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ON NORMAL CHARACTERIZATIONS BY THE DISTRIBUTION OF LINEAR FORMS, ASSUMING FINITE VARIANCE

Barry C. ARNOLD
Dean L. ISAACSON

Departments of Statistics and Mathematics Iowa State University, Ames, IA, U.S.A.

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If X_1 and X_2 are independent and identically distributed (i. i. d.) with finite variance, then $(X_1+X_2)/\sqrt{2}$ has the same distribution as X_1 if and only if X_1 is normal with mean zero (Pólya [9]). The idea of using linear combinations of i. i. d. random variables to characterize the normal has since been extended to the case where $\sum_{i=1}^{\infty} a_i X_i$ has the same distribution as X_1. In particular if at least two of the a_i's are non-zero and X_1 has finite variance, then Laha and Lukacs [8] showed that X_1 is normal. They also [7] established the same result without the assumption of finite variance. The purpose of this note is to present a different and easier proof of the characterization under the assumption of finite variance. The idea of the proof follows closely the approach used by Pólya in [9]. The same technique is also used to give a characterization of the exponential distribution.

Linear forms   characteristic functions
symmetric random variables

1. Introduction

Suppose X_1 and X_2 are independent identically distributed (i. i. d.) random variables (r. v.) with a common normal distribution with mean 0. It is evident that under these circumstances $(X_1+X_2)/\sqrt{2}$ and X_1 have identical distributions, written $(X_1+X_2)/\sqrt{2} \overset{d}{=} X_1$. In 1923 Polya noted that under the assumption of finite variance, this property, i.e. $(X_1+X_2)/\sqrt{2} \overset{d}{=} X_1$, characterizes the normal distribution. Subsequent work centered on generalizing this result and if possible eliminating the restriction of finite variances. In the more general setting one considered a sequence $\{X_i\}_{i=1}^{\infty}$ of i. i. d. r. v. and infinite linear combinations $\boldsymbol{a}'\boldsymbol{X} = \sum_{i=1}^{\infty} a_i X_i$, $\boldsymbol{b}'\boldsymbol{X} = \sum_{i=1}^{\infty} b_i X_i$. The problem of characterizations of the normal distribution based on $\boldsymbol{a}'\boldsymbol{X} \overset{d}{=} \boldsymbol{b}'\boldsymbol{X}$ for distinct vectors $\boldsymbol{a}, \boldsymbol{b}$ is well summarized in Kagan, Linnik, and Rao [6]. The problem is considerably simplified if it is required that the vector $\boldsymbol{b}'$ be of the form $(1, 0, 0, 0, \ldots)$. The standard work on characterizing the normal based on $\boldsymbol{a}'\boldsymbol{X} \overset{d}{=} X_1$ is that of Laha and Lukacs. They (Laha and

Lukacs [7]) show without moment conditions that if $\sum_{i=1}^{\infty} a_i^2 = 1$ and $\boldsymbol{a}'\boldsymbol{X} =^{\mathrm{d}} X_1$, non-degenerate, then X_1 is normally distributed. They first prove that the distribution of X must be infinitely divisible and then, using the Lévy-Khintchine representation of such distributions, carefully eliminate the possibility of a non-normal distribution. Eaton [3] has done related work on the multivariate normal and non-normal symmetric stable distributions using a similar technique (for the non-normal stable distributions it is found that identical distribution of X_1 and two other linear forms is required).

Laha and Lukacs [8] considered the possibility of simplifying the argument used in their earlier paper [7] by assuming the existence of a variance. An alternative representation of such infinitely divisible laws is then available but the proof remains somewhat parallel to that used when the variance was not assumed to exist. It is not as technically difficult but still it has not been reduced to a "classroom exercise." A convenient reference for proofs of the above results is Ghurye and Olkin's recent survey paper [4]. They also include an extension of Eaton's multivariate result.

It is the purpose of this note to present a proof, in the finite variance case, which is at a quite elementary level. The proof presented is really quite similar to that proposed originally by Pólya [9] in the simple case where $(X_1+X_2)/\sqrt{2} =^{\mathrm{d}} X_1$ and it is plausible that he was aware of its possible extension to the more general case.

2. The characterization

Throughout this section we will consider an i. i. d. sequence $\{X_i\}_{i=1}^{\infty}$ of non-degenerate random variables and will denote their common characteristic function by φ.

Theorem. *If* $\mathrm{E}(X_i^2)$ *exists and, for some vector a with at least two non-zero coordinates,* $\sum_{i=1}^{\infty} a_i X_i =^{\mathrm{d}} X_1$, *then* X_1 *has a normal distribution.*

Proof. First observe that if $\sum_{i=1}^{\infty} a_i X_i =^{\mathrm{d}} X_1$ then the same is true of the symmetrized i. i. d. sequence $\{Y_i\}_{i=1}^{\infty}$ where $Y_i =^{\mathrm{d}} X_i - X_i'$; $X_i =^{\mathrm{d}} X_i'$; X_i, X_i' independent. These Y_i's have common mean 0 and variance τ^2. The characteristic function $\psi(t)$ of the Y_i's satisfies $\psi(t) = \varphi(t)\varphi(-t)$. Since $\sum_{i=1}^{\infty} a_i Y_i =^{\mathrm{d}} Y_1$, it follows that

$$\psi(t) = \prod_{i=1}^{\infty} \psi(ta_i). \tag{1}$$

The characteristic function of the symmetric random variable Y_i is real and so taking logarithms we find

$$\log \psi(t) = -\tfrac{1}{2}\tau^2 t^2 + \mathrm{o}(|t|^2) \quad (t \to 0) \tag{2}$$

since $\mathbf{E}(Y)=0$ and $\mathbf{E}(Y^2)=\tau^2$. Iteration of (1) leads to, for any n,

$$\log \psi(t)=\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}\log \psi(ta_{i_1}a_{i_2}\cdots a_{i_n}). \tag{3}$$

Now since $\sum_{i=1}^{\infty} a_i Y_i \stackrel{d}{=} Y_1$, their second moments must be equal so we have $\sum_{i=1}^{\infty} a_i^2=1$. Let $m^*=\max_i a_i^2$. Clearly $m^*<1$ (since a has at least two non-zero coordinates). Now fix $t\neq 0$ and $\varepsilon>0$ and choose n sufficiently large so that

$$0<|s|^2\leq m^{*n}t^2 \Rightarrow |\log \psi(s)+\tfrac{1}{2}\tau^2 s^2|<\varepsilon s^2 \tag{4}$$

which is possible by (2). Using (3), (4) and the fact that $\sum_{i=1}^{\infty} a_i^2=1$, we have,

$$\begin{aligned}|\log \psi(t)+\tfrac{1}{2}\tau^2t^2| &= \left|\log \psi(t)+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}\tfrac{1}{2}a_{i_1}^2a_{i_2}^2\cdots a_{i_n}^2t^2\tau^2\right| \\ &\leq \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}\Big|\log \psi(ta_{i_1}a_{i_2}\cdots a_{i_n}) \\ &\qquad +\tfrac{1}{2}\tau^2(ta_{i_1}a_{i_2}\cdots a_{i_n})^2\Big| \\ &<\varepsilon \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty} t^2a_{i_1}^2a_{i_2}^2\cdots a_{i_n}^2=\varepsilon t^2\end{aligned}$$

where the last inequality follows since $(ta_{i_1}a_{i_2}\cdots a_{i_n})^2\leq m^{*n}t^2$. Since ε was arbitrary, it follows that $\psi(t)=\exp(-\frac{1}{2}\tau^2t^2)$ and so Y_1 is normal $(0,\tau^2)$. Recall that $Y_1=X_1-X_1'$ where X_1, X_1' are independent, so it follows by Cramer's normal composition theorem that X_1 is normal.

Note. The variance of X_1 is necessarily $\tau^2/2$ but its mean is not necessarily determined. If $\sum_{i=1}^{\infty} a_i=1$ (which requires at least one of the a_i's to be negative since $\sum_{i=1}^{\infty} a_i^2=1$), then the mean of X_1 could be any real number. If $\sum_{i=1}^{\infty} a_i\neq 1$, then, in order to have $\sum_{i=1}^{\infty} a_iX_i \stackrel{d}{=} X_1$, we must have $\mathbf{E}(X_1)=0$. This last possibility was not explicitly mentioned by Laha and Lukacs [8].

3. Related characterizations

If one considers non-negative random variables whose distribution has a right derivative at 0 one may verify that, for a sequence $X_1, X_2, \ldots$ of such i. i. d. r. v. and a vector $(a_1, a_2, \ldots)$ with at least two non-zero entries, $\min_i a_iX_i \stackrel{d}{=} X_1$ if and only if X_1 has an exponential distribution. The proof is a close parallel to that given here. The proof for $a_1=a_2=2$ was presented by Arnold [1] and independently by Gupta [5]. Arnold and Isaacson [2] treated the case of a finite sequence $X_1, X_2, \ldots, X_n$ and it is that proof which readily generalizes to an infinite sequence and which suggested the above proof for Laha and Lukacs' normal characterization.

Finally we mention a related characterization problem which appears to be more difficult to resolve. Suppose that $(A_1, A_2, \ldots)$ is a non-degenerate random sequence independent of the X_i's satisfying $\sum_{i=1}^{\infty} A_i^2 = 1$ a.s. and such that $\sum_{i=1}^{\infty} A_i X_i \stackrel{d}{=} X_1$. Can we in this case conclude that X_1 is normally distributed? Such a phenomenon would arise if one considered a sample of non-degenerate random size N. If the X_i's are i. i. d. normal $(0, \sigma^2)$, then $\sum_{i=1}^{N} X_i/\sqrt{N} \stackrel{d}{=} X_1$. Initially one might assume variances existed but, in the light of Laha and Lukacs [7], this might be unnecessary. In the random sample size case, we are led to consider the functional equation

$$\varphi(t) = \sum_{n=1}^{\infty} p_n[\varphi(t/\sqrt{n})]^n, \tag{5}$$

Clearly (5) admits solutions of the form $\varphi(t) = \exp(-\frac{1}{2}t^2\sigma^2)$ but we have been unable to show that other solutions are not possible.

References

[1] B.C. Arnold, Two characterizations of the exponential distribution using order statistics (1971) Unpublished.

[2] B.C. Arnold and D. Isaacson, On solutions to $\min(X, Y) \stackrel{d}{=} aX$ and $\min(X, Y) \stackrel{d}{=} aX \stackrel{d}{=} bY$, Z. Wahr. Verw. Geb. 35 (1976) 115–119.

[3] M.L. Eaton, Characterization of distributions by the identical distribution of linear forms, J. Appl. Prob. 3 (1966) 481–494.

[4] S.G. Ghurye and I. Olkin, Identically distributed linear forms and the normal distribution, Adv. Appl. Prob. 5 (1973) 138–152.

[5] R.C. Gupta, A characteristic property of the exponential distribution, Sankhya, Series B 35 (1973) 365–366.

[6] A.M. Kagan, Y.V. Linnik and C.R. Rao, Characterization Problems in Mathematical Statistics (Wiley, New York, 1973)

[7] R.G. Laha and E. Lukacs, On a linear form whose distribution is identical with that of a monomial, Pac. J. Math. 15, (1965) 207–214.

[8] R.G. Laha and E. Lukacs, On linear forms and stochastic integrals, Bull. Inst. Internat. Statist. 41 (1965) 828–840.

[9] G. Pólya, Herleitung des Gausther Fehlergesetzes aus einer Funktionalgleichung, Math. Zeitschrift 18 (1923) 96–108.