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# Statistical $L_p$-approximation by double Gauss–Weierstrass singular integral operators

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ABSTRACT

In this paper, we study statistical $L_p$-approximation properties of the double Gauss–Weierstrass singular integral operators which do not need to be positive. Also, we present a non-trivial example showing that our statistical $L_p$-approximation is stronger than the ordinary one.



## 1. Introduction

In this paper we study some statistical approximation properties of a certain family of linear operators which do not need to be positive. Recall that, for such operators, the classical Korovkin theorem does not work due to non-positivity. However, we show that it is possible to approximate (both in the ordinary sense and the statistical sense) to functions by these operators. Readers will find some related studies for the ordinary approximation in the papers [1–9] and the references cited therein. We also give an example of why we need statistical approximation instead of ordinary approximation. In the literature there are many papers about statistical approximation (see, e.g., [10–17]).

First of all, we give some basic definitions and notations used in the present paper.

Let $A := [a_{jn}],\ j, n = 1, 2, \ldots,$ be an infinite summability matrix and assume that, for a given sequence $x = (x_n)_{n\in\mathbb{N}}$, the series $\sum_{n=1}^{\infty} a_{jn}x_n$ converges for every $j \in \mathbb{N}$. Then, by the $A$-transform of $x$, we mean the sequence $Ax = ((Ax)_j)_{j\in\mathbb{N}}$ such that, for every $j \in \mathbb{N}$,

$$(Ax)_j := \sum_{n=1}^{\infty} a_{jn}x_n.$$

A summability matrix $A$ is said to be regular (see [18]) if, for every $x = (x_n)_{n\in\mathbb{N}}$ for which $\lim_{n\to\infty} x = L$, we get $\lim_{j\to\infty}(Ax)_j = L$. Now, fix a non-negative regular summability matrix $A$. In [19], Freedman and Sember introduced a convergence method, the so-called $A$-statistical convergence, as in the following way. A given sequence $x = (x_n)_{n\in\mathbb{N}}$ is said to be $A$-statistically convergent to $L$ if, for every $\varepsilon > 0$,

$$\lim_{j\to\infty} \sum_{n:|x_n-L|\geq\varepsilon} a_{nj} = 0.$$

This limit is denoted by $st_A - \lim_n x_n = L$.

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Observe that, if $A = C_1 = [c_{jn}]$, the Cesáro matrix of order one defined to be $c_{jn} = 1/j$ if $1 \leq n \leq j$, and $c_{jn} = 0$ otherwise, then $C_1$-statistical convergence coincides with the concept of statistical convergence, which was first introduced by Fast [20]. In this case, we use the notation $st - \lim$ instead of $st_{C_1} - \lim$ (see Section 5 for this situation). Notice that every convergent sequence is $A$-statistically convergent to the same value for any non-negative regular matrix $A$; however, the converse is not always true. Not all properties of convergent sequences hold true for $A$-statistical convergence (or statistical convergence). For instance, although it is well known that a subsequence of a convergent sequence is convergent, this is not always true for $A$-statistical convergence. Another example is that every convergent sequence must be bounded; however, it does not need to be bounded of an $A$-statistically convergent sequence. Of course, with these properties, the use of $A$-statistical convergence in the approximation theory provides us many advantages.

## 2. Construction of the operators

Consider the set $\mathbb{D}$ given by

$$\mathbb{D} := \left\{(s,t) \in \mathbb{R}^2 : s^2 + t^2 \leq \pi^2\right\}.$$

As usual, by $L_p(\mathbb{D})$ we denote the space of all functions $f$ defined on $\mathbb{D}$ for which

$$\iint_{\mathbb{D}} |f(x,y)|^p \,\mathrm{d}x\mathrm{d}y < \infty, \quad 1 \leq p < \infty.$$

In this case, the $L_p$-norm of a function $f$ in $L_p(\mathbb{D})$, denoted by $\|f\|_p$, is given by

$$\|f\|_p = \left(\iint_{\mathbb{D}} |f(x,y)|^p \,\mathrm{d}x\mathrm{d}y\right)^{1/p}.$$

Throughout the paper, for $r \in \mathbb{N}$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we use

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-m} & \text{if } j = 1,2,\ldots,r, \\ 1 - \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j} j^{-m} & \text{if } j = 0 \end{cases} \tag{2.1}$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1,2,\ldots, \; m \in \mathbb{N}. \tag{2.2}$$

We observe that

$$\sum_{j=0}^{r} \alpha_{j,r} = 1 \quad \text{and} \quad -\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j} = (-1)^r\binom{r}{0}. \tag{2.3}$$

Assume now that $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers. Setting

$$\lambda_n := \frac{1}{\pi\left(1 - \mathrm{e}^{-\pi^2/\xi_n^2}\right)} \quad \left(\lambda_n \to \frac{1}{\pi}, \text{ as } \xi_n \to 0\right), \tag{2.4}$$

we introduce the following double smooth Gauss–Weierstrass singular integral operators:

$$W_{r,n}^{[m]}(f;x,y) = \frac{\lambda_n}{\xi_n^2}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(\iint_{\mathbb{D}} f\,(x+sj, y+tj)\,\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right), \tag{2.5}$$

where $(x,y) \in \mathbb{D}$, $n, r \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $f \in L_p(\mathbb{D})$, $1 \leq p < \infty$.

**Remarks.** • The operators $W_{r,n}^{[m]}$ are not in general positive. For example, consider the non-negative function $\varphi(u,v) = u^2 + v^2$ and also take $r = 2$, $m = 3$, $x = 0$ and $y = 0$ in (2.5).
- It is not hard to see that the operators $W_{r,n}^{[m]}$ preserve the constant functions in two variables.
- We observe, for any $\alpha > 0$, that

$$\iint_{\mathbb{D}} \mathrm{e}^{-(s^2+t^2)/\alpha}\mathrm{d}s\mathrm{d}t = \alpha\pi\left(1 - \mathrm{e}^{-\pi^2/\alpha}\right). \tag{2.6}$$

- Let $k \in \mathbb{N}_0$. Then, it holds, for each $\ell = 0, 1, \ldots, k$ and for every $n \in \mathbb{N}$, that

$$\iint_{\mathbb{D}} s^{k-\ell} t^{\ell} \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 2\gamma_{n,k} B\left(\dfrac{k-\ell+1}{2}, \dfrac{\ell+1}{2}\right) & \text{if } k \text{ is even,} \end{cases} \tag{2.7}$$

where $B(a, b)$ denotes the Beta function, and

$$\gamma_{n,k} := \int_0^{\pi} \rho^{k+1} \mathrm{e}^{-\rho^2/\xi_n^2} \mathrm{d}\rho = \frac{\xi_n^{k+2}}{2} \left\{ \Gamma\left(1 + \frac{k}{2}\right) - \Gamma\left(1 + \frac{k}{2}, \left(\frac{\pi}{\xi_n}\right)^2\right) \right\}, \tag{2.8}$$

where $\Gamma(\alpha, z) = \int_z^{\infty} t^{\alpha-1} \mathrm{e}^{-t} \mathrm{d}t$ is the incomplete gamma function and $\Gamma$ is the gamma function.

## 3. Estimates for the operators (2.5)

For $f \in L_p(\mathbb{D})$ and $2\pi$-periodic per coordinate, the $r$th (double) $L_p$-modulus of smoothness of $f$ is given by (see, e.g., [21])

$$\omega_r(f; h)_p := \sup_{\sqrt{u^2+v^2} \le h} \left\| \Delta_{u,v}^r(f) \right\|_p < \infty, \quad h > 0,\ 1 \le p < \infty, \tag{3.1}$$

where

$$\Delta_{u,v}^r\left(f(x, y)\right) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x + ju, y + jv). \tag{3.2}$$

Throughout this paper we use the notation

$$\partial^{m-\ell,\ell} f(x, y) := \frac{\partial^m f(x, y)}{\partial^{m-\ell} x \partial^{\ell} y} \quad \text{for } \ell = 0, 1, \ldots, m.$$

We assume that

$$f \in C_{\pi}^{(m)}(\mathbb{D}), \tag{3.3}$$

the space of functions $2\pi$-periodic per coordinate, having $m$ times continuous partial derivatives with respect to the variables $x$ and $y$, $m \in \mathbb{N}_0$.

### 3.1. Estimates in the case of $m \in \mathbb{N}$

In this subsection, we only consider the case of $m \in \mathbb{N}$.
For $r \in \mathbb{N}$ and $f$ satisfying (3.3), let

$$H_{r,n}^{[m]}(x, y) := W_{r,n}^{[m]}(f; x, y) - f(x, y) - \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left( \sum_{k=1}^{m} \frac{\delta_{k,r}^{[m]}}{k!} \sum_{\ell=0}^{k} \binom{k}{k-\ell} s^{k-\ell} t^{\ell} \partial^{k-\ell,\ell} f(x, y) \right) \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t.$$

By (2.7), since, for every $r, n, m \in \mathbb{N}$,

$$\begin{aligned} &\iint_{\mathbb{D}} \left( \sum_{k=1}^{m} \frac{\delta_{k,r}^{[m]}}{k!} \sum_{\ell=0}^{k} \binom{k}{k-\ell} s^{k-\ell} t^{\ell} \partial^{k-\ell,\ell} f(x, y) \right) \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t \\ &\quad = 2 \sum_{i=1}^{[m/2]} \frac{\delta_{2i,r}^{[m]} \gamma_{n,2i}}{(2i)!} \left\{ \sum_{\ell=0}^{2i} \binom{2i}{2i-\ell} \partial^{2i-\ell,\ell} f(x, y) B\left(\frac{2i-\ell+1}{2}, \frac{\ell+1}{2}\right) \right\}, \end{aligned}$$

where $[\cdot]$ is the integral part, we have

$$\begin{aligned} H_{r,n}^{[m]}(x, y) &= W_{r,n}^{[m]}(f; x, y) - f(x, y) - \frac{2\lambda_n}{\xi_n^2} \sum_{i=1}^{[m/2]} \frac{\delta_{2i,r}^{[m]} \gamma_{n,2i}}{(2i)!} \\ &\quad \times \left\{ \sum_{\ell=0}^{2i} \binom{2i}{2i-\ell} \partial^{2i-\ell,\ell} f(x, y) B\left(\frac{2i-\ell+1}{2}, \frac{\ell+1}{2}\right) \right\}, \end{aligned} \tag{3.4}$$

where $\gamma_{n,k}$ is given by (2.8). Now we get the next result.

**Lemma 3.1.** *For every $r, n, m \in \mathbb{N}$ and for all $f$ satisfying (3.3), we have*

$$H_{r,n}^{[m]}(x,y) = \frac{\lambda_n}{\xi_n^2 (m-1)!} \sum_{\ell=0}^{m} \iint_{\mathbb{D}} \left( \int_0^1 (1-w)^{m-1} \Delta_{sw,tw}^{r} \left( \partial^{m-\ell,\ell} f(x,y) \right) \mathrm{d}w \right) \binom{m}{m-\ell} s^{m-\ell} t^{\ell} \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t.$$

**Proof.** Let $(x,y) \in \mathbb{D}$ be fixed. By Taylor's formula, one can obtain that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} \left( f(x+js, y+jt) - f(x,y) \right) = \sum_{k=1}^{m} \frac{\delta_{k,r}^{[m]}}{k!} \sum_{\ell=0}^{k} \binom{k}{k-\ell} s^{k-\ell} t^{\ell} \partial^{k-\ell,\ell} f(x,y) + \frac{1}{(m-1)!} \int_0^1 (1-w)^{m-1} \varphi_{x,y}^{[m]}(w; s, t) \mathrm{d}w,$$

where

$$\begin{aligned} \varphi_{x,y}^{[m]}(w; s, t) &:= \sum_{j=0}^{r} \alpha_{j,r}^{[m]} j^m \left\{ \sum_{\ell=0}^{m} \binom{m}{m-\ell} s^{m-\ell} t^{\ell} \partial^{m-\ell,\ell} f(x+jsw, y+jtw) \right\} \\ &\quad - \delta_{m,r}^{[m]} \sum_{\ell=0}^{m} \binom{m}{m-\ell} s^{m-\ell} t^{\ell} \partial^{m-\ell,\ell} f(x,y) \\ &= \sum_{\ell=0}^{m} \binom{m}{m-\ell} s^{m-\ell} t^{\ell} \Delta_{sw,tw}^{r} \left( \partial^{m-\ell,\ell} f(x,y) \right). \end{aligned}$$

Then, by (3.4), we get

$$\begin{aligned} H_{r,n}^{[m]}(x,y) &= W_{r,n}^{[m]}(f; x, y) - f(x,y) - \frac{2\lambda_n}{\xi_n^2} \sum_{i=1}^{[m/2]} \frac{\delta_{2i,r}^{[m]} \gamma_{n,2i}}{(2i)!} \left\{ \sum_{\ell=0}^{2i} \binom{2i}{2i-\ell} \partial^{2i-\ell,\ell} f(x,y) B\left( \frac{2i-\ell+1}{2}, \frac{\ell+1}{2} \right) \right\} \\ &= \frac{\lambda_n}{\xi_n^2 (m-1)!} \sum_{\ell=0}^{m} \iint_{\mathbb{D}} \left( \int_0^1 (1-w)^{m-1} \Delta_{sw,tw}^{r} \left( \partial^{m-\ell,\ell} f(x,y) \right) \mathrm{d}w \right) \binom{m}{m-\ell} s^{m-\ell} t^{\ell} \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t, \end{aligned}$$

which completes the proof. □

**Theorem 3.2.** *Let $m, r \in \mathbb{N}$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in C_{\pi}^{(m)}(\mathbb{D})$. Then the following inequality*

$$\left\| H_{r,n}^{[m]} \right\|_p \le \frac{C \xi_n^m}{\left( 1 - \mathrm{e}^{-\pi^2/\xi_n^2} \right)^{\frac{1}{p}}} \left( \sum_{\ell=0}^{m} \omega_r \left( \partial^{m-\ell,\ell} f, \xi_n \right)_p^p \right)^{\frac{1}{p}}$$

*holds for some positive constant $C$ depending on $m, p, q, r$.*

**Proof.** By Lemma 3.1, we first obtain that

$$\begin{aligned} \left| H_{r,n}^{[m]}(x,y) \right|^p &\le \frac{\lambda_n^p}{\xi_n^{2p} \left( (m-1)! \right)^p} \left\{ \sum_{\ell=0}^{m} \iint_{\mathbb{D}} \left( \int_0^1 (1-w)^{m-1} \left| \Delta_{sw,tw}^{r} \left( \partial^{m-\ell,\ell} f(x,y) \right) \right| \mathrm{d}w \right) \right. \\ &\quad \left. \times \binom{m}{m-\ell} |s|^{m-\ell} |t|^{\ell} \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t \right\}^p \\ &= \frac{C_1}{\xi_n^{2p} \left( 1 - \mathrm{e}^{-\pi^2/\xi_n^2} \right)^p} \left\{ \sum_{\ell=0}^{m} \iint_{\mathbb{D}} \left( \int_0^1 (1-w)^{m-1} \left| \Delta_{sw,tw}^{r} \left( \partial^{m-\ell,\ell} f(x,y) \right) \right| \mathrm{d}w \right) \right. \\ &\quad \left. \times \binom{m}{m-\ell} |s|^{m-\ell} |t|^{\ell} \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t \right\}^p, \end{aligned}$$

where

$$C_1 := \frac{1}{\pi^p \left( (m-1)! \right)^p}.$$

Hence, we get

$$\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \le \frac{C_1}{\xi_n^{2p}\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)^p} \iint_{\mathbb{D}} \left(\iint_{\mathbb{D}} u_{x,y}(s,t)\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)^p \mathrm{d}x\mathrm{d}y,$$

where

$$u_{x,y}(s,t) = \sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}|t|^{\ell} \left(\int_0^1 (1-w)^{m-1} \left|\Delta_{sw,tw}^{r}\left(\partial^{m-\ell,\ell} f(x,y)\right)\right| \mathrm{d}w\right). \tag{3.5}$$

Then, using the Hölder's inequality for double integrals and also considering (2.6), we may write that

$$\begin{aligned}
\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y &\le \frac{C_1}{\xi_n^{2p}\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)^p} \iint_{\mathbb{D}} \left(\iint_{\mathbb{D}} u_{x,y}^p(s,t)\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y \left(\iint_{\mathbb{D}} \mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)^{\frac{p}{q}} \\
&= C_1 \frac{\left\{\pi\xi_n^2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)\right\}^{\frac{p}{q}}}{\xi_n^{2p}\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)^p} \iint_{\mathbb{D}} \left(\iint_{\mathbb{D}} u_{x,y}^p(s,t)\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y \\
&:= \frac{C_2}{\xi_n^2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \iint_{\mathbb{D}} \left(\iint_{\mathbb{D}} u_{x,y}^p(s,t)\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where

$$C_2 := C_1\pi^{\frac{p}{q}} = \frac{1}{\pi\left((m-1)!\right)^p}.$$

We now estimate $u_{x,y}^p(s,t)$. Observe that

$$\begin{aligned}
u_{x,y}(s,t) &\le \sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}|t|^{\ell} \left(\int_0^1 \left|\Delta_{sw,tw}^{r}\left(\partial^{m-\ell,\ell} f(x,y)\right)\right|^p \mathrm{d}w\right)^{\frac{1}{p}} \left(\int_0^1 (1-w)^{q(m-1)}\mathrm{d}w\right)^{\frac{1}{q}} \\
&:= C_3 \sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}|t|^{\ell} \left(\int_0^1 \left|\Delta_{sw,tw}^{r}\left(\partial^{m-\ell,\ell} f(x,y)\right)\right|^p \mathrm{d}w\right)^{\frac{1}{p}},
\end{aligned}$$

where

$$C_3 := \frac{1}{(q(m-1)+1)^{\frac{1}{q}}}.$$

Hence, we have

$$u_{x,y}^p(s,t) \le C_3^p \left(\sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}|t|^{\ell} \left(\int_0^1 \left|\Delta_{sw,tw}^{r}\left(\partial^{m-\ell,\ell} f(x,y)\right)\right|^p \mathrm{d}w\right)^{\frac{1}{p}}\right)^p,$$

which gives

$$\begin{aligned}
u_{x,y}^p(s,t) &\le C_3^p \left\{\sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}|t|^{\ell} \left(\int_0^1 \left|\Delta_{sw,tw}^{r}\left(\partial^{m-\ell,\ell} f(x,y)\right)\right|^p \mathrm{d}w\right)\right\} \left(\sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}|t|^{\ell}\right)^{\frac{p}{q}} \\
&= C_3^p \left(|s|+|t|\right)^{\frac{mp}{q}} \left\{\sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}|t|^{\ell} \left(\int_0^1 \left|\Delta_{sw,tw}^{r}\left(\partial^{m-\ell,\ell} f(x,y)\right)\right|^p \mathrm{d}w\right)\right\}.
\end{aligned}$$

Letting

$$C_4 := C_2C_3^p = \frac{1/\left((m-1)!\right)^p}{\pi\left(q(m-1)+1\right)^{\frac{p}{q}}},$$

and combining the above results, we get

$$\begin{aligned}\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \leq\ & \frac{C_4}{\xi_n^2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \iint_{\mathbb{D}} \left\{ \iint_{\mathbb{D}} \left[ (|s|+|t|)^{\frac{mp}{q}} \sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}\,|t|^{\ell} \right.\right. \\ & \left.\left. \times \left( \int_0^1 \left| \Delta_{sw,tw}^{r}\left(\partial^{m-\ell,\ell} f(x,y)\right)\right|^p \mathrm{d}w \right) \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \right] \mathrm{d}s\mathrm{d}t \right\} \mathrm{d}x\mathrm{d}y \\ =\ & \frac{C_4}{\xi_n^2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \iint_{\mathbb{D}} \left\{ (|s|+|t|)^{\frac{mp}{q}} \sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}\,|t|^{\ell} \right. \\ & \left. \times \left[ \int_0^1 \left( \iint_{\mathbb{D}} \left| \Delta_{sw,tw}^{r}\left(\partial^{m-\ell,\ell} f(x,y)\right)\right|^p \mathrm{d}x\mathrm{d}y \right) \mathrm{d}w \right] \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \right\} \mathrm{d}s\mathrm{d}t,\end{aligned}$$

which implies that

$$\begin{aligned}\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \leq\ & \frac{C_4}{\xi_n^2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \iint_{\mathbb{D}} \left\{ (|s|+|t|)^{\frac{mp}{q}} \sum_{\ell=0}^{m} \binom{m}{m-\ell} |s|^{m-\ell}\,|t|^{\ell} \right. \\ & \left. \times \left[ \int_0^1 \omega_r\left(\partial^{m-\ell,\ell} f, w\sqrt{s^2+t^2}\right)_p^p \mathrm{d}w \right] \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \right\} \mathrm{d}s\mathrm{d}t.\end{aligned}$$

Then, we have

$$\begin{aligned}\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \leq\ & \frac{4C_4}{\xi_n^2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \iint_{\mathbb{D}_1} \left\{ (s+t)^{\frac{mp}{q}} \sum_{\ell=0}^{m} \binom{m}{m-\ell} s^{m-\ell} t^{\ell} \right. \\ & \left. \times \left[ \int_0^1 \omega_r\left(\partial^{m-\ell,\ell} f, w\sqrt{s^2+t^2}\right)_p^p \mathrm{d}w \right] \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \right\} \mathrm{d}s\mathrm{d}t,\end{aligned}$$

where

$$\mathbb{D}_1 := \left\{ (s,t) \in \mathbb{R}^2 : 0 \leq s \leq \pi \text{ and } 0 \leq t \leq \sqrt{\pi^2 - s^2} \right\}. \tag{3.6}$$

Now, using the fact that

$$\omega_r(f, \lambda h)_p \leq (1+\lambda)^r \omega_r(f,h)_p \quad \text{for any } h, \lambda > 0 \text{ and } p \geq 1, \tag{3.7}$$

we get

$$\begin{aligned}\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \leq\ & \frac{4C_4}{\xi_n^2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \sum_{\ell=0}^{m} \binom{m}{m-\ell} \omega_r\left(\partial^{m-\ell,\ell} f, \xi_n\right)_p^p \iint_{\mathbb{D}_1} \left\{ (s+t)^{\frac{mp}{q}} s^{m-\ell} t^{\ell} \right. \\ & \left. \times \left[ \int_0^1 \left(1 + \frac{w\sqrt{s^2+t^2}}{\xi_n}\right)^{rp} \mathrm{d}w \right] \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \right\} \mathrm{d}s\mathrm{d}t,\end{aligned}$$

and hence

$$\begin{aligned}\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \leq\ & \frac{C_5}{\xi_n\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \sum_{\ell=0}^{m} \binom{m}{m-\ell} \omega_r\left(\partial^{m-\ell,\ell} f, \xi_n\right)_p^p \iint_{\mathbb{D}_1} \left\{ (s+t)^{\frac{mp}{q}} s^{m-\ell} t^{\ell} \right. \\ & \left. \times \left[ \left(1 + \frac{\sqrt{s^2+t^2}}{\xi_n}\right)^{rp+1} - 1 \right] \frac{\mathrm{e}^{-(s^2+t^2)/\xi_n^2}}{\sqrt{s^2+t^2}} \right\} \mathrm{d}s\mathrm{d}t,\end{aligned}$$

where

$$C_5 := \frac{4C_4}{rp+1} = \left(\frac{4}{rp+1}\right) \frac{1/\left((m-1)!\right)^p}{\pi\left(q(m-1)+1\right)^{\frac{p}{q}}}.$$

Therefore, we obtain that

$$\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \leq \frac{C_5}{\xi_n\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \left\{\int_0^{\pi} \rho^{mp}\left(\left(1+\frac{\rho}{\xi_n}\right)^{rp+1}-1\right)\mathrm{e}^{-\rho^2/\xi_n^2}\mathrm{d}\rho\right\}$$
$$\times \sum_{\ell=0}^{m}\binom{m}{m-\ell}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_p^p\left(\int_0^{\pi/2}(\cos\theta+\sin\theta)^{\frac{mp}{q}}\cos^{m-\ell}\theta\sin^{\ell}\theta\mathrm{d}\theta\right)$$
$$\leq \frac{C_5}{\xi_n\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \left\{\int_0^{\pi} \rho^{mp}\left(1+\frac{\rho}{\xi_n}\right)^{rp+1}\mathrm{e}^{-\rho^2/\xi_n^2}\mathrm{d}\rho\right\}$$
$$\times \sum_{\ell=0}^{m}\binom{m}{m-\ell}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_p^p\left(\int_0^{\pi/2}(\cos\theta+\sin\theta)^{\frac{mp}{q}}\cos^{m-\ell}\theta\sin^{\ell}\theta\mathrm{d}\theta\right).$$

Using the fact that $0\leq\sin\theta+\cos\theta\leq 2$ for $\theta\in[0,\frac{\pi}{2}]$, we get

$$\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \leq \frac{C_6}{\xi_n\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \left(\int_0^{\pi} \rho^{mp}\left(1+\frac{\rho}{\xi_n}\right)^{rp+1}\mathrm{e}^{-\rho^2/\xi_n^2}\mathrm{d}\rho\right)$$
$$\times \sum_{\ell=0}^{m}\binom{m}{m-\ell}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_p^p\left(\int_0^{\pi/2}\cos^{m-\ell}\theta\sin^{\ell}\theta\mathrm{d}\theta\right),$$

where

$$C_6 := 2^{\frac{mp}{q}}C_5 = \left(\frac{2^{\frac{mp}{q}+2}}{rp+1}\right)\frac{1/\left((m-1)!\right)^p}{\pi\left(q(m-1)+1\right)^{\frac{p}{q}}}.$$

If we take $u=\rho/\xi_n$, then we see that

$$\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \leq \frac{C_6\xi_n^{mp}}{2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)}\left(\int_0^{\pi/\xi_n} u^{mp}(1+u)^{rp+1}\mathrm{e}^{-u^2}\mathrm{d}u\right)$$
$$\times\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}B\left(\frac{m-\ell+1}{2},\frac{\ell+1}{2}\right)\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_p^p\right\},$$

which implies that

$$\iint_{\mathbb{D}} \left|H_{r,n}^{[m]}(x,y)\right|^p \mathrm{d}x\mathrm{d}y \leq \frac{C_6\xi_n^{mp}}{2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)}\left(\int_0^{\infty}(1+u)^{(m+r)p+1}\mathrm{e}^{-u^2}\mathrm{d}u\right)$$
$$\times\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}B\left(\frac{m-\ell+1}{2},\frac{\ell+1}{2}\right)\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_p^p\right\}$$
$$=\frac{C_7\xi_n^{mp}}{\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)}\sum_{\ell=0}^{m}\binom{m}{m-\ell}B\left(\frac{m-\ell+1}{2},\frac{\ell+1}{2}\right)\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_p^p,$$

where

$$C_7 := \left(\frac{2^{\frac{mp}{q}+1}}{rp+1}\right)\frac{1/\left((m-1)!\right)^p}{\pi\left(q(m-1)+1\right)^{\frac{p}{q}}}\left(\int_0^{\infty}(1+u)^{(m+r)p+1}\mathrm{e}^{-u^2}\mathrm{d}u\right).$$

Therefore the last inequality yields that

$$\left\|H_{r,n}^{[m]}\right\|_p \leq \frac{C\xi_n^m}{\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}}\left(\sum_{\ell=0}^{m}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_p^p\right)^{\frac{1}{p}},$$

where

$$C := C(m,p,q,r)$$
$$= \frac{1/(m-1)!}{\pi^{\frac{1}{p}}\,(q(m-1)+1)^{\frac{1}{q}}}\left(\frac{2^{\frac{mp}{q}+1}}{rp+1}\right)^{\frac{1}{p}}\left(\int_0^{\infty}(1+u)^{(m+r)p+1}\,\mathrm{e}^{-u^2}\mathrm{d}u\right)^{\frac{1}{p}}$$
$$\times\left\{\max_{\ell=0,1,\ldots,m}\binom{m}{m-\ell}B\left(\frac{m-\ell+1}{2},\frac{\ell+1}{2}\right)\right\}^{\frac{1}{p}}.$$

The theorem is proved. □

We also get the next result.

**Theorem 3.3.** *Let $m,r\in\mathbb{N}$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $f\in C_{\pi}^{(m)}(\mathbb{D})$. Then the following inequality*

$$\left\|W_{r,n}^{[m]}(f)-f\right\|_p \le \frac{C\xi_n^m}{\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}}\left(\sum_{\ell=0}^{m}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_p^p\right)^{\frac{1}{p}}+B\lambda_n\sum_{i=1}^{[m/2]}\xi_n^{2i}$$

*holds for some positive constants $B$, $C$ depending on $m,p,q,r$; $B$ also depends on $f$.*

**Proof.** By (3.4) and the subadditivity of the $L_p$-norm, we get

$$\left\|W_{r,n}^{[m]}(f)-f\right\|_p \le \left\|H_{r,n}^{[m]}\right\|_p+\frac{2\lambda_n}{\xi_n^2}\sum_{i=1}^{[m/2]}\frac{\delta_{2i,r}^{[m]}\gamma_{n,2i}}{(2i)!}\left\{\sum_{\ell=0}^{2i}\binom{2i}{2i-\ell}\left\|\partial^{2i-\ell,\ell}f\right\|_p B\left(\frac{2i-\ell+1}{2},\frac{\ell+1}{2}\right)\right\}$$
$$\le \left\|H_{r,n}^{[m]}\right\|_p+\lambda_n\sum_{i=1}^{[m/2]}\frac{\delta_{2i,r}^{[m]}\xi_n^{2i}}{(i+1)\cdots(2i)}\left\{\sum_{\ell=0}^{2i}\binom{2i}{2i-\ell}\left\|\partial^{2i-\ell,\ell}f\right\|_p B\left(\frac{2i-\ell+1}{2},\frac{\ell+1}{2}\right)\right\}.$$

Now, by putting

$$B := \max_{1\le i\le[m/2]}\left\{\frac{\delta_{2i,r}^{[m]}}{(i+1)\cdots(2i)}\sum_{\ell=0}^{2i}\binom{2i}{2i-\ell}\left\|\partial^{2i-\ell,\ell}f\right\|_p B\left(\frac{2i-\ell+1}{2},\frac{\ell+1}{2}\right)\right\},$$

we get

$$\left\|W_{r,n}^{[m]}(f)-f\right\|_p \le \left\|H_{r,n}^{[m]}\right\|_p+B\lambda_n\sum_{i=1}^{[m/2]}\xi_n^{2i},$$

by Theorem 3.2; the claim is now proved. □

The following result gives an estimation in the cases of $p=1$ and $m\in\mathbb{N}$.

**Theorem 3.4.** *Let $m,r\in\mathbb{N}$ and $f\in C_{\pi}^{(m)}(\mathbb{D})$. Then, we have*

$$\left\|H_{r,n}^{[m]}\right\|_1 \le \frac{D\xi_n^m}{\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)}\sum_{\ell=0}^{m}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_1$$

*for some positive constant $D$ depending on $m,r$.*

**Proof.** By Lemma 3.1, we observe that

$$\left|H_{r,n}^{[m]}(x,y)\right| \le \frac{\lambda_n}{\xi_n^2(m-1)!}\sum_{\ell=0}^{m}\iint_{\mathbb{D}}\left(\int_0^1(1-w)^{m-1}\left|\Delta_{sw,tw}^{r}\left(\partial^{m-\ell,\ell}f(x,y)\right)\right|\mathrm{d}w\right)$$
$$\times\binom{m}{m-\ell}|s|^{m-\ell}\,|t|^{\ell}\,\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t.$$

Then, we get

$$\begin{aligned}\left\|H_{r,n}^{[m]}\right\|_1 &\le \frac{\lambda_n}{\xi_n^2 (m-1)!}\iint_{\mathbb{D}}\left\{\sum_{\ell=0}^{m}\iint_{\mathbb{D}}\left(\int_0^1 (1-w)^{m-1}\left|\Delta_{sw,tw}^r\left(\partial^{m-\ell,\ell}f(x,y)\right)\right|\mathrm{d}w\right)\right.\\ &\quad\left.\times\binom{m}{m-\ell}|s|^{m-\ell}|t|^{\ell}\,\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right\}\mathrm{d}x\mathrm{d}y\\ &= \frac{\lambda_n}{\xi_n^2 (m-1)!}\sum_{\ell=0}^{m}\left\{\iint_{\mathbb{D}}\left[\int_0^1 (1-w)^{m-1}\left(\iint_{\mathbb{D}}\left|\Delta_{sw,tw}^r\left(\partial^{m-\ell,\ell}f(x,y)\right)\right|\mathrm{d}x\mathrm{d}y\right)\mathrm{d}w\right]\right.\\ &\quad\left.\times\binom{m}{m-\ell}|s|^{m-\ell}|t|^{\ell}\,\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right\},\end{aligned}$$

which gives that

$$\begin{aligned}\left\|H_{r,n}^{[m]}\right\|_1 &\le \frac{\lambda_n}{\xi_n^2 (m-1)!}\sum_{\ell=0}^{m}\left\{\iint_{\mathbb{D}}\left[\int_0^1 (1-w)^{m-1}\omega_r\left(\partial^{m-\ell,\ell}f, w\sqrt{s^2+t^2}\right)_1\mathrm{d}w\right]\right.\\ &\quad\left.\times\binom{m}{m-\ell}|s|^{m-\ell}|t|^{\ell}\,\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right\}\\ &= \frac{4\lambda_n}{\xi_n^2 (m-1)!}\sum_{\ell=0}^{m}\left\{\iint_{\mathbb{D}_1}\left[\int_0^1 (1-w)^{m-1}\omega_r\left(\partial^{m-\ell,\ell}f, w\sqrt{s^2+t^2}\right)_1\mathrm{d}w\right]\right.\\ &\quad\left.\times\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right\},\end{aligned}$$

where the set $\mathbb{D}_1$ is given by (3.6). Now, using (3.7), we deduce that

$$\begin{aligned}\left\|H_{r,n}^{[m]}\right\|_1 &\le \frac{4\lambda_n}{\xi_n^2 (m-1)!}\sum_{\ell=0}^{m}\binom{m}{m-\ell}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_1\\ &\quad\times\iint_{\mathbb{D}_1}\left(\int_0^1 (1-w)^{m-1}\left(1+\frac{w\sqrt{s^2+t^2}}{\xi_n}\right)^r\mathrm{d}w\right)s^{m-\ell}t^{\ell}\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\\ &\le \frac{4\lambda_n}{\xi_n^2 (m-1)!}\sum_{\ell=0}^{m}\binom{m}{m-\ell}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_1\\ &\quad\times\left\{\iint_{\mathbb{D}_1}\left(\int_0^1\left(1+\frac{w\sqrt{s^2+t^2}}{\xi_n}\right)^r\mathrm{d}w\right)s^{m-\ell}t^{\ell}\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right\}\\ &\le \frac{D'}{\xi_n\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)}\sum_{\ell=0}^{m}\binom{m}{m-\ell}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_1\left\{\iint_{\mathbb{D}_1}\left(1+\frac{\sqrt{s^2+t^2}}{\xi_n}\right)^{r+1}s^{m-\ell}t^{\ell}\frac{\mathrm{e}^{-(s^2+t^2)/\xi_n^2}}{\sqrt{s^2+t^2}}\mathrm{d}s\mathrm{d}t\right\},\end{aligned}$$

where

$$D' = \frac{4}{\pi\,(r+1)\,(m-1)!}.$$

Hence, we conclude that

$$\begin{aligned}\left\|H_{r,n}^{[m]}\right\|_1 &\le \frac{D'}{\xi_n\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)}\left(\int_0^{\pi}\left(1+\frac{\rho}{\xi_n}\right)^{r+1}\rho^m\mathrm{e}^{-\rho^2/\xi_n^2}\mathrm{d}\rho\right)\\ &\quad\times\sum_{\ell=0}^{m}\binom{m}{m-\ell}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_1\left(\int_0^{\pi/2}\cos^{m-\ell}\theta\,\sin^{\ell}\theta\,\mathrm{d}\theta\right)\\ &= \frac{D'\xi_n^m}{\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)}\left(\int_0^{\pi/\xi_n}(1+u)^{r+1}u^m\mathrm{e}^{-u^2}\mathrm{d}u\right)\\ &\quad\times\sum_{\ell=0}^{m}\binom{m}{m-\ell}\omega_r\left(\partial^{m-\ell,\ell}f,\xi_n\right)_1 B\left(\frac{m-\ell+1}{2},\frac{\ell+1}{2}\right).\end{aligned}$$

Now, taking

$$D := D' \left( \int_0^{\infty} (1+u)^{m+r+1} u^m \mathrm{e}^{-u^2} \mathrm{d}u \right) \max_{\ell=0,1,\ldots,m} \left\{ \binom{m}{m-\ell} B\left( \frac{m-\ell+1}{2}, \frac{\ell+1}{2} \right) \right\},$$

we get

$$\left\| H_{r,n}^{[m]} \right\|_1 \le \frac{D\xi_n^m}{\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \sum_{\ell=0}^{m} \omega_r \left( \partial^{m-\ell,\ell} f, \xi_n \right)_1,$$

which completes the proof. □

Furthermore, we get the next result.

**Theorem 3.5.** *Let $m, r \in \mathbb{N}$ and $f \in C_{\pi}^{(m)}(\mathbb{D})$. Then*

$$\left\| W_{r,n}^{[m]}(f) - f \right\|_1 \le \frac{D\xi_n^m}{\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)} \sum_{\ell=0}^{m} \omega_r \left( \partial^{m-\ell,\ell} f, \xi_n \right)_1 + E\lambda_n \sum_{i=1}^{[m/2]} \xi_n^{2i}$$

*holds for some positive constants $D, E$ depending on $m, r$; $E$ also depends on $f$.*

**Proof.** By (3.4) and the subadditivity of the $L_1$-norm, we get

$$\begin{aligned} \left\| W_{r,n}^{[m]}(f) - f \right\|_1 &\le \left\| H_{r,n}^{[m]} \right\|_1 + \frac{2\lambda_n}{\xi_n^2} \sum_{i=1}^{[m/2]} \frac{\delta_{2i,r}^{[m]} \gamma_{n,2i}}{(2i)!} \left\{ \sum_{\ell=0}^{2i} \binom{2i}{2i-\ell} \left\| \partial^{2i-\ell,\ell} f \right\|_1 B\left( \frac{2i-\ell+1}{2}, \frac{\ell+1}{2} \right) \right\} \\ &\le \left\| H_{r,n}^{[m]} \right\|_1 + \lambda_n \sum_{i=1}^{[m/2]} \frac{\delta_{2i,r}^{[m]} \xi_n^{2i}}{(i+1)\cdots(2i)} \left\{ \sum_{\ell=0}^{2i} \binom{2i}{2i-\ell} \left\| \partial^{2i-\ell,\ell} f \right\|_1 B\left( \frac{2i-\ell+1}{2}, \frac{\ell+1}{2} \right) \right\}. \end{aligned}$$

Now, by setting

$$E := \max_{1 \le i \le [m/2]} \left\{ \frac{\delta_{2i,r}^{[m]}}{(i+1)\cdots(2i)} \sum_{\ell=0}^{2i} \binom{2i}{2i-\ell} \left\| \partial^{2i-\ell,\ell} f \right\|_1 B\left( \frac{2i-\ell+1}{2}, \frac{\ell+1}{2} \right) \right\},$$

we get

$$\left\| W_{r,n}^{[m]}(f) - f \right\|_1 \le \left\| H_{r,n}^{[m]} \right\|_1 + E\lambda_n \sum_{i=1}^{[m/2]} \xi_n^{2i},$$

by Theorem 3.4; the claim is now proved. □

### 3.2. Estimates in the case of $m = 0$

We now focus on the estimation in the case of $m = 0$. We first get the following result.

**Theorem 3.6.** *Let $r \in \mathbb{N}$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for every $f \in L_p(\mathbb{D})$ and $2\pi$-periodic per coordinate, the following inequality*

$$\left\| W_{r,n}^{[0]}(f) - f \right\|_p \le \frac{K\omega_r (f, \xi_n)_p}{\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}},$$

*holds for some positive constant $K$ depending on $p, r$.*

**Proof.** By (2.1), (2.3) and (2.5), we may write that

$$\begin{aligned} W_{r,n}^{[0]}(f; x, y) - f(x, y) &= \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left\{ \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} (f(x+sj, y+tj) - f(x, y)) \right\} \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t \\ &= \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left[ \sum_{j=1}^{r} \left( (-1)^{r-j} \binom{r}{j} f(x+sj, y+tj) \right) + (-1)^r \binom{r}{0} f(x, y) \right] \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t \\ &= \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left\{ \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x+sj, y+tj) \right\} \mathrm{e}^{-(s^2+t^2)/\xi_n^2} \mathrm{d}s\mathrm{d}t. \end{aligned}$$

Also, by (3.2), we get

$$W_{r,n}^{[0]}(f;x,y)-f(x,y)=\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}}\Delta_{s,t}^r\left(f(x,y)\right)\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t,$$

which implies that

$$\left|W_{r,n}^{[0]}(f;x,y)-f(x,y)\right|\le\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}}\left|\Delta_{s,t}^r\left(f(x,y)\right)\right|\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t. \tag{3.8}$$

Hence, we get

$$\iint_{\mathbb{D}}\left|W_{r,n}^{[0]}(f;x,y)-f(x,y)\right|^p\mathrm{d}x\mathrm{d}y\le\frac{\lambda_n^p}{\xi_n^{2p}}\iint_{\mathbb{D}}\left(\iint_{\mathbb{D}}\left|\Delta_{s,t}^r\left(f(x,y)\right)\right|\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)^p\mathrm{d}x\mathrm{d}y.$$

Now, we obtain from Hölder's inequality for double integrals that

$$\begin{aligned}\iint_{\mathbb{D}}\left|W_{r,n}^{[0]}(f;x,y)-f(x,y)\right|^p\mathrm{d}x\mathrm{d}y\le\frac{\lambda_n^p}{\xi_n^{2p}}\iint_{\mathbb{D}}\Bigg\{&\left(\iint_{\mathbb{D}}\left|\Delta_{s,t}^r\left(f(x,y)\right)\right|^p\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)\\&\times\left(\iint_{\mathbb{D}}\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)^{p/q}\Bigg\}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then, using (2.6), we may write that

$$\begin{aligned}\iint_{\mathbb{D}}\left|W_{r,n}^{[0]}(f;x,y)-f(x,y)\right|^p\mathrm{d}x\mathrm{d}y&\le\frac{\lambda_n^p}{\xi_n^{2p}}\left(\pi\xi_n^2\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)\right)^{\frac{p}{q}}\iint_{\mathbb{D}}\left(\iint_{\mathbb{D}}\left|\Delta_{s,t}^r\left(f(x,y)\right)\right|^p\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y\\&=\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}}\left(\iint_{\mathbb{D}}\left|\Delta_{s,t}^r\left(f(x,y)\right)\right|^p\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Thus, by (3.7), we have

$$\begin{aligned}\iint_{\mathbb{D}}\left|W_{r,n}^{[0]}(f;x,y)-f(x,y)\right|^p\mathrm{d}x\mathrm{d}y&\le\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}}\omega_r\left(f,\sqrt{s^2+t^2}\right)_p^p\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\\&=\frac{4\lambda_n}{\xi_n^2}\iint_{\mathbb{D}_1}\omega_r\left(f,\sqrt{s^2+t^2}\right)_p^p\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t\\&\le\frac{4\lambda_n}{\xi_n^2}\omega_r\left(f,\xi_n\right)_p^p\iint_{\mathbb{D}_1}\left(1+\frac{\sqrt{s^2+t^2}}{\xi_n}\right)^{rp}\mathrm{e}^{-(s^2+t^2)/\xi_n^2}\mathrm{d}s\mathrm{d}t,\end{aligned}$$

where $\mathbb{D}_1$ is given by (3.6). After some calculations, we deduce that

$$\begin{aligned}\iint_{\mathbb{D}}\left|W_{r,n}^{[0]}(f;x,y)-f(x,y)\right|^p\mathrm{d}x\mathrm{d}y&\le\frac{4\lambda_n}{\xi_n^2}\omega_r\left(f,\xi_n\right)_p^p\int_0^{\pi/2}\int_0^{\pi}\left(1+\frac{\rho}{\xi_n}\right)^{rp}\mathrm{e}^{-\rho^2/\xi_n^2}\rho\mathrm{d}\rho\mathrm{d}\theta\\&=\frac{2\omega_r\left(f,\xi_n\right)_p^p}{1-\mathrm{e}^{-\pi^2/\xi_n^2}}\int_0^{\pi/\xi_n}(1+u)^{rp}\mathrm{e}^{-u^2}u\mathrm{d}u\\&\le\frac{2\omega_r\left(f,\xi_n\right)_p^p}{1-\mathrm{e}^{-\pi^2/\xi_n^2}}\left(\int_0^{\infty}(1+u)^{rp+1}\mathrm{e}^{-u^2}\mathrm{d}u\right).\end{aligned}$$

Therefore, we have

$$\left\|W_{r,n}^{[0]}(f)-f\right\|_p\le\frac{K\omega_r\left(f,\xi_n\right)_p}{\left(1-\mathrm{e}^{-\pi^2/\xi_n^2}\right)^{1/p}},$$

where

$$K:=K(p,r)=\left(2\int_0^{\infty}(1+u)^{rp+1}\mathrm{e}^{-u^2}\mathrm{d}u\right)^{\frac{1}{p}}.$$

The theorem is proved. □

Finally, we give an estimation in the case of $p=1$ and $m=0$.

**Theorem 3.7.** *For every $f \in L_1(\mathbb{D})$ and $2\pi$-periodic per coordinate, we have*

$$\left\| W_{r,n}^{[0]}(f) - f \right\|_1 \leq \frac{L\omega_r(f, \xi_n)_1}{1 - e^{-\pi^2/\xi_n^2}}$$

*for some positive constant $L$ depending on $r$.*

**Proof.** By (3.8), we easily observe that

$$\begin{aligned} \left\| W_{r,n}^{[0]}(f) - f \right\|_1 &= \iint_{\mathbb{D}} \left| W_{r,n}^{[0]}(f; x, y) - f(x, y) \right| dxdy \\ &\leq \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left( \iint_{\mathbb{D}} \left| \Delta_{s,t}^r (f(x, y)) \right| e^{-(s^2+t^2)/\xi_n^2} dsdt \right) dxdy \\ &= \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left( \iint_{\mathbb{D}} \left| \Delta_{s,t}^r (f(x, y)) \right| dxdy \right) e^{-(s^2+t^2)/\xi_n^2} dsdt \\ &\leq \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \omega_r \left( f, \sqrt{s^2+t^2} \right)_1 e^{-(s^2+t^2)/\xi_n^2} dsdt \\ &= \frac{4\lambda_n}{\xi_n^2} \iint_{\mathbb{D}_1} \omega_r \left( f, \sqrt{s^2+t^2} \right)_1 e^{-(s^2+t^2)/\xi_n^2} dsdt, \end{aligned}$$

where $\mathbb{D}_1$ is given by (3.6). Now, using (3.7), we get

$$\begin{aligned} \left\| W_{r,n}^{[0]}(f) - f \right\|_1 &\leq \frac{4\lambda_n \omega_r(f, \xi_n)_1}{\xi_n^2} \iint_{\mathbb{D}_1} \left( 1 + \frac{\sqrt{s^2+t^2}}{\xi_n} \right)^r e^{-(s^2+t^2)/\xi_n^2} dsdt \\ &= \frac{4\lambda_n \omega_r(f, \xi_n)_1}{\xi_n^2} \int_0^{\pi/2} \int_0^{\pi} \left( 1 + \frac{\rho}{\xi_n} \right)^r e^{-\rho^2/\xi_n^2} \rho d\rho d\theta \\ &= \frac{2\omega_r(f, \xi_n)_1}{1 - e^{-\pi^2/\xi_n^2}} \int_0^{\pi/\xi_n} (1+u)^r e^{-u^2} udu \\ &\leq \frac{2\omega_r(f, \xi_n)_1}{1 - e^{-\pi^2/\xi_n^2}} \left( \int_0^{\infty} (1+u)^{r+1} e^{-u^2} du \right). \end{aligned}$$

Then, the last inequality yields that

$$\left\| W_{r,n}^{[0]}(f) - f \right\|_1 \leq \frac{L\omega_r(f, \xi_n)_1}{1 - e^{-\pi^2/\xi_n^2}},$$

where

$$L := L(r) = 2 \int_0^{\infty} (1+u)^{r+1} e^{-u^2} du.$$

The proof is completed. □

## 4. Statistical $L_p$-approximation by the operators (2.5)

By the right continuity of $\omega_r(f; \cdot)_p$ at zero, we first get the next result.

**Lemma 4.1.** *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers for which*

$$st_A - \lim_n \xi_n = 0 \tag{4.1}$$

*holds. Then, for every $f \in C_{\pi}^{(m)}(\mathbb{D})$, $m \in \mathbb{N}_0$, we have*

$$st_A - \lim_n \omega_r(f; \xi_n)_p = 0, \quad 1 \leq p < \infty. \tag{4.2}$$

*4.1. Statistical $L_p$-approximation in the case of $m \in \mathbb{N}$*

The following result is a direct consequence of Theorems 3.3 and 3.5.

**Corollary 4.2.** *Let $1 \le p < \infty$ and $m \in \mathbb{N}$. Then, for every $f \in C_{\pi}^{(m)}(\mathbb{D})$, we have*

$$\left\|W_{r,n}^{[m]}(f) - f\right\|_p \le \frac{M_1 \xi_n^m}{\left(1 - \mathrm{e}^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}} \left\{\sum_{\ell=0}^{m} \left(\omega_r\left(\partial^{m-\ell,\ell} f, \xi_n\right)_p\right)^p\right\}^{\frac{1}{p}} + M_2 \lambda_n \sum_{i=1}^{[m/2]} \xi_n^{2i}$$

*for some positive constants $M_1$, $M_2$ depending on $m, p, q, r$, where*

$$M_1 := \begin{cases} D \text{ (as in Theorem 3.5)} & \text{if } p = 1 \\ C \text{ (as in Theorem 3.3)} & \text{if } 1 < p < \infty \text{ with } (1/p) + (1/q) = 1 \end{cases}$$

*and*

$$M_2 := \begin{cases} E \text{ (as in Theorem 3.5)} & \text{if } p = 1 \\ B \text{ (as in Theorem 3.3)} & \text{if } 1 < p < \infty \text{ with } (1/p) + (1/q) = 1. \end{cases}$$

Now we can give our first statistical $L_p$-approximation result.

**Theorem 4.3.** *Let $m, r \in \mathbb{N}$ and $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers for which (4.1) holds. Then, for any $f \in C_{\pi}^{(m)}(\mathbb{D})$, we have*

$$st_A - \lim_n \left\|W_{r,n}^{[m]}(f) - f\right\|_p = 0. \tag{4.3}$$

**Proof.** From (4.1) and Lemma 4.1, we may write that

$$st_A - \lim_n \frac{\xi_n^m}{\left(1 - \mathrm{e}^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}} = 0,$$

$$st_A - \lim_n \left(\omega_r\left(\partial^{m-\ell,\ell} f, \xi_n\right)_p\right)^p = 0 \quad \text{for each } \ell = 0, 1, \ldots, m$$

and

$$st_A - \lim_n \xi_n^{2i} = 0 \quad \text{for each } i = 1, 2, \ldots, \left[\frac{m}{2}\right].$$

The above results clearly imply that

$$st_A - \lim_n \frac{\xi_n^m}{\left(1 - \mathrm{e}^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}} \left\{\sum_{\ell=0}^{m} \left(\omega_r\left(\partial^{m-\ell,\ell} f, \xi_n\right)_p\right)^p\right\}^{\frac{1}{p}} = 0 \tag{4.4}$$

and

$$st_A - \lim_n \lambda_n \sum_{i=1}^{[m/2]} \xi_n^{2i} = 0. \tag{4.5}$$

Now, for a given $\varepsilon > 0$, define the following sets:

$$S := \left\{n \in \mathbb{N} : \left\|W_{r,n}^{[m]}(f) - f\right\|_p \ge \varepsilon\right\},$$

$$S_1 := \left\{n \in \mathbb{N} : \frac{\xi_n^m}{\left(1 - \mathrm{e}^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}} \left\{\sum_{\ell=0}^{m} \left(\omega_r\left(\partial^{m-\ell,\ell} f, \xi_n\right)_p\right)^p\right\}^{\frac{1}{p}} \ge \frac{\varepsilon}{2M_1}\right\},$$

$$S_2 := \left\{n \in \mathbb{N} : \lambda_n \sum_{i=1}^{[m/2]} \xi_n^{2i} \ge \frac{\varepsilon}{2M_2}\right\}.$$

Then, it follows from Corollary 4.2 that

$$S \subseteq S_1 \cup S_2,$$

which implies, for every $j \in \mathbb{N}$, that

$$\sum_{n \in S} a_{jn} \le \sum_{n \in S_1} a_{jn} + \sum_{n \in S_2} a_{jn}.$$

Now, taking the limit as $j \to \infty$ in both sides of the last inequality and also using (4.4), (4.5), we conclude that

$$\lim_j \sum_{n \in S} a_{jn} = 0,$$

which gives (4.3). Hence, the proof is completed. □

### 4.2. *Statistical $L_p$-approximation in the case of $m = 0$*

In this subsection, we first combine Theorems 3.6 and 3.7 as follows.

**Corollary 4.4.** *Let $1 \le p < \infty$ and $r \in \mathbb{N}$. Then, for every $f \in L_p(\mathbb{D})$ and $2\pi$-periodic per coordinate, we have*

$$\left\| W_{r,n}^{[0]}(f) - f \right\|_p \le \frac{N \omega_r (f, \xi_n)_p}{\left(1 - e^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}}$$

*for some positive constant $N$ depending on $p$, $r$, where*

$$N := \begin{cases} L \text{ (as in Theorem 3.7)} & \text{if } p = 1 \\ K \text{ (as in Theorem 3.6)} & \text{if } 1 < p < \infty \text{ with } (1/p) + (1/q) = 1. \end{cases}$$

Now we can state our second statistical $L_p$-approximation result.

**Theorem 4.5.** *Let $r \in \mathbb{N}$ and $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers for which (4.1) holds. Then, for any $f \in L_p(\mathbb{D})$ and $2\pi$-periodic per coordinate, we have*

$$st_A - \lim_n \left\| W_{r,n}^{[0]}(f) - f \right\|_p = 0. \tag{4.6}$$

**Proof.** Letting

$$T_1 := \left\{ n \in \mathbb{N} : \left\| W_{r,n}^{[0]}(f) - f \right\|_p \ge \varepsilon \right\}$$

and

$$T_2 := \left\{ n \in \mathbb{N} : \frac{\omega_r (f, \xi_n)_p}{\left(1 - e^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}} \ge \frac{\varepsilon}{N} \right\},$$

it follows from Corollary 4.4 that, for every $\varepsilon > 0$,

$$T_1 \subseteq T_2.$$

Hence, for each $j \in \mathbb{N}$, we have

$$\sum_{n \in T_1} a_{jn} \le \sum_{n \in T_2} a_{jn}.$$

Now, letting $j \to \infty$ in the last inequality and considering Lemma 4.1, and also using the fact that

$$st_A - \lim_n \frac{\omega_r (f, \xi_n)_p}{\left(1 - e^{-\pi^2/\xi_n^2}\right)^{\frac{1}{p}}} = 0,$$

we obtain that

$$\lim_j \sum_{n \in T_1} a_{jn} = 0,$$

which proves (4.6). □

## 5. Concluding remarks

In this section, we give some special cases of our approximation results obtained in the previous section.

In particular, we first consider the case of $A = C_1$, the Cesáro matrix of order one. In this case, from Theorems 4.3 and 4.5 we have the following result immediately.

**Corollary 5.1.** *Let $m \in \mathbb{N}_0$, $r \in \mathbb{N}$, and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers for which*

$$st - \lim_n \xi_n = 0$$

*holds. Then, for all $f \in C_\pi^{(m)}(\mathbb{D})$, we have*

$$st - \lim_n \left\| W_{r,n}^{[m]}(f) - f \right\|_p = 0.$$

The second result is the case of $A = I$, the identity matrix. Then, the next approximation theorem is a direct consequence of Theorems 4.3 and 4.5.

**Corollary 5.2.** *Let $m \in \mathbb{N}_0$, $r \in \mathbb{N}$, and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers for which*

$$\lim_n \xi_n = 0$$

*holds. Then, for all $f \in C_\pi^{(m)}(\mathbb{D})$, the sequence $\{W_{r,n}^{[m]}(f)\}_{n\in\mathbb{N}}$ is uniformly convergent to $f$ with respect to the $L_p$-norm.*

Finally, define a sequence $(\xi_n)_{n\in\mathbb{N}}$ as follows:

$$\xi_n := \begin{cases} 1, & \text{if } n = k^2,\ k = 1, 2, \ldots \\ \dfrac{1}{1+n}, & \text{otherwise.} \end{cases} \tag{5.1}$$

Then, observe that $st - \lim_n \xi_n = 0$. So, if we use this sequence $(\xi_n)_{n\in\mathbb{N}}$ in the definition of the operator $W_{r,n}^{[m]}$, then we obtain from Corollary 5.1 (or, Theorems 4.3 and 4.5) that $st - \lim_n \|W_{r,n}^{[m]}(f) - f\|_p = 0$ holds for all $f \in C_\pi^{(m)}(\mathbb{D})$, $1 \leq p < \infty$. However, since the sequence $(\xi_n)_{n\in\mathbb{N}}$ given by (5.1) is non-convergent, the classical $L_p$-approximation to a function $f$ by the operators $W_{r,n}^{[m]}(f)$ is impossible; i.e., Corollary 5.2 fails for these operators. We should remark that Theorems 4.3 and 4.5, and Corollary 5.1 are also valid when $\lim \xi_n = 0$ because every convergent sequence is $A$-statistically convergent, and so statistically convergent. But, as in the above example, our theorems still work although $(\xi_n)_{n\in\mathbb{N}}$ is non-convergent. Therefore, this non-trivial example clearly shows that our statistical $L_p$-approximation results in Theorems 4.3 and 4.5, and also in Corollary 5.1, are stronger than Corollary 5.2.

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