_2 \sum_{i=1}^{3} \big[x_1^i, x_2\big] v_i - \delta \sum_{j=1}^{2} \big[x_1^j, x_3\big] u_j \quad \left(\operatorname{mod} \Omega_{18}^5\right). \tag{2}$$

Expanding out the commutators in Eq. (2), we observe that $x_1^2 x_3 v_3$ is the unique term in Eq. (2) that has the highest exponent of x_1. Since $\operatorname{char}(\mathbb{F}) \neq 3$, this can happen only if $v_3 \in \Omega_{18}$. Thus, $x_1^2 x_3 v_3 \in \Omega_{18}^5$. The highest exponent of x_1 in Eq. (2) then appears in $x_1 x_3 v_2$ and $x_1 x_4 u_2$. So, these terms have to cancel out with each other. We deduce that $u_2 \in \Omega_{18}$ and $v_2 \in \Omega_{18}^2$. Now Eq. (2) reduces to $\beta_2 x_3 v_1 \equiv \delta x_4 u_1 \pmod{\Omega_{18}^5}$. This implies that $u_1 \in \Omega_{18}^2$ and $v_1 \in \Omega_{18}^3$. So, if we write

$$\begin{aligned}
f(x_2) &\equiv \beta_2 x_2 + u_0 + x_1 u_1 + x_1^2 u_2 + x_1^3 u_3 + x_1^4 u_4 \quad \left(\operatorname{mod} \Omega_{18}^5\right), \\
f(x_3) &\equiv \delta x_3 + v_0 + x_1 v_1 + x_1^2 v_2 + x_1^3 v_3 + x_1^4 v_4 \quad \left(\operatorname{mod} \Omega_{18}^5\right),
\end{aligned}$$

then each u_i and v_i is a linear combination of (possibly trivial) PBW monomials that do not involve x_1, u_0 has weight at least 2, v_0 has weight at least 3, weight of $x_1 v_1 + x_1^2 v_2 + x_1^3 v_3 + x_1^4 v_4$ is at least 4, and weight of $x_1u_1 + x_1^2u_2$ is at least 3. Thus,

$$-\alpha_1^3\delta x_6 = f\big([x_3, x_2]\big) \equiv \beta_2 \sum_{i=1}^{4}\big[x_1^i, x_2\big]v_i - \delta \sum_{j=1}^{3}\big[x_1^j, x_3\big]u_j \pmod{\Omega_{18}^6}. \tag{3}$$

Arguing as in Eq. (2), we deduce that Eq. (3) reduces to the following:

$$-\alpha_1^3\delta x_6 \equiv \beta_2 x_3 v_1 - \delta x_4 u_j \pmod{\Omega_{18}^6}.$$

The latter is possible only if $\alpha_1 = 0$ or $\delta = 0$. Since we have already proved that $\beta_1 = 0$ it follows that $\delta = 0$ which is a contradiction.

The argument in the proof of Lemma 3.2 shows, for $\mathrm{char}(\mathbb{F}) = 3$, that the map $x_1 \mapsto x_1$, $x_2 \mapsto x_2 + x_1^3$ can be extended to an isomorphism between Ω_{17} and Ω_{18}. □

4.5. Family (5)

$$K_{23} = \big\langle x_1, x_2, x_3; x_4, x_5; x_6 \,\big|\, [x_1, x_2] = x_4,\ [x_1, x_4] = x_6,\ [x_1, x_3] = x_5,\ [x_2, x_3] = x_6\big\rangle;$$
$$K_{25} = \big\langle x_1, x_2, x_3; x_4, x_5; x_6 \,\big|\, [x_1, x_2] = x_4,\ [x_1, x_4] = x_6,\ [x_1, x_3] = x_5\big\rangle.$$

Lemma 4.6. *The algebras Ω_{23} and Ω_{25} are not isomorphic.*

Proof. Suppose that $\mathrm{char}\,\mathbb{F} \neq 3$ and assume, by contradiction, that $f : \Omega_{25} \to \Omega_{23}$ is an isomorphism. For $i = 1, 2, 3$, write $f(x_i)$ as a linear combination of PBW monomials and assume that $\alpha_{i_1,\dots,i_n}$, $\beta_{i_1,\dots,i_n}$, and $\gamma_{i_1,\dots,i_n}$ are the coefficients of $x_{i_1}\cdots x_{i_n}$ in $f(x_1)$, $f(x_2)$, and $f(x_3)$, respectively. Then

$$0 = f\big([x_2, x_3]\big) \equiv (\beta_1\gamma_2 - \beta_2\gamma_1)x_4 + (\beta_1\gamma_3 - \beta_3\gamma_1)x_5 \pmod{\Omega_{23}^3},$$

and hence $\beta_1\gamma_2 - \beta_2\gamma_3 = \beta_1\gamma_3 - \beta_3\gamma_1 = 0$. Since f induces an isomorphism between $\Omega_{25}/(\Omega_{25})^2$ and $\Omega_{23}/(\Omega_{23})^2$, we obtain that $\beta_1 = \gamma_1 = 0$. Note that

$$f(x_5) = f\big([x_1, x_3]\big) \equiv \alpha_1\gamma_2 x_4 + \alpha_1\gamma_3 x_5 \pmod{\Omega_{23}^3}.$$

Thus, $0 = f([x_1, x_5]) \equiv \alpha_1^2\gamma_2 x_6 \pmod{(\Omega_{23})^4}$, which gives that $\gamma_2 = 0$. Now we calculate $f([x_2, x_3])$ modulo $(\Omega_{23})^4$ to show that $\beta_2\gamma_3 = 0$. As $[x_4, \Omega_{23}] \leqslant (\Omega_{23})^4$, we have, modulo $(\Omega_{23})^4$, that

$$\begin{aligned} f\big([x_2, x_3]\big) \equiv{}& \beta_2\gamma_3 x_6 + \beta_2\gamma_{1,1}(x_6 - 2x_1x_4) - \beta_2\gamma_{1,2}x_2x_4 - \beta_2\gamma_{1,3}x_3x_4 \\ &+ 2(\beta_{1,1}\gamma_3 - \beta_3\gamma_{1,1})x_1x_5 + (\beta_{1,2}\gamma_3 - \beta_3\gamma_{1,2})x_2x_5 + (\beta_{1,3}\gamma_3 - \beta_3\gamma_{1,3})x_3x_5. \end{aligned}$$

This gives that $\beta_2\gamma_3 = 0$ which implies that the images $f(x_2)$, $f(x_2)$, $f(x_2)$ are linearly dependent modulo $(\Omega_{23})^2$, which is a contradiction. □

4.6. Family (6)

We consider the following Lie algebras:

$$K_9 = \big\langle x_1, x_2, x_3; x_4; x_5, x_6 \,\big|\, [x_1, x_2] = x_4,\ [x_1, x_4] = x_5,\ [x_2, x_4] = x_6\big\rangle;$$
$$\begin{aligned} K_{24}(\varepsilon) = \big\langle x_1, x_2, x_3; x_4; x_5, x_6 \,\big|\, &[x_1, x_2] = x_4,\ [x_1, x_4] = x_5,\ [x_1, x_3] = \varepsilon x_6, \\ &[x_2, x_4]=x_6,\ [x_2, x_3] = x_5\big\rangle. \end{aligned}$$

The family $K_{24}(\varepsilon)$ is a parametric family of Lie algebras such that $K_{24}(\varepsilon_1) \cong K_{24}(\varepsilon_2)$ if and only if there is a $\nu \in \mathbb{F}$ such that $\varepsilon_1 \nu^2 = \varepsilon_2$ (see [dG]).

Lemma 4.7. *The algebras Ω_9 and $\Omega_{24}(\varepsilon)$ are not isomorphic, for all $\varepsilon \in \mathbb{F}$. Furthermore, $\Omega_{24}(\varepsilon_1) \cong \Omega_{24}(\varepsilon_2)$ if and only if $K_{24}(\varepsilon_1) \cong K_{24}(\varepsilon_2)$, for every $\varepsilon_1, \varepsilon_2 \in \mathbb{F}$.*

Proof. Since $x_3 \in Z(\Omega_9) \setminus (\Omega_9)^2$, we have that $Z(\Omega_9) \nleqslant (\Omega_9)^2$. Let $\varepsilon \in \mathbb{F}$. We claim that $Z(\Omega_{24}(\varepsilon)) \leqslant (\Omega_{24}(\varepsilon))^2$. Let $z \in Z(\Omega_{24}(\varepsilon))$ and write z as a linear combination of PBW monomials in which $\alpha_{i_1,\dots,i_n}$ denotes the coefficient of $x_1 \cdots x_n$. First we compute, modulo $(\Omega_{24}(\varepsilon))^3$, that $0 = [z, x_3] \equiv \alpha_1 \varepsilon x_6 + \alpha_2 x_5$, which gives that $\alpha_2 = 0$. Then,

$$[z, x_2] \equiv \alpha_1 x_4 + (-\alpha_3 - \alpha_{1,1}) x_5 + 2\alpha_{1,1} x_1 x_4 + \alpha_{1,2} x_2 x_4 + \alpha_{1,3} x_3 x_4 \quad \bigl(\mathrm{mod}\ \Omega_{24}^4(\varepsilon)\bigr),$$

which shows that $\alpha_1 = \alpha_3 = 0$. Hence $z \in (\Omega_{24}(\varepsilon))^2$, and so $Z(\Omega_{24}(\varepsilon)) \leqslant (\Omega_{24}(\varepsilon))^2$, as claimed. This implies that $\Omega_9 \ncong \Omega_{24}(\varepsilon)$.

Let us now prove the second assertion of the lemma. Without loss of generality, we assume that $\varepsilon_2 \neq 0$. Let $K = K_{24}(\varepsilon_2)$ and $\Omega = \Omega(K)$. Suppose that $f : \Omega_{24}(\varepsilon_1) \to \Omega$ is an algebra isomorphism. As usual, for $i = 1, 2, 3$, we write the images $f(x_i)$ as linear combinations of PBW monomials, and let $\alpha_{i_1,\dots,i_n}$, $\beta_{i_1,\dots,i_n}$ and $\gamma_{i_1,\dots,i_n}$ denote the coefficients of $x_{i_1} \cdots x_{i_n}$ in $f(x_1)$, $f(x_2)$, and $f(x_3)$, respectively. Since $f([x_1, x_3])$, $f([x_2, x_3]) \in \Omega^3$, we deduce that $\alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$ and that $\beta_1\gamma_2 - \beta_2\gamma_1 = 0$. Since the images $f(x_1)$, $f(x_2)$, $f(x_2)$ are linearly independent modulo Ω^2, this gives that $\gamma_1 = \gamma_2 = 0$. Set $\delta = \alpha_1\beta_2 - \alpha_2\beta_1$. Since $f(x_1)$, $f(x_2)$, and $f(x_3)$ are linearly independent modulo Ω^2, we have $\delta \neq 0$. Further,

$$f(x_4) = \bigl[f(x_1), f(x_2)\bigr] \equiv \delta[x_1, x_2] = \delta x_4 \quad \bigl(\mathrm{mod}\ \Omega^3\bigr).$$

Thus, $f([x_1, x_4]) \equiv \alpha_1\delta x_5 + \alpha_2\delta x_6$ (mod Ω^4). So, modulo Ω^4, we have,

$$\begin{aligned} f\bigl([x_2, x_3]\bigr) \equiv{}& (\beta_1\gamma_4 + \beta_2\gamma_3 + \beta_2\gamma_{1,1}) x_5 + (\varepsilon_2\beta_1\gamma_3 - \beta_1\gamma_{2,2} + \beta_2\gamma_{1,2} + \beta_2\gamma_4) x_6 \\ &+ (\beta_1\gamma_{1,2} - 2\beta_2\gamma_{1,1}) x_1 x_4 + (2\beta_1\gamma_{2,2} - \beta_2\gamma_{1,2}) x_2 x_4 + (\beta_1\gamma_{2,3} - \beta_2\gamma_{1,3}) x_3 x_4. \end{aligned}$$

As $f([x_1, x_4]) = f([x_2, x_3])$ we obtain the following equations:

$$\beta_1\gamma_4 + \beta_2\gamma_3 + \beta_2\gamma_{1,1} = \alpha_1\delta; \tag{4}$$

$$\varepsilon_2\beta_1\gamma_3 - \beta_1\gamma_{2,2} + \beta_2\gamma_{1,2} + \beta_2\gamma_4 = \alpha_2\delta; \tag{5}$$

$$\beta_1\gamma_{1,2} - 2\beta_2\gamma_{1,1} = 0; \tag{6}$$

$$2\beta_1\gamma_{2,2} - \beta_2\gamma_{1,2} = 0; \tag{7}$$

$$\beta_1\gamma_{2,3} - \beta_2\gamma_{1,3} = 0. \tag{8}$$

Now we use the relation $\varepsilon_1[x_2, x_4] = [x_1, x_3]$. So, modulo Ω^4, we have

$$\begin{aligned} f\bigl([x_2, x_4]\bigr) \equiv{}& \beta_1\delta x_5 + \beta_2\delta x_6, \\ f\bigl([x_1, x_3]\bigr) \equiv{}& (\alpha_1\gamma_4 + \alpha_2\gamma_3 + \alpha_2\gamma_{1,1}) x_5 + (\varepsilon_2\alpha_1\gamma_3 - \alpha_1\gamma_{2,2} + \alpha_2\gamma_{1,2} + \alpha_2\gamma_4) x_6 \\ &+ (\alpha_1\gamma_{1,2} - 2\alpha_2\gamma_{1,1}) x_1 x_4 + (2\alpha_1\gamma_{2,2} - \alpha_2\gamma_{1,2}) x_2 x_4 + (\alpha_1\gamma_{2,3} - \alpha_2\gamma_{1,3}) x_3 x_4. \end{aligned}$$

We get the following equations:

$$\alpha_1\gamma_4 + \alpha_2\gamma_3 + \alpha_2\gamma_{1,1} = \varepsilon_1\beta_1\delta; \tag{9}$$

$$\varepsilon_2\alpha_1\gamma_3 - \alpha_1\gamma_{2,2} + \alpha_2\gamma_{1,2} + \alpha_2\gamma_4 = \varepsilon_1\beta_2\delta; \tag{10}$$

$$\alpha_1\gamma_{1,2} - 2\alpha_2\gamma_{1,1} = 0; \tag{11}$$

$$2\alpha_1\gamma_{2,2} - \alpha_2\gamma_{1,2} = 0; \tag{12}$$

$$\alpha_1\gamma_{2,3} - \alpha_2\gamma_{1,3} = 0. \tag{13}$$

Eqs. (6) and (11) imply that $\gamma_{1,1} = \gamma_{1,2} = 0$. Similarly $\gamma_{2,2} = \gamma_{2,3} = \gamma_{1,3} = 0$. Thus the system of equations above are reduced to the following:

$$\beta_1\gamma_4 + \beta_2\gamma_3 = \alpha_1\delta; \tag{14}$$

$$\varepsilon_2\beta_1\gamma_3 + \beta_2\gamma_4 = \alpha_2\delta; \tag{15}$$

$$\alpha_1\gamma_4 + \alpha_2\gamma_3 = \varepsilon_1\beta_1\delta; \tag{16}$$

$$\varepsilon_2\alpha_1\gamma_3 + \alpha_2\gamma_4 = \varepsilon_1\beta_2\delta. \tag{17}$$

Set

$$\begin{aligned}
p_1 &= (-1/2)\varepsilon_1\beta_2\delta^{-2} - (1/2)\varepsilon_2\alpha_1\gamma_3\delta^{-3};\\
p_2 &= (1/2)\alpha_1\gamma_4\delta^{-3} + \alpha_2\gamma_3\delta^{-3};\\
p_3 &= \alpha_2\delta^{-2} - (1/2)\beta_2\gamma_4\delta^{-3};\\
p_4 &= (-1/2)\alpha_1\delta^{-2} - (1/2)\beta_2\gamma_3\delta^{-3}.
\end{aligned}$$

We can check that

$$\begin{aligned}
&p_1(\beta_1\gamma_4 + \beta_2\gamma_3 - \alpha_1\delta) + p_2(\varepsilon_2\beta_1\gamma_3 + \beta_2\gamma_4 - \alpha_2\delta)\\
&\quad + p_3(\alpha_1\gamma_4 + \alpha_2\gamma_3 - \varepsilon_1\beta_1\delta) + p_4(\varepsilon_2\alpha_1\gamma_3 + \alpha_2\gamma_4 - \varepsilon_1\beta_2\delta) = \varepsilon_1 - \varepsilon_2\gamma_3^2\delta^{-2}.
\end{aligned}$$

Thus, considering that $\delta \neq 0$, the equation $\varepsilon_1 = \varepsilon_2\gamma_3^2\delta^{-2}$ follows from Eqs. (14)–(17). However this implies that $K_{24}(\varepsilon_1) \cong K_{24}(\varepsilon_2)$ as claimed. □

Acknowledgment

The second author would like to thank the Centro de Álgebra da Universidade de Lisboa for hosting his visit in April 2010 while this work was completed.

References

[CKL] J. Chun, T. Kajiwara, J. Lee, Isomorphism theorem on low dimensional Lie algebras, Pacific J. Math. 214 (1) (2004) 17–21.

[D] J. Dixmier, Enveloping Algebras, revised reprint of the 1977 translation, Grad. Stud. Math., vol. 11, American Mathematical Society, Providence, RI, 1996, xx+379 pp.

[E] B. Eick, Computing automorphism groups and testing isomorphisms for modular group algebras, J. Algebra 320 (11) (2008) 3895–3910.

[MI] B. Eick, ModIsom — a GAP package, version 1.1, May 2009, www-public.tu-bs.de:8080/~beick/soft/modisom.

[GAP] The GAP group, GAP — Groups, Algorithms, and Programming, version 4.4.12, 2008, http://www.gap-system.org.

[dG] W.A. de Graaf, Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2, J. Algebra 309 (2) (2007) 640–653.
[KL] G.R. Krause, T.H. Lenagan, Growth of Algebras and Gelfand–Kirillov Dimension, revised edition, Grad. Stud. Math., vol. 22, American Mathematical Society, Providence, RI, 2000, x+212 pp.
[M] P. Malcolmson, Enveloping algebras of simple three-dimensional Lie algebras, J. Algebra 146 (1992) 210–218.
[R] D. Riley, The dimension subalgebra problem for enveloping algebras of Lie superalgebras, Proc. Amer. Math. Soc. 123 (10) (1995) 2975–2980.
[RU] D. Riley, H. Usefi, The isomorphism problem for universal enveloping algebras of Lie algebras, Algebr. Represent. Theory 10 (6) (2007) 517–532.
[Sch] C. Schneider, A computer-based approach to the classification of nilpotent Lie algebras, Experiment. Math. 14 (2) (2005) 153–160.