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Multidimensional Cooley–Tukey Algorithms Revisited

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The representation theory of Abelian groups is used to obtain an algebraic divide-and-conquer algorithm for computing the finite Fourier transform. The algorithm computes the Fourier transform of a finite Abelian group in terms of the Fourier transforms of an arbitrary subgroup and its quotient. From this algebraic algorithm a procedure is derived for obtaining concrete factorizations of the Fourier transform matrix in terms of smaller Fourier transform matrices, diagonal multiplications, and permutations. For cyclic groups this gives as special cases the Cooley–Tukey and Good–Thomas algorithms. For groups with several generators, the procedure gives a variety of multidimensional Cooley–Tukey type algorithms. This method of designing multidimensional fast Fourier transform algorithms gives different data flow patterns from the standard "row–column" approaches. We present some experimental evidence that suggests that in hierarchical memory environments these data flows are more efficient.

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1. INTRODUCTION

This study was motivated by two problems:

Problem 1. Design fast Fourier transform algorithms that respect crystallographic group symmetries.

Problem 2. Design multidimensional Fourier transforms that provide the best match for a hierarchical and/or distributed memory computing environment.

Problem 1 has been studied in [1, 2]. Problem 2 has such an extensive literature that we cannot hope to survey it here. Our study of both of these problems led us to the use of non-standard twiddle factors in the Cooley–Tukey algorithm. Our route to these results required an algebraic reformulation of the Cooley–Tukey algorithm. After completing our program, it became clear that the algebraic structure we were using would provide information about the Fourier transform on non-Abelian groups. In our examination of the algorithmic non-Abelian Fourier transform literature, the earliest results seem to be those of Beth [4], but the results of Clausen [5, 6], Diaconis and Rockmore [7], and Rockmore [11–13] seem to be the closest to the results in this paper. However, even though we were led to the same tools, we have used them to study different problems. Since we are dealing with the Abelian case, we can ask and answer more refined questions than have yet occurred in the non-Abelian literature.

With these preambles aside we can begin our technical discussion. Let $\mathbf{Z}/A\mathbf{Z}$ denote the integers modulo A, which we will denote by A as a group. If we assume $A = MN$, then $M\mathbf{Z}/A\mathbf{Z}$ is a subgroup of A which is isomorphic to $\mathbf{Z}/H\mathbf{Z}$ and will be denoted by B. Then A/B is isomorphic to $\mathbf{Z}/M\mathbf{Z}$ and will be denoted by C. We may identify the B-cosets of A with C and we will make this explicit by using $[c]$ to denote the B-coset corresponding to $c \in C$. We will map C into A by ξ: $C \to A$ by requiring $\xi(c) \in [c]$ and $\xi(c)$ to be a coset representative. We will always assume that $\xi(0) = 0$. Then every element $a \in A$ may be written uniquely as $a = \xi(c) + b$, $c \in C$, $b \in B$. And so we have a 1–1 mapping S_ξ: $S \to \xi(C) \times B$ defined by $S_\xi(a) = (\xi(c), b)$, where $a = \xi(c) + b$. Let $0, 1', \dots, (M-1)'$ denote the ordered elements of C, and $0, M, \dots, (N-1)M$ the ordered elements of B. This defines an ordering on $\xi(C)$ and we order the elements $(\xi(c), b)$ in $\xi(C) \times B$ lexicographically. This defines a new order on the elements of A or a permutation of the elements of A which we will denote by $P(\xi)$. The Cooley–Tukey algorithm may now be stated as follows. Choose the coset representative that assigns to $m' \in$

$\mathbf{Z}/M\mathbf{Z}$, $0 \le m < M$, the element m in $\mathbf{Z}/A\mathbf{Z}$. Let $F(\cdot)$ denote the Fourier transform of the group in the bracket. Then

$$F(A)^{-1} = \left(F(C)^{-1} \otimes I\right)T\left(I \otimes F(B)^{-1}\right)P$$

where P is the permutation determined by the above coset mapping ξ and T is a diagonal matrix called the twiddle factor.

The first main result of this paper is the following: let $\xi\colon C \to A$ be any coset representatives for A/B and let $P(\xi)$ be the corresponding permutation matrix, then there exists a diagonal matrix $T(\xi)$ such that

$$F(A)^{+1} = \left(F(C)^{-1} \otimes I\right)T(\xi)\left(I \otimes F(B)^{-1}\right)P(\xi).$$

The Good-Thomas algorithm is a special case of this result which exists if M and N are relatively prime. Then A is isomorphic to $C \times B$ as groups and if we use this fact to construct our coset representatives ξ and the permutation $P(\xi)$, then $T(\xi)$ becomes the identity matrix.

It is natural to ask what happens if we remove the assumptions that A is a cyclic group, but merely assume that A is Abelian and B is a subgroup of A. The first thing to observe is that the Cooley–Tukey row-column method for dealing with general Abelian groups involves an implicit assumption about how A and B are related. We will now make this explicit.

Assume, for instance, that A is a 2-primary group and so we may write

$$A = \mathbf{Z}/2^{k_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/2^{k_L}\mathbf{Z}$$

which we will call a *presentation* of A. Of course, A may have distinct presentations. We will say that a subgroup $B \subset A$ is *coherently presentable* if A has a presentation in which

$$B = 2^{l_1}\mathbf{Z}/2^{k_1}\mathbf{Z} \times \cdots \times 2^{l_L}\mathbf{Z}/2^{k_L}\mathbf{Z}.$$

It is easy to see that the usual row–column Cooley–Tukey algorithm requires that B be coherently presentable in A. However, not all groups are coherently presentable. For example, if

$$A = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$$

and

$$B = \{(0,0),(1,2),(0,4),(1,6)\},$$

B is *not* coherently presentable in A and so the classical technique cannot be applied. However, our general result still holds.

THEOREM 1.1. *Let A be a finite Abelian group, let B be a subgroup of A, and let $A/B = C$. Let $\xi(c)$, $c \in C$, be a coset representative of c and let $\eta(c_1, c_2) = \xi(c_1) + \xi(c_2) - \xi(c_1 + c_2)$. Then there exists a permutation matrix $P(\xi)$ and a diagonal matrix $T(\eta)$ such that*

$$F(A)^{-1} = \left(F(C)^{-1} \otimes I\right)T(\eta)\left(I \otimes F(B)^{-1}\right)P(\xi).$$

The paper is organized into two parts. In the first part (Sections 2–4) the basic theoretical foundation for our approach is developed, culminating in a proof of Theorem 1.1. In the second part, the remainder of the paper, we develop a practical method for designing multidimensional Fourier transform algorithms based on introducing coordinates in the theoretical discussion in the first part. More specifically, in Section 5 we restate the theoretical results of the previous sections in terms of a concrete procedure for factoring the Fourier transform matrix. Then in Section 6 we apply this procedure to illustrate the theory for several one-dimensional and two-dimensional examples. In this section we find formulas for a one- and two-dimensional Cooley–Tukey algorithm. We close in Section 7 by giving some preliminary results of some computer experiments that suggest that this approach may have practical value in implementing large multidimensional Fourier transforms on machines with hierarchical or distributed memory.

2. ALGEBRAIC PRELIMINARIES

Let $\mathbf{C}$ denote the complex numbers and $\mathbf{C}^{\times}$ the multiplicative group $\mathbf{C} - \{0\}$. A homomorphism χ of a finite Abelian group into $\mathbf{C}^{\times}$ is called a *character*. If $L^2(A)$ denotes the vector space of complex-valued functions on the group A, we may consider $\chi \in L^2(A)$. We now define a unitary representation ρ of A, the regular representation, on $L^2(A)$ by

$$(\rho(a)f)(x) = f(x - a) \qquad f \in L^2(A).$$

Note that if χ is a character of A then

$$(\rho(a)\chi)(x) = \chi(x - a) = \chi(-a)\chi(x).$$

Hence $\chi(x)$ is an eigenvector of $\rho(a)$ for all $a \in A$ and the eigenvalue of $\rho(a)$ on χ is $\chi(-a)$. It is well known that the set of characters forms an Abelian group, denoted by $\hat{A}$, which is isomorphic to A. Hence the set of characters forms an orthonormal basis of $L^2(A)$ relative to which $\rho(a)$ is a diagonal matrix for each $a \in A$.

For each $a \in A$ define $\delta_a \in L^2(A)$ by

$$\delta_a(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise}. \end{cases}$$

Once we order the elements, the functions δ_a, $a \in A$, are an ordered basis of $L^2(A)$ called a δ-basis. Given an isomorphism $A \to \hat{A}$, $a \mapsto \hat{a}$, the Fourier transform of A, $F(A)$, is the linear transform of $F(A)$: $L^2(A) \to L^2(A)$ given by $F(A)\delta_a = \hat{a}$. Now let $f \in L^2(A)$, then $f = \sum_{a \in A} \alpha_a \delta_a$, $\alpha_a \in \mathbf{C}$, and $f = \sum \beta_a \hat{a}$. Since $F(A)\delta_a = \hat{a}$, we have

$$F(A)^{-1} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{L-1} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{L-1} \end{pmatrix}$$

or $F(A)^{-1}$ determines the orthogonal projection of a vector onto the orthonormal basis $\hat{a}$, $\hat{a} \in \hat{A}$.

Now let $A = \mathbf{Z}/n\mathbf{Z}$ and let $\delta_0, \delta_1, \ldots, \delta_{n-1}$ be a δ-basis. Then

$$(\rho(a)\delta_x)(y) = \delta_x(y - a).$$

Now $\delta_x(y - a) = 0$ unless $y - a = x$ or $y = x + a$ when it is 1. Hence $\rho(a)\delta_x = \delta_{x+a}$. Thus working with column vectors relative to the above basis,

$$\rho(1) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_0 \\ \vdots \\ a_{n-2} \end{pmatrix},$$

so that

$$\rho(1) = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{pmatrix} = S_n.$$

One verifies that if $a, x \in \mathbf{Z}/n\mathbf{Z}$ then $a \mapsto \hat{a}$ given by

$$\hat{a}(x) = e^{2\pi i a x/n}$$

defines an isomorphism of $\mathbf{Z}/n\mathbf{Z}$ with the group of n characters on $\mathbf{Z}/n\mathbf{Z}$. Then, with respect to a δ-basis we find

$$F(\mathbf{Z}/n\mathbf{Z}) = F_n = (e^{2\pi iax/n})_{0\leq a,x<n}.$$

Noting that

$$S_n\begin{pmatrix} 1 \\ e^{2\pi ia/n} \\ \vdots \\ e^{2\pi ia(n-1)/n} \end{pmatrix} = e^{-2\pi ia/n}\begin{pmatrix} 1 \\ e^{2\pi ia/n} \\ \vdots \\ e^{2\pi ia(n-1)/n} \end{pmatrix},$$

we have

$$S_nF_n = F_n\begin{pmatrix} e^{-2\pi i0/n} & & & \\ & e^{-2\pi i1/n} & & \\ & & \ddots & \\ & & & e^{-2\pi i(n-1)/n} \end{pmatrix}.$$

Hence, F_n^{-1} is the orthogonal projection onto the eigenvectors of S_n.

In our later work the following generalizations of this result will be essential. For $\alpha \neq 0$, let

$$S_n(\alpha) = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & \alpha \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{pmatrix},$$

let $\beta^n = \alpha$, and let $\omega = e^{2\pi i/n}$. By an easy computation

$$(S_n(\alpha))^n = \alpha I_n,$$

where I_n is the $n \times n$ identity matrix. Hence the eigenvalues of $S_n(\alpha)$ are $\beta\omega^k$, $0 \leq k < n$. Now if

$$S_n(\alpha)\begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \beta\omega^{-1}\begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

we obtain, by recursion,

$$x_{n-1} = \frac{\beta}{\alpha}\omega^{-1}, x_{n-2} = \frac{\beta^2}{\alpha}\omega^{-2}, \dots, x_0 = 1.$$

By some elementary matrix operations, we have

$$S_n(\alpha)D(\alpha)F_n = D(\alpha)F_nE(\alpha)$$

where

$$D(\alpha) = \begin{pmatrix} 1 & & & \\ & \frac{\beta^{n-1}}{\alpha} & & \\ & & \ddots & \\ & & & \frac{\beta}{\alpha} \end{pmatrix}$$

and

$$E(\alpha) = \beta\begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix}.$$

Hence the orthogonal projection onto the eigenvectors of $S_n(\alpha)$ is given by $(D(\alpha)F_n)^{-1} = F_n^{-1}D(\alpha)^{-1}$. Note

$$D(\alpha) = \begin{pmatrix} 1 & & & \\ & \beta & & \\ & & \ddots & \\ & & & \beta^{n-1} \end{pmatrix}.$$

LEMMA 2.1. *Let*

$$S_n(\alpha_1, \dots, \alpha_n) = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & \alpha_1 \\ \alpha_2 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \alpha_3 & 0 & \cdot & \cdot & \cdot & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & \cdot & \cdot & \cdot & 0 & \alpha_n & 0 \end{pmatrix}, \qquad \prod \alpha_i = \alpha \neq 0.$$

Then $S_n(\alpha_1, \dots, \alpha_n)$ is diagonally similar to $S_n(\alpha)$.

Proof. By direct computation,

$$S_n(\alpha_1,\dots,\alpha_n)\begin{pmatrix} 1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_2\cdots\alpha_n \end{pmatrix} = S_n(\alpha,\alpha_2,\dots,\alpha_2\cdots\alpha_n)$$

and

$$\begin{pmatrix} 1 & & & \\ & \alpha_2^{-1} & & \\ & & \ddots & \\ & & & (\alpha_2\cdots\alpha_n)^{-1} \end{pmatrix} S_n(\alpha,\alpha_2,\dots,\alpha_2\cdots\alpha_n) = S_n(\alpha). \quad ∎$$

The above discussion may be summarized as follows:

THEOREM 2.2. *Let*

$$S_n(\alpha_1,\dots,\alpha_n) = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & \alpha_1 \\ \alpha_2 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \alpha_3 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & \alpha_n & 0 \end{pmatrix}, \qquad \prod \alpha_i = \alpha \neq 0.$$

Then there exists a diagonal matrix $D(\alpha_1,\dots,\alpha_n)$ *such that*

$$S_n(\alpha_1,\dots,\alpha_n)D(\alpha_1,\dots,\alpha_n)F_n = D(\alpha_1,\dots,\alpha_n)F_nE(\alpha)$$

where

$$E(\alpha) = \beta\begin{pmatrix} e^{-2\pi i0/n} & & & \\ & e^{-2\pi i1/n} & & \\ & & \ddots & \\ & & & e^{-2\pi i(n-1)/n} \end{pmatrix}.$$

Now let

$$C = \mathbf{Z}/d_1\mathbf{Z} \times \cdots \times \mathbf{Z}/d_t\mathbf{Z}.$$

Then $L^2(C)$ may be identified with

$$L^2(\mathbf{Z}/d_1\mathbf{Z}) \otimes \cdots \otimes L^2(\mathbf{Z}/d_t\mathbf{Z})$$

and with this identification

$$\rho(C) = \rho(\mathbf{Z}/d_i\mathbf{Z}) \otimes \cdots \otimes \rho(\mathbf{Z}/d_t\mathbf{Z})$$

or

$$\rho(C) = \left\{S_{d_1}^{j_1} \otimes \cdots \otimes S_{d_t}^{j_t} | 0 \le j_i < d_i, i = 1, \dots, t\right\}.$$

Then $F(C) = F_{d_1} \otimes \cdots \otimes F_{d_t}$ and

$$F(C)^{-1}\rho(C)F(C)$$

is a tensor product of diagonal matrices.

Now let

$$\rho_\chi(C) = S_{d_1}\left(\alpha_1^{(1)}, \dots, \alpha_{d_1}^{(1)}\right)^{j_1} \otimes \cdots \otimes S_{d_t}\left(\alpha_1^{(t)}, \dots, \alpha_{d_t}^{(t)}\right)^{j_t}.$$

Then we have proved the theorem.

THEOREM 2.3. *$F(C)^{-1}E^{-1}\rho_\chi(C)EF(C)$ is a tensor product of diagonal matrices with E a tensor product of diagonal matrices.*

3. CHARACTER SUBSPACES

Let $\hat{a} \in \hat{A}$, then $\hat{a}$ restricted to B, $\hat{a}|_B$, is a homomorphism of B to $\mathbf{C}^\times$ and so $\hat{a}|_B \in \hat{B}$. Hence the restriction mapping defines a homomorphism of $\hat{A}$ into $\hat{B}$ with kernel $\hat{K}$. Now $\hat{k} \in \hat{K}$ maps B to 1 and so $\hat{k}$ induces a homomorphism of $A/B = C$ to $\mathbf{C}^\times$. Hence $\hat{K}$ may be identified with a subgroup of $\hat{C}$. But if $\hat{c} \in \hat{C}$,

$$A \to A/B \xrightarrow{\hat{c}} \mathbf{C}^\times$$

is a character of A in $\hat{K}$. Hence $\hat{K}$ may be identified with $\hat{C}$ and so by a counting argument the restriction mapping maps $\hat{A}$ onto $\hat{B}$ and so $\hat{A}/\hat{K}$ is isomorphic to $\hat{B}$.

DEFINITION 3.1. Let $\chi \in \hat{B}$ and $V_\times \subset L^2(A)$ be defined by

$$V_\times = \left\{f \in L^2(A) | \rho(b)f = \chi(-b)f, b \in B\right\}.$$

$V_\times$ is called a B character subspace.

LEMMA 3.1. *For* $a \in A$, $\chi \in \hat{B}$,

$$\rho(a)V_\chi = V_\chi.$$

Proof. For $a \in A$ and $b \in B$,

$$\rho(b)\big((\rho(a)V_\chi\big) = \rho(a)\big(\rho(b)V_\chi\big)$$
$$= \chi(-b)\rho(a)V_\chi. \quad \blacksquare$$

LEMMA 3.2. *Let* $\hat{a}_1, \dots, \hat{a}_l$, $l = |\hat{K}|$, *be the elements of* $\hat{A}$ *such that* $\hat{a}_\alpha|_B = \chi$, $\alpha = 1, \dots, l$. *Then* $\hat{a}_1, \dots, \hat{a}_l$ *is an orthonormal basis of* V_χ.

Proof. Since $\hat{A}/\hat{K} = \hat{B}$, the elements of $\hat{A}$, which when restricted to B equal χ, are a $\hat{K}$ coset. Orthonormality follows from the fact that unequal characters are orthogonal. $\blacksquare$

Since the $\hat{K}$ cosets exhaust $\hat{A}$ we have the following result.

THEOREM 3.3.

1. $L^2(A) = \oplus_{\chi \in B} V_\chi$;
2. $\dim V_\chi = \text{order } \hat{K} = \text{order } C$;
3. *If* $\hat{a} \in \hat{A}$ *is a* $\hat{K}$*-coset representative of* χ, *then* $\hat{a} + \hat{K}$ *is an orthonormal basis of* V_χ.

In this language the idea of the Cooley–Tukey algorithm is the following. We are given a vector X in a δ-basis and $\hat{a} \in \hat{A}$ in this δ-basis and we want to compute efficiently all the dot products $\langle X, \hat{a} \rangle$, $\hat{a} \in \hat{A}$. We can do this in two stages.

Stage 1. Compute the projection of X onto V_χ, $\chi \in \hat{B}$, and denote it by X_χ.

Stage 2. Let $\hat{a}_\chi + \hat{K}$ be the characters of A in V_χ. Compute the dot product of X_χ with each of these characters.

This may be seen more explicitly as follows. In the Introduction we saw that the Cooley–Tukey algorithm can be formulated as

$$F(A)^{-1} = \big(F(C)^{-1} \otimes I\big)T\big(I \otimes F(B)^{-1}\big)P$$

where P is a permutation matrix and T is a diagonal matrix. Let Q denote the permutation matrix such that

$$Q^{-1}\big(I \otimes F(C)^{-1}\big)Q = F(C)^{-1} \otimes I.$$

Since T is diagonal, $Q^{-1}T'Q = T$, where T' is diagonal. Hence,

$$Q^{-1}\left(I \otimes F(C)^{-1}\right)T'Q\left(I \otimes F(B)^{-1}\right)P = F(A)^{-1}.$$

Then $Y = Q(I \otimes F(B)^{-1})PX$ is the computation of Stage 1, $((I \otimes F(C)^{-1})T')Y$ is the computation of Stage 2, and Q^{-1} returns the output in the appropriate order.

4. AN ALGEBRAIC DIVIDE-AND-CONQUER ALGORITHM

Let A be a finite Abelian group, B a subgroup, and $C = A/B$. For $c \in C$, let $[c]$ denote the corresponding B-coset, and for each $c \in C$, let $\xi(c) \in [c] \subset A$ be a coset representative. Then $a \in A$ can be uniquely written as

$$a = \xi(c) + b \qquad c \in C, b \in B.$$

This determines a mapping $S_\xi\colon A \to \xi(C) \times B$ and by abuse of notation a mapping

$$S_\xi\colon L^2(A) \to L^2(\xi(C)) \otimes L^2(B).$$

Now $\rho(A)$ restricted to B induces an action of B on $L^2(\xi(C)) \otimes L^2(B)$ which is given by $I \otimes \rho(B)$ where $\rho(B)$ is the regular representation of B on $L^2(B)$. Now for $\chi \in \hat{B}$, V_χ has been defined by

$$V_\chi = \left\{ f \in L^2(A) \mid \rho(b)f = \chi(-b)f, b \in B \right\}.$$

Now consider $\chi \in L^2(B)$. Then $S_\xi(V_\chi) = L^2(\xi(C)) \otimes \chi$ because

$$(1 \otimes \rho(b))(y \otimes \chi) = y \otimes \rho(b)\chi = \chi(b)y \otimes \chi$$

and $\dim L^2(\xi(C)) \otimes \chi = \dim V_\chi$.

We have seen that V_χ is $\rho(A)$ invariant and so we may define $\rho_X = \rho|_{V_\chi}$. Our task is to compute ρ_χ and its diagonalizing matrix. We can be guided in this task by noting that ρ_χ is the representation of A obtained by inducing the representation χ of B to A. This tells us that we can establish a natural correspondence

$$\Delta_\xi\colon L^2(C) \to V_\chi$$

related to $\rho(C)$, the regular representation of C. This we will now proceed directly to do, without explicitly using the theory of induced representations.

For $g(c) \in L^2(C)$ define $\Delta_\xi g = \tilde{g} \in L^2(A)$ as follows: Let $a = \xi(c) + b$ and define

$$\tilde{g}(a) = \chi(b)g(c).$$

We must verify that $\tilde{g} \in V_\chi$ and that $\Delta_\xi \colon L^2(C) \to V_\chi$ is an isomorphism. Now for $a = \xi(c) + b$ and $b_1 \in B$

$$\rho(b_1)\tilde{g}(a) = \chi(b - b_1)g(c) = \chi(-b_1)\tilde{g}(a)$$

and so $\tilde{g} \in V_\chi$. Assume $\chi(b)g(c) = 0$. Since $|\chi(b)| = 1$, this means that $g(c) = 0$ for all c and Δ_ξ is an isomorphism.

Hence we may view ρ_χ as a linear transformation of $L^2(C)$ which we will need to describe in detail. To do this we will need the following definition. Let $c_1, c_2 \in C$ and define

$$\eta \colon C \times C \to B$$

by

$$\eta(c_1, c_2) = \xi(c_1) + \xi(c_2) - \xi(c_1 + c_2).$$

(In the language of group cohomology, given B and C, η is the 2-cocycle that determines A.) Since $\xi(0) = 0$, $\eta(c, -c) = \xi(c) + \xi(-c)$ or $-\xi(c) = \xi(-c) - \eta(c, -c)$.

THEOREM 4.1. *Let ρ_χ acting on $L^2(C)$ also be denoted by ρ_χ. Let $a = \xi(c) + b$ and $x \in C$. Then if δ_x is the δ-basis of $L^2(C)$,*

$$\rho_\chi(\xi(c) + b)\delta_x = \chi(-\eta(x, c) - b)\delta_{x+c}.$$

Proof. For $c_1 \in C$,

$$\begin{aligned}\big(\rho_\chi(\xi(c) + b)\delta_x\big)(c_1) &= \big(\rho_\chi(\xi(c) + b)\tilde{\delta}_x\big)(\xi(c_1)) \\ &= \tilde{\delta}_x(\xi(c_1) - \xi(c) - b) \\ &= \tilde{\delta}_x(\xi(c_1 - c))\chi(\eta(c_1, -c) - \eta(c, -c) - b).\end{aligned}$$

But $\tilde{\delta}_x(\xi(c_1 - c)) = \delta_x(c_1 - c) = \delta_{x+c}$. But, then everything is zero unless $c_1 = x + c$ and so

$$\begin{aligned}\rho_\chi(\xi(c) + b)\delta_x &= \chi(\eta(x + c, -c) - \eta(c, -c) - b)\delta_{x+c} \\ &= \chi(-\eta(x, c) - b)\delta_{x+c}. \quad \blacksquare\end{aligned}$$

The crucial thing is that this theorem demonstrates that ρ_χ relative to the δ-basis of C is a matrix whose only non-zero entries occur exactly at the 1's of $\rho(C)$. This shows that we have finally arrived at the material presented in Section 3.

Now let $C = \mathbf{Z}/d_1\mathbf{Z} \times \cdots \times \mathbf{Z}/d_t\mathbf{Z}$. Then for each $\chi \in \hat{B}$, $\rho_\chi(A)$ is a representation of the form

$$S_{d_1}\left(\alpha_1^{(1)}, \dots, \alpha_{d_1}^{(1)}\right)^{j_1} \otimes \cdots \otimes S_{d_t}\left(\alpha_1^{(t)}, \dots, \alpha_{d_t}^{(t)}\right)^{j_t}.$$

Then we may apply Theorem 2.3 to obtain the diagonalizing matrix of $\rho_\chi(A)$ to be of the form $EF(C)$ where E is a tensor product of diagonal matrices. Hence, $F(C)^{-1}E^{-1}$ is the projection of V_χ onto the eigenvectors of $\rho_\chi(A)$. Pulling all this together we have

$$F(A)^{-1} = S_\xi^{-1}\left(F(C)^{-1} \otimes I\right)E^*\left(I \otimes F(B)^{-1}\right)\Big)S_\xi$$

where E^* is a tensor product of diagonal matrices.

5. COORDINATES AND A CONCRETE PROCEDURE

The preceding material provides the foundation for the uniform derivation of a wide variety of concrete Cooley–Tukey type algorithms for computing the finite Fourier transform. The key transition is to introduce coordinates so that the computational procedures may actually be calculated. We introduce coordinates in the following way.

By the fundamental theorem of Abelian groups any finite Abelian group A has a basis $\langle a_1, a_2, \dots, a_t\rangle$. That is, there exists elements $a_i \in A$ of order n_i such that A is the direct product of the cyclic subgroups generated by the a_i. In the terminology of Section 1, we say that A has the presentation

$$A = \mathbf{Z}/n_1\mathbf{Z} \times \mathbf{Z}/n_2\mathbf{Z} \times \cdots \times \mathbf{Z}/n_t\mathbf{Z}.$$

And so, each element $a \in A$ has a coordinate representation

$$a = (\alpha_1, \alpha_2, \dots, \alpha_t)$$

with $0 \le \alpha_i < n_i$, meaning that

$$a = \sum \alpha_i a_i.$$

In this presentation of A we order the coordinates lexicographically and thus order A. Now relative to a compatibly ordered δ-basis of $L^2(A)$, we write

$$F(A) = F_{n_1} \otimes F_{n_2} \otimes \cdots \otimes F_{n_t}$$

where $F_n = (e^{2\pi irs})_{0 \le r,s < n}$. (Note that this defines an isomorphism $A \to \hat{A}$.)

Now given an Abelian group A and a subgroup B consider the exact sequence

$$0 \to B \to A \overset{\xi}{\underset{\leftarrow}{\to}} C \to 0,$$

where $\xi(c)$, $c \in C$, is a choice of coset representatives of B in A. Choose presentations of A, B, and $C = A/B$. The main theorem of the preceding sections can now be interpreted as a method for finding a matrix factorization of $F(A)$ in terms of $F(B)$ and $F(C)$.

We state this as a procedure to factor $F(A)$ by applying the following steps to a δ-basis of $L^2(A)$.

1. Permute the input to form the cosets A/B according to the choice of representatives ξ. This amounts to reordering a δ-basis of $L^2(A)$ corresponding to the isomorphism $L^2(A) = L^2(\xi(C)) \otimes L^2(B)$ relative to the orders defined by the chosen presentations. This is the permutation $P(\xi)$.

2. Compute $F(B)$ on each of these cosets.

3. Form the character spaces V_χ, $\chi \in \hat{B}$, by collecting the vectors computed in Step 2 corresponding to each character χ. (These are in fact the image of a δ-basis of C.)

4. Multiply each of the basis vectors by an appropriate scalar. This is the diagonal matrix $T'(\eta)$, the "twiddle factors."

5. Compute $F(C)$ on each of the character spaces.

6. Permute the output to obtain $F(A)$. This is the permutation Q.

The theory developed earlier guarantees that for any choice of presentations and choice of coset representative ξ there is a choice of twiddle factors in Step 4 and permutation in the last step for which this procedure produces a factorization of $F(A)$. The theory actually provides more. Theorem 4.1 enables us to calculate the twiddle factors directly from the 2-cocycle η, defined in Section 4, since as we remarked in Step 3 the basis we find for the character spaces is the image of a δ-basis for C.

As we will see in the next section, not only can we apply this procedure for specific choices to obtain an algorithm, but also in cases where the steps of this procedure can be parametrized we can obtain formulas for whole classes of factorizations. For example, we will obtain formulas for F_n in terms of F_r and F_s where $n = rs$.

We will now summarize the results of the earlier sections using particular choices of basis elements. The results will be states in a concrete form that can be used in the calculation of examples and descriptions of algorithms.

Let A be an Abelian group of order n, B be a subgroup of order m, and $C = A/B$ be of order $n/m = d$. Let ξ: $C \to A$ be a choice of coset representatives of A/B.

Let $b \to \hat{b}$ be an isomorphism from $B \to \hat{B}$. The following notation is used for the action of $F(B)$ on a B coset defined by the coset representative $\xi(c)$.

$$F(B)\,\delta_{b+\xi(c)} = \sum_{\beta \in B} \hat{b}(\beta)\,\delta_{\beta+\xi(c)} = \delta^{b}_{\xi(c)}.$$

A simple calculation shows that $\delta^{b}_{\xi(c)}$ is in the character space $V_{\hat{b}}$ and that $\{\delta^{b}_{\xi(c)} | c \in C\}$ is a basis for $V_{\hat{b}}$.

$$\begin{aligned}
\rho(b')\,\delta^{b}_{\xi(c)} &= \rho(b') \sum_{\beta \in B} \hat{b}(\beta)\,\delta_{\beta+\xi(c)} \\
&= \sum_{\beta \in B} \hat{b}(\beta)\,\delta_{\beta+b'+\xi(c)} \\
&= \sum_{\beta' \in B} \hat{b}(\beta' - b')\,\delta_{\beta'+\xi(c)} \\
&= \hat{b}(-b')\,\delta^{b}_{\xi(c)}.
\end{aligned}$$

The basis, $\{\delta^{b}_{\xi(c)} | c \in C\}$, is the image of the δ-basis for $L^2(C)$ under the map, Δ_ξ, used in Theorem 4.1, which mapped $f \in L^2(C)$ to $\tilde{f} \in V_{\hat{b}}$. Sometimes the image basis will be called a $\tilde{\delta}$-basis.

The following lemma shows the effect of a change of coset representatives on a $\tilde{\delta}$-basis.

LEMMA 5.1. *Let ξ: $C \to A$ and ξ': $C \to A$ be two choices of coset representatives for A/B. Then the change of basis matrix $\{\delta^{b}_{\xi(c)} | c \in C\} \to \{\delta^{b}_{\xi'(c)} | c \in C\}$ is a diagonal matrix whose diagonal elements are characters of B.*

Proof. Since $\xi(c) \equiv \xi'(c) \bmod B$, $\xi(c) = \xi'(c) + b'$, $b' \in B$, and the previous calculation shows that $\delta^{b}_{\xi(c)} = \hat{b}(-b')\delta^{b}_{\xi'(c)}$. ∎

In a similar fashion to the computation that showed that $\delta^b_{\xi(c)}$ was in V_b, we can compute that action of $\rho(a)$, $a \in A$, on the $\tilde{\delta}$-basis. Write $a = \xi(c) + b'$. Then

$$\begin{aligned}
\rho(a)\,\delta^b_{\xi(c_i)} &= \rho(\xi(c))\,\rho(b')\,\delta^b_{\xi(c_i)} \\
&= \hat{b}(-b')\,\rho(\xi(c))\,\delta^b_{\xi(c_i)} \\
&= \hat{b}(-b') \sum_{\beta \in B} b\hat{\ }(\beta)\,\delta_{\beta + \xi(c_i) + \xi(c)} \\
&= \hat{b}(-b') \sum_{\beta \in B} \hat{b}(\beta)\,\delta_{\beta + \xi(x_i + c) + \eta(c_i, c)} \\
&= \hat{b}(-b')\hat{b}(-\eta(c_i, c)) \sum_{\beta \in B} \hat{b}(\beta)\,\delta_{\beta + \xi(c_i + c)} \\
&= \hat{b}(-\eta(c_i, c) - b)\,\delta^b_{\xi(c_i + c)}.
\end{aligned}$$

Assume that $a \equiv a' \bmod B$. Then $\rho_{\hat{b}}(a) \equiv \rho_{\hat{b}}(a') \bmod \hat{b}$, meaning that their action on $V_{\hat{b}}$ is equivalent up to $\hat{b}(b')$ for some $b' \in B$. From this observation we see that $\rho_{\hat{b}}$ is equivalent to the regular representation of C acting on $L^2(C)$. More specifically,

$$\rho_{\hat{b}}(\xi(c_i))\rho_{\hat{b}}(\xi(c_j)) = \hat{b}(-\eta(c_i, c_j))\rho_{\hat{b}}(\xi(c_i + c_j)).$$

We now use this equation to construct a matrix representation of $\rho_{\hat{b}}$ with respect to a $\tilde{\delta}$-basis.

First assume that C is cyclic, of order d, with generator c. The matrix representing $\rho_{\hat{b}}(\xi(c))$ with respect to the basis $\{\delta^b_{\xi(jc)} | 0 \le j < d\}$ is

$$R = S_d\big(\hat{b}(-\eta((d-1)c, c)), \dots, \hat{b}(-\eta(c, c))\big),$$

and with respect to this basis

$$\rho_{\hat{b}}(\xi(jc)) = \rho_{\hat{b}}(j\xi(c) + b') = \hat{b}(-b')R^j$$

for some $b' \in B$.

By Lemma 2.1 R is diagonally similar to

$$\begin{aligned}
&S_d\big(\hat{b}(-\eta((d-1)c, c) \cdots \hat{b}(-\eta(c, c)))\big) \\
&\qquad = S_d\big(\hat{b}(-\eta((d-1)c, c) - \cdots - \eta(c, c))\big).
\end{aligned}$$

Therefore B is diagonally similar to $S_d(\hat{b}(b'))$ for

$$b' = -\eta((d-1)c, c) - \cdots - \eta(c, c).$$

Moreover, $S_d(\hat{b}(b'))$ is diagonally similar to βS_d, where $\beta^d = \hat{b}(b')$.

Assume that C is a direct sum and that η is compatible with the direct sum. Under these assumptions we show that $\rho_{\hat{b}}(\xi(C)) = \rho_{\hat{b}}(\xi(C_1)) \otimes \rho_{\hat{b}}(\xi(C_2))$.

LEMMA 5.2. *Assume* $C = C_1 \times C_2$ *and that* $\eta(c_1, c_2) = 0$ *for all* $c_1 \in C_1$ *and* $c_2 \in C_2$. *Then* $\rho_{\hat{b}}(C) = \rho_{\hat{b}}(C_1) \otimes \rho_{\hat{b}}(C_2)$.

Proof. Define $\delta^b_{\xi(c_1)} \otimes \delta^b_{\xi(c_2)} = \delta^b_{\xi(c_1+c_2)}$. Then

$$\begin{aligned}
\rho_{\hat{b}}(c_1' + c_2')\,\delta^b_{\xi(c_1+c_2)} &= \delta^b_{\xi(c_1+c_2)+\xi(c_1'+c_2')} \\
&= \delta^b_{\xi(c_1)+\xi(c_2)+\xi(c_1')+\xi(c_2')} \\
&= \hat{b}(\eta(c_1, c_1'))\hat{b}(\eta(c_2, c_2'))\,\delta^b_{\xi(c_1+c_1')+\xi(c_2+c_2')} \\
&= \rho_{\hat{b}}(c_1')\,\delta^b_{\xi(c_1)} \otimes \rho_{\hat{b}}(c_2')\,\delta^b_{\xi(c_2)}. \quad \blacksquare
\end{aligned}$$

Remark. The assumptions of the previous lemma can be satisfied by choosing $\xi(c_1 + c_2) = \xi(c_1) + \xi(c_2)$ for $c_1 \in C_1$ and $c_2 \in C_2$.

THEOREM 5.3. *Let* $\langle c_1, \ldots, c_t \rangle = \mathbf{Z}/d_1\mathbf{Z} \times \cdots \times \mathbf{Z}/d_t\mathbf{Z}$ *be a presentation for* $C = A/B$. *Then*

$$\rho_{\hat{b}}(A) = \left\{\hat{b}(b(j_1, \ldots, j_t))S_{d_1}\left(\alpha_1^{(1)}, \ldots, \alpha_{d_1}^{(1)}\right)^{j_1} \otimes \cdots \otimes S_{d_t}\left(\alpha_1^{(t)}, \ldots, \alpha_{d_t}^{(t)}\right)^{j_t}\right\}$$

where $b(j_1, \ldots, j_t)$ *is the element of* B *such that*

$$\xi(j_1c_1 + \cdots + j_tc_t) = j_1\xi(c_1) + \cdots + j_t\xi(c) + b(j_1, \ldots, j_t)$$

and

$$S_{d_i}\left(\alpha_1^{(i)}, \ldots, \alpha_{d_i}^{(i)}\right) = \rho_{\hat{b}}(\xi(c_i)).$$

COROLLARY 5.4. *There exists a diagonal matrix* T *such that*

$$\rho_{\hat{b}}(A) = T\left\{\hat{b}(b(j_1, \ldots, j_t))S_{d_1}^{j_1} \otimes \cdots \otimes S_{d_t}^{j_t} | 0 \le j_1 < d_1, \ldots, 0 \le j_t < d_t\right\}T^{-1}$$

6. APPLICATIONS

We now apply the concrete procedure in Section 5 to obtain various factorizations of the Fourier transform.

6.1. *One-Dimensional Examples*

We will consider three factorizations for F_6. Let $A = \mathbf{Z}/6\mathbf{Z} = \{0, 1, 2, 3, 4, 5\}$ and choose $B = \mathbf{Z}/2\mathbf{Z} = \{0, 3\}$. Then $C = \mathbf{Z}/3\mathbf{Z} = \{0', 1', 2'\}$. We calculate factorizations based on three different choices of the cross section $\xi: C \to A$. The first is a "natural" choice which leads to the standard one-dimensional Cooley–Tukey algorithm, and we pause to derive a formula in this case. The second choice is the case where ξ is actually a group homomorphism, which can be chosen if A is the direct product of B and C. In the cyclic case this can be done if the order of B and C are relatively prime. This choice is the basis of the Good–Thomas algorithm where the twiddle factors are trivial. Finally, we compute a case where ξ is arbitrarily chosen.

A standard form of the Cooley–Tukey factorization is given in [8]:

THEOREM 6.1.

$$F_{rs} = (F_r \otimes I_s) T_s^{sr} (I_r \otimes F_s) L_r^{sr},$$

where T_s^{sr} is a diagonal and L_r^{sr} is a permutation matrix.

More explicitly, if $\omega = e^{2\pi i/rs}, T_s^{rs} = \oplus_{i=0}^{r-1} (D_s^{rs})^i$, the direct sum of powers of the diagonal matrix $D_s^{rs} = \operatorname{diag}\{1, \omega, \dots, \omega^{s-1}\}$. And we have

DEFINITION 6.1 (Stride Permutation). Let x be a vector of length m and y a vector of length n. Then

$$L_n^{mn}(x \otimes y) = y \otimes x.$$

The notation indicates that elements of a vector of length mn are loaded into n segments, each at stride n. If $x = (x_0, x_1, \dots, x_{mn-1})$ then

$$L_n^{mn} x = (x_0, x_n, \dots, x_{(m-1)n}, \dots, x_{n-1}, x_{2n-1}, \dots, x_{mn-1}).$$

We remark that $L_s^{rst} L_t^{rst} = L_{st}^{rst}$, and hence $L_n^N L_{N/n}^N = I_N$.

The load-stride permutation is the permutation that commutes the factors of the tensor product.

THEOREM 6.2. *If A is an $m \times m$ matrix, and B is an $n \times n$ matrix, then*

$$L_n^{mn}(A \otimes B) = (B \otimes A) L_n^{mn}.$$

Equivalently,

$$B \otimes A = L_n^{mn}(A \otimes B) L_n^{mn}.$$

6.1.1. *A Natural* ξ. Now a "natural" cross section $\xi: C \to A$ is given by $\xi(j') = j$. With these choices let us apply the procedure.

Step 1. $P(\xi)$. Permuting the δ-basis of $L^2(A)$, we obtain the partition

$$\{\delta_0, \dots, \delta_5\} = \{\delta_0, \delta_3\} \vee \{\delta_1, \delta_4\} \vee \{\delta_2, \delta_5\},$$

where $\vee$ denotes the disjoint union.

We write the corresponding permutation matrix as

$$P = \text{perm}\{0, 3, 1, 4, 2, 5\} = L_3^6,$$

where $\text{perm}\{n_0, n_1, \dots, n_t\}$ denotes the permutation $i \mapsto n_i$.

Step 2. $F(B) = F_2$. Applying $F(B) = F_2$ to each of these cosets we obtain

$$\delta_0^0 = \delta_0 + \delta_3$$
$$\delta_0^3 = \delta_0 - \delta_3$$
$$\delta_1^0 = \delta_1 + \delta_4$$
$$\delta_1^3 = \delta_1 - \delta_4$$
$$\delta_2^0 = \delta_2 + \delta_5$$
$$\delta_2^3 = \delta_2 - \delta_5.$$

Step 3. V_χ. Gathering these vectors at stride 2 we form the two character spaces of B.

$$V_0 = \langle \delta_0^0, \delta_1^0, \delta_2^0 \rangle$$
$$V_3 = \langle \delta_0^3, \delta_1^3, \delta_2^3 \rangle.$$

Step 4. $T'(\eta)$. Now to find the appropriate twiddle factors we must compute the regular representation of A restricted to these character spaces. We can do this directly or appeal to Theorem 4.1. In the first instance we have

$$\begin{aligned}
\rho(1)\,\delta_0^0 &= \rho(1)(\delta_0 + \delta_3) \\
&= \delta_1 + \delta_4 \\
&= \delta_1^0 \\
\rho(1)\,\delta_1^0 &= \rho(1)(\delta_1 + \delta_4) \\
&= \delta_2 + \delta_5 \\
&= \delta_2^0 \\
\rho(1)\,\delta_2^0 &= \rho(1)(\delta_2 + \delta_5) \\
&= \delta_3 + \delta_0 \\
&= \delta_0^0 \\
\rho(1)\,\delta_0^3 &= \rho(1)(\delta_0 - \delta_3) \\
&= \delta_1 - \delta_4 \\
&= \delta_1^3 \\
\rho(1)\,\delta_1^3 &= \rho(1)(\delta_1 - \delta_4) \\
&= \delta_2 - \delta_5 \\
&= \delta_2^3 \\
\rho(1)\,\delta_2^3 &= \rho(1)(\delta_2 - \delta_5) \\
&= \delta_3 - \delta_0 \\
&= -\delta_0^3.
\end{aligned}$$

And so

$$\rho|_{V_0}(1) = \rho_0(1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = S_3$$

and

$$\rho|_{V_3}(1) = \rho_3(1) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = S_3(-1).$$

Now we compute $\rho_j(1)$ directly using Theorem 4.1. Recall that ρ restricted to the character space V_χ can be viewed as acting on $L^2(C)$. If $a = \xi(c) + b$ and $x \in C$, then if δ'_x is the δ-basis of $L^2(C)$,

$$\rho_j(a)\,\delta'_x = \chi_j(-\eta(x,c) - b)\,\delta'_{x+c},$$

and thus, in the case at hand,

$$\rho_j(1)\,\delta'_x = \chi_j(-\eta(x,1'))\,\delta'_{x+1'}.$$

From the "natural" choice of coset representatives, the 2-cocycle η is particularly simple. In fact, $\eta(x,1')$ is zero except when $x = 2'$ and then $\eta(2',1') = 3$. Thus,

$$\rho_j(1) = \begin{pmatrix} 0 & 0 & \chi_j(-3) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

which confirms the result already calculated.

Now in order for F_3 to diagonalize these matrices we must multiply by a diagonal, the twiddle factors. From the discussion in Section 2 we have

$$T' = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \omega & \\ & & & & & \omega^2 \end{pmatrix},$$

where $\omega = e^{2\pi i/6}$.

Step 5. $F(C)$. Now, noting that $\omega^2 = e^{2\pi i/3}$, apply F_3 in each of these spaces to obtain

$$\nu_0 = \delta_0^0 + \delta_1^0 + \delta_2^0$$

$$\nu_1 = \delta_0^0 + \omega^2\delta_1^0 + \omega^4\delta_2^0$$

$$\nu_2 = \delta_0^0 + \omega^4\delta_1^0 + \omega^4\delta_2^0$$

$$\nu_3 = \delta_0^3 + \omega\delta_1^3 + \omega^2\delta_2^3$$

$$\nu_4 = \delta_0^3 + \omega^3\delta_1^3 + \delta_2^3$$

$$\nu_5 = \delta_0^3 + \omega^5\delta_1^3 + \omega^4\delta_2^3.$$

Step 6. Q. Now that ν_j's are the characters of A in some order. So to find the output permutation, all we need to do is to determine which ones they are, relative to the initial order of $\hat{A}$, or the order carried over by the mapping $a \mapsto \hat{a}$ and our choice of the order on A. We have chosen the characters of A, $\hat{a}(x) = e^{2\pi i a x/6} = \omega^{ax}$ for $a \in A$. So, the most straightforward way to determine which character is ν_j is to evaluate it at the generator 1 of A. We have from Step 2

$$\delta_j^k(1) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, from Step 5,

$$\nu_0(1) = 1 = \chi_0(1)$$

$$\nu_1(1) = \omega^2 = \chi_2(1)$$

$$\nu_2(1) = \omega^4 = \chi_4(1)$$

$$\nu_3(1) = \omega^1 = \chi_1(1)$$

$$\nu_4(1) = \omega^3 = \chi_3(1)$$

$$\nu_5(1) = \omega^5 = \chi_5(1).$$

And so, the required permutation is a gather at stride 3:

$$Q = \text{perm}\{0, 3, 1, 4, 2, 5\} = L_3^6.$$

To summarize, by this procedure we have obtained the factorization

$$F_6 = L_3^6 (I_2 \otimes F_3) T' L_2^6 (I_3 \otimes F_2) L_3^6.$$

This can be brought to our standard form,

$$F_6 = (F_3 \otimes I_2) T (I_3 \otimes F_2) L_3^6,$$

where $T = \text{diag}\{1, 1, 1, \omega, 1, \omega^2\}$, by observing the following: $(L_2^6)^{-1} = L_3^6$,

$$T = L_3^6 T' L_2^6$$

and

$$F_3 \otimes I_2 = L_3^6 (I_2 \otimes F_3) L_2^6.$$

6.1.2. 1-*D Cooley–Tukey*. Let $A \cong \mathbf{Z}/mn\mathbf{Z}$, and let $B < A = \{0, m, \dots, (n-1)m\}$. $B \cong \mathbf{Z}/n\mathbf{Z}$ and $C = A/B \cong \mathbf{Z}/m\mathbf{Z}$. Choose $\{0, 1, \dots, m-1\}$ as coset representatives for A/B.

LEMMA 6.3. *Let $\hat{b} \in \hat{B}$ and let ρ be the regular representation of A. Then, for $0 \le j < m$,*

$$\rho(j) = S_m\big(\hat{b}(m)\big) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \hat{b}(-m) \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & & \\ & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^j,$$

with respect to the basis $\{\delta_0^b, \ldots, \delta_{m-1}^b\}$.

Lemma 6.3 implies the following lemma.

LEMMA 6.4. *Let $\chi \in \hat{A}$ be a character such that $\chi|_B = \hat{b}$. Then*

$$\operatorname{diag}\{\chi(0), \chi(1), \ldots, \chi(m-1)\} \begin{pmatrix} 0 & 0 & \cdots & 0 & \chi(m) \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & & \\ & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\times \operatorname{diag}\{\chi(0), \chi(-1), \ldots, \chi(-(m-1))\}$$

$$= \chi(-1) \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & & \\ & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

LEMMA 6.5. *Let $\chi_1, \ldots, \chi_n$ be characters in A that restrict to the characters in B, $D_i = D(\chi_i(1))$, and $T = \oplus_{i=0}^{m-1} D_i$. Then by Theorem* 3.3 $L^2(A) = \oplus_{i=0}^{n-1} V_{\chi_i}$ *and*

$$\rho(1) = \bigoplus_{i=1}^{m-1} S_m(\chi_i(m)) = T^{-1}\left(\bigoplus_{i=0}^{m-1} \chi_i(1) S_m\right) T.$$

THEOREM 6.6. *Using this decomposition we observe that the Fourier transform matrix for $\mathbf{Z}/mn\mathbf{Z}$ can be factored,*

$$F_{mn} = Q(\chi)(I_m \otimes F_n) T (F_m \otimes I_n),$$

where $Q(\chi)$ is a permutation determined by the choice of characters in $\hat{A}$ that restrict to the characters in $\hat{B}$.

The permutation $Q(\chi)$ can be determined by comparing the order of the eigenvalues of $\rho(1)$ obtained by $F(A)$ to those obtained by this factorization. Diagonalizing S_m by $F(C)$ results in a matrix whose diagonal elements are $\hat{k}(1)$ for the characters in $\hat{K}$ (see Section 3 for the definition of $\hat{K}$). Since the χ_i are a set of coset representatives for $\hat{A}/\hat{K}$, the resulting diagonal elements obtained from the factorization in the theorem, $\chi_i(1)\hat{k}(1)$, are the characters of A evaluated at 1.

If χ_j is chosen to be $e^{2\pi ij/mn}$, then $Q(\chi)$ is a stride permutation and the resulting formula is the standard decimation in frequency algorithm [8].

6.1.3. *A Splitting* ξ. Returning to our example of F_6, there is another choice of ξ for which η is even simpler. In fact, since 2 and 3 are relatively prime we can choose ξ to be a homomorphisms $\xi: C \to A$. Under these circumstances we say that ξ is a splitting of the sequence:

$$0 \to B \to A \overset{\overset{\xi}{\leftarrow}}{\to} C \to 0.$$

If ξ is homomorphism, $\eta \equiv 0$ and, in Step 4, all the twiddle factors will be 1. So, let

$$\xi(0') = 0$$

$$\xi(1') = 4$$

$$\xi(2') = 2.$$

It is easy to see that with this set of choices ξ is a homomorphism.

Step 1. $P(\xi)$.

$$\{\delta_0, \dots, \delta_5\} = \{\delta_0, \delta_3\} \vee \{\delta_4, \delta_1\} \vee \{\delta_2, \delta_5\}.$$

Write $P = \text{perm}\{0, 3, 4, 1, 2, 5\}$.

Step 2. $F(B) = F_2$. Let

$$\delta_0^0 = \delta_0 + \delta_3$$

$$\delta_0^3 = \delta_0 - \delta_3$$

$$\delta_1^0 = \delta_4 + \delta_1$$

$$\delta_1^3 = \delta_4 - \delta_1$$

$$\delta_2^0 = \delta_2 + \delta_5$$

$$\delta_2^3 = \delta_2 - \delta_5.$$

Step 3. V_χ.

$$V_0 = \langle \delta_0^0, \delta_1^0, \delta_2^0 \rangle$$
$$V_3 = \langle \delta_0^3, \delta_1^3, \delta_2^3 \rangle.$$

Step 4. $T'(\eta)$.

$$\begin{aligned}
\rho(1)\,\delta_0^0 &= \rho(1)(\delta_0 + \delta_3) \\
&= \delta_1 + \delta_4 \\
&= \delta_1^0 \\
\rho(1)\,\delta_1^0 &= \rho(1)(\delta_4 + \delta_1) \\
&= \delta_5 + \delta_2 \\
&= \delta_2^0 \\
\rho(1)\,\delta_2^0 &= \rho(1)(\delta_2 + \delta_5) \\
&= \delta_3 + \delta_0 \\
&= \delta_0^0 \\
\rho(1) &= \rho(1)(\delta_0 - \delta_3) \\
&= \delta_1 - \delta_4 \\
&= -\delta_1^3 \\
\rho(1)\,\delta_1^3 &= \rho(1)(\delta_4 - \delta_1) \\
&= \delta_5 - \delta_2 \\
&= -\delta_2^3 \\
\rho(1)\,\delta_2^3 &= \rho(1)(\delta_2 - \delta_5) \\
&= \delta_3 - \delta_0 \\
&= -\delta_0^3.
\end{aligned}$$

And so,

$$\rho_0(1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = S_3$$

and

$$\rho_3(1) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = -S_3.$$

Now we compute $\rho_j(1)$ directly using Theorem 4.1. Here $1 = \xi(1') + 3$ and $\eta \equiv 0$ so

$$\rho_j(1)\,\delta'_x = \chi_j(-3)\,\delta'_{x+1'}.$$

Thus,

$$\rho_j(1) = \chi_j(-3) S_3,$$

which confirms the result already calculated. Since F_3 diagonalizes $\rho_j(1)$, all the twiddle factors T' are ones.

Step 5. $F(C)$. Now apply F_3 in each of these spaces to obtain

$$\nu_0 = \delta_0^0 + \delta_1^0 + \delta_2^0$$

$$\nu_1 = \delta_0^0 + \omega^1\delta_1^0 + \omega^4\delta_2^0$$

$$\nu_2 = \delta_0^0 + \omega^4\delta_1^0 + \omega^4\delta_2^0$$

$$\nu_3 = \delta_0^3 + \delta_1^3 + \delta_2^3$$

$$\nu_4 = \delta_0^3 + \omega^2\delta_1^3 + \omega^4\delta_2^3$$

$$\nu_5 = \delta_0^3 + \omega^4\delta_1^3 + \omega^2\delta_2^3.$$

Step 6. Q. Now again, the ν_j's are characters of A in some order. We have

$$\delta_j^k(1) = \begin{cases} 1 & \text{if } j = 1 \text{ and } k = 0 \\ -1 & \text{if } j = 1 \text{ and } k = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\nu_0(1) = 1 = \chi_0(1)$$

$$\nu_1(1) = \omega^2 = \chi_2(1)$$

$$\nu_2(1) = \omega^4 = \chi_4(1)$$

$$\nu_3(1) = \omega^3 = \chi_3(1)$$

$$\nu_4(1) = \omega^5 = \chi_5(1)$$

$$\nu_5(1) = \omega^1 = \chi_1(1).$$

Writing $Q = \text{perm}\{0, 2, 4, 3, 5, 1\}$, we obtain the factorization

$$F_6 = Q(I_2 \otimes F_3) L_2^6 (I_3 \otimes F_2) P.$$

To obtain our standard form we write $Q = Q' L_3^6$ and

$$F_6 = Q'(F_3 \otimes I_2)(I_3 \otimes F_2) P,$$

where $Q' = \text{perm}\{0, 3, 2, 5, 4, 1\}$.

6.1.4. *An Arbitrary* ξ. Finally, we compute a factorization for one more choice of coset representatives:

$$\begin{aligned} \xi(0') &= 0 \\ \xi(1') &= 4 \\ \xi(2') &= 5. \end{aligned}$$

Step 1. $P(\xi)$.

$$\{\delta_0, \ldots, \delta_5\} = \{\delta_0, \delta_3\} \vee \{\delta_4, \delta_1\} \vee \{\delta_5, \delta_2\}.$$

And $P = \text{perm}\{0, 3, 4, 1, 5, 2\}$.

Step 2. $F(B) = F_2$. All we need for this computation are the values of δ_j^k at $1 \in A$:

$$\delta_j^k(1) = \begin{cases} 1 & \text{if } j = 1 \text{ and } k = 0 \\ -1 & \text{if } j = 1 \text{ and } k = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Step 3. V_χ.

$$\begin{aligned} V_0 &= \langle \delta_0^0, \delta_1^0, \delta_2^0 \rangle \\ V_3 &= \langle \delta_0^3, \delta_1^3, \delta_2^3 \rangle. \end{aligned}$$

Step 4. $T'(\eta)$. We compute this directly from Theorem 4.1. Since $\chi_0 \equiv 1$ we always have $\rho_0(1) = S_3$. Since $1 = \xi(1') + 3$, for the other character space V_3 we have

$$\rho_3(1) = \chi_3(-3)\begin{pmatrix} 0 & 0 & \chi_3(-\eta(2',1')) \\ \chi_3(-\eta(0',1')) & 0 & 0 \\ 0 & \chi_3(-\eta(1',1')) & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

or

$$\rho_3(1) = S_3(1,-1,1).$$

We know that $S_3(1,-1,1)$ is diagonally similar to $S_3(-1)$, with similarity transform diag$\{1,-1,-1\}$. Thus,

$$T' = \text{diag}\{1,1,1,1,-\omega,-\omega^2\} = \text{diag}\{1,1,1,1,\omega^4,\omega^5\}.$$

Step 5. $F(C)$.

$$\nu_0 = \delta_0^0 + \delta_1^0 + \delta_2^0$$

$$\nu_1 = \delta_0^0 + \omega^2\delta_1^0 + \omega^4\delta_2^0$$

$$\nu_2 = \delta_0^0 + \omega^4\delta_1^0 + \omega^4\delta_2^0$$

$$\nu_3 = \delta_0^3 + \omega^4\delta_1^3 + \omega^5\delta_2^3$$

$$\nu_4 = \delta_0^3 + \delta_1^3 + \omega\delta_2^3$$

$$\nu_5 = \delta_0^3 + \omega^2\delta_1^3 + \omega\delta_2^3.$$

Step 6. Q. To find the required output permutation we evaluate the ν_j's at 1. Thus,

$$\nu_0(1) = 1 = \chi_0(1)$$

$$\nu_1(1) = \omega^2 = \chi_2(1)$$

$$\nu_2(1) = \omega^4 = \chi_4(1)$$

$$\nu_3(1) = -\omega^4 = \omega^1 = \chi_1(1)$$

$$\nu_4(1) = -1 = \omega^3 = \chi_3(1)$$

$$\nu_5(1) = -\omega^2 = \omega^5 = \chi_5(1).$$

And so, the required permutation in Step 6 is a gather at stride 3, $Q = L_3^6$.

6.2. *Two-Dimensional Examples*

We now compute two two-dimensional (2-D) examples. The first is a coherent presentation with a natural choice of ξ. Just as in the one-dimensional (1-D) case, this leads to a general formula for a 2-D factorization. Then we look at the example given in the Introduction of a presentation that is not coherently presentable. This case is not only potentially interesting for algorithm design, but, as in crystallographic FFT's [1, 2], it arises naturally when the subgroup B is determined by other features of the problem.

6.2.1. *A Coherent Case*. Let $A = \mathbf{A}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$:

$$A = \{(0,0),(0,1),\ldots,(0,3),(1,0),(1,1),\ldots,(3,2),(3,3)\}.$$

Let $B < A$,

$$B = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} = \{(0,0),(0,2),(2,0),(2,2)\}.$$

Then C can be presented as $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$,

$$C = \{(0',0'),(0',1'),(1',0'),(1',1')\}.$$

We can choose natural orbit representatives so that ξ simply removes the primes, $\xi(j',k') = (j,k)$.

Since our presentations are coherent, everything from the one-dimensional case carried over by using the tensor product in each step. We write $L^2(\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}) = L^2(\mathbf{Z}/4\mathbf{Z}) \otimes L^2(\mathbf{Z}/4\mathbf{Z})$ with δ-basis $\delta_{ij} = \delta_i \otimes \delta_j$ for $0 \leq i, j < 4$. And the regular representation of A is the tensor product of the regular representations of $\mathbf{Z}/4\mathbf{Z}$,

$$\rho(k,l) = \rho(k) \otimes \rho(l)$$

or

$$\rho(k,l)\,\delta_{ij} = \delta_{i+k,\,j+l}.$$

We obtain the four characters of B by restriction of the characters of A to B, $\chi_{00} = \chi_0 \cdot \chi_0$, $\chi_{02} = \chi_0 \cdot \chi_2$, $\chi_{20} = \chi_2 \cdot \chi_0$, and $\chi_{22} = \chi_2 \cdot \chi_2$.

Step 1. $P(\xi)$. The choice of the natural ξ partitions the δ-basis:

$$\begin{aligned}\{\delta_{00},\ldots,\delta_{33}\} = {} & \{\delta_{00},\delta_{02},\delta_{20},\delta_{22}\}\\ & \vee\{\delta_{01},\delta_{03},\delta_{21},\delta_{23}\}\\ & \vee\{\delta_{10},\delta_{12},\delta_{30},\delta_{32}\}\\ & \vee\{\delta_{11},\delta_{13},\delta_{31},\delta_{33}\}.\end{aligned}$$

With lexicographic order on the indices we have

$$P = \text{perm}\{00, 02, 20, 22, 01, 03, 21, 23, 10, 12, 30, 32, 11, 13, 31, 33\}.$$

Step 2. $F(B) = F_2 \otimes F_2$. Apply $F_2 \otimes F_2$ to each of these cosets. For example, for the first coset we have

$$\delta_{00}^{00} = \delta_{00} + \delta_{02} + \delta_{20} + \delta_{22}$$

$$\delta_{00}^{02} = \delta_{00} - \delta_{02} + \delta_{20} - \delta_{22}$$

$$\delta_{00}^{20} = \delta_{00} + \delta_{02} - \delta_{20} - \delta_{22}$$

$$\delta_{00}^{22} = \delta_{00} - \delta_{02} - \delta_{20} + \delta_{23}.$$

To obtain the general case, first compute the 1-D case

$$\delta_0^0 = \delta_0 + \delta_2$$

$$\delta_0^2 = \delta_0 - \delta_2$$

$$\delta_1^0 = \delta_1 + \delta_3$$

$$\delta_1^2 = \delta_1 - \delta_3.$$

It can be readily verified that the image of $F_2 \otimes F_2$ is given by

$$\delta_{kl}^{ij} = \delta_k^i \otimes \delta_l^j,$$

ordered by lexicographic order on the multiindex (k, l, i, j). In order to find the output permutation in Step 6 we will need to know the values of the δ_{kl}^{ij} on the generators of A. Since, from the 1-D case case,

$$\delta_k^i(0) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\delta_k^i(1) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$\begin{aligned} \delta_{kl}^{ij}(0,1) &= \delta_k^i(0) \otimes \delta_l^j(1) \\ &= \begin{cases} 1 & \text{if } k = 0 \text{ and } l = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\delta_{kl}^{ij}(1,0) = \delta_k^i(1) \otimes \delta_l^j(0)$$
$$= \begin{cases} 1 & \text{if } k = 1 \text{ and } l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Step 3. Form V_χ. Bases for the character spaces can be found by gathering the δ_{kl}^{ij} at stride 4.

$$V_{00} = \langle \delta_{00}^{00}, \delta_{01}^{00}, \delta_{10}^{00}, \delta_{11}^{0} \rangle$$
$$V_{02} = \langle \delta_{00}^{02}, \delta_{01}^{02}, \delta_{210}^{0}, \delta_{11}^{02} \rangle$$
$$V_{20} = \langle \delta_{00}^{20}, \delta_{01}^{20}, \delta_{10}^{20}, \delta_{11}^{20} \rangle$$
$$V_{22} = \langle \delta_{00}^{22}, \delta_{01}^{22}, \delta_{10}^{22}, \delta_{11}^{22} \rangle.$$

Or

$$V_{ij} = V_i \otimes V_j.$$

Step 4. $T'(\eta)$. Since

$$\rho | V_{ij} = \rho | V_i \otimes V_j$$

or

$$\rho_{ij} = \rho_i \otimes \rho_j$$

we can readily compute ρ_{ij} and the required twiddle factors from the 1-D case. Indeed,

$$\rho_{00}(0,1) = I_2 \otimes S_2$$
$$\rho_{00}(1,0) = S_2 \otimes I_2$$
$$\rho_{02}(0,1) = I_2 \otimes S_2(-1)$$
$$\rho_{02}(1,0) = S_2 \otimes I_2$$
$$\rho_{20}(0,1) = I_2 \otimes S_2$$
$$\rho_{20}(1,0) = S_2(-1) \otimes I_2$$
$$\rho_{22}(0,1) = I_2 \otimes S_2(-1)$$
$$\rho_{22}(1,0) = S_2(-1) \otimes I_2,$$

and therefore the required twiddle factors are

$$T' = (I_2 \otimes I_2) \oplus (I_2 \otimes W_2) \oplus (W_2 \otimes I_2) \oplus (W_2 \otimes W_2),$$

where

$$W_2 = \text{diag}\{1, i\}.$$

Step 5. $F(C) = F_2 \otimes F_2$. Apply $F(C)$.

$$\begin{aligned}
\nu_{00}^{00} &= \delta_{00}^{00} + \delta_{01}^{00} + \delta_{10}^{00} + \delta_{11}^{00}\\
\nu_{00}^{01} &= \delta_{00}^{00} - \delta_{01}^{00} + \delta_{10}^{00} - \delta_{11}^{00}\\
\nu_{00}^{10} &= \delta_{00}^{00} + \delta_{01}^{00} - \delta_{10}^{00} - \delta_{11}^{00}\\
\nu_{00}^{11} &= \delta_{00}^{00} - \delta_{01}^{00} - \delta_{10}^{00} + \delta_{11}^{00}\\
\nu_{02}^{00} &= \delta_{00}^{02} + i\delta_{01}^{02} + \delta_{10}^{02} + i\delta_{11}^{02}\\
\nu_{02}^{01} &= \delta_{00}^{02} - i\delta_{01}^{02} + \delta_{10}^{02} - i\delta_{11}^{02}\\
\nu_{02}^{10} &= \delta_{00}^{02} + i\delta_{01}^{02} - \delta_{10}^{02} - i\delta_{11}^{02}\\
\nu_{02}^{11} &= \delta_{00}^{02} - i\delta_{01}^{02} - \delta_{10}^{02} + i\delta_{11}^{02}\\
\nu_{20}^{00} &= \delta_{00}^{20} + \delta_{01}^{20} + i\delta_{10}^{20} + i\delta_{11}^{20}\\
\nu_{20}^{01} &= \delta_{00}^{20} - \delta_{01}^{20} + i\delta_{10}^{20} - i\delta_{11}^{20}\\
\nu_{20}^{10} &= \delta_{00}^{20} + \delta_{01}^{20} - i\delta_{10}^{20} - i\delta_{11}^{20}\\
\nu_{20}^{11} &= \delta_{00}^{20} - \delta_{01}^{20} - i\delta_{10}^{20} + i\delta_{11}^{20}\\
\nu_{22}^{00} &= \delta_{00}^{22} + i\delta_{01}^{22} + i\delta_{10}^{22} - \delta_{11}^{22}\\
\nu_{22}^{01} &= \delta_{00}^{22} - i\delta_{01}^{22} + i\delta_{10}^{22} + \delta_{11}^{22}\\
\nu_{22}^{10} &= \delta_{00}^{22} + i\delta_{01}^{22} - i\delta_{10}^{22} + \delta_{11}^{22}\\
\nu_{22}^{11} &= \delta_{00}^{22} - i\delta_{01}^{22} - i\delta_{10}^{22} - \delta_{11}^{22}.
\end{aligned}$$

Step 6. Q. Now the ν_{kl}^{ij}'s are characters of A in some order. To find the output permutation we need to determine which ones they are. We can do this by evaluating them on the generators $(0,1)$ and $(1,0)$ of A. Now $A = \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$ and $\hat{A} = \widehat{\mathbf{Z}/4\mathbf{Z}} \times \widehat{\mathbf{Z}/4\mathbf{Z}}$. So the characters of A in the chosen order are

$$\chi_{kl}(x,y) = \chi_k(x) \cdot \chi_l(y) = i^{kx+ly}$$

for $0 \le k, l, x, y < 4$.

Thus, referring to Steps 2 and 5, we have

$$\begin{aligned}
\nu_{00}^{00}(0,1) &= 1 \\
\nu_{00}^{00}(1,0) &= 1 \Rightarrow \nu_{00}^{00} = \chi_{00} \\
\nu_{00}^{01}(0,1) &= -1 \\
\nu_{00}^{01}(1,0) &= 1 \Rightarrow \nu_{00}^{01} = \chi_{02} \\
\nu_{00}^{10}(0,1) &= 1 \\
\nu_{00}^{10}(1,0) &= -1 \Rightarrow \nu_{00}^{10} = \chi_{20} \\
\nu_{00}^{11}(0,1) &= -1 \\
\nu_{00}^{11}(1,0) &= -1 \Rightarrow \nu_{00}^{11} = \chi_{22}
\end{aligned}$$

$$\begin{aligned}
\nu_{02}^{00}(0,1) &= i \\
\nu_{02}^{00}(1,0) &= 1 \Rightarrow \nu_{02}^{00} = \chi_{01} \\
\nu_{02}^{01}(0,1) &= -i \\
\nu_{02}^{01}(1,0) &= 1 \Rightarrow \nu_{02}^{01} = \chi_{03} \\
\nu_{02}^{10}(0,1) &= i \\
\nu_{02}^{10}(1,0) &= -1 \Rightarrow \nu_{02}^{10} = \chi_{21} \\
\nu_{02}^{11}(0,1) &= -i \\
\nu_{02}^{11}(1,0) &= -1 \Rightarrow \nu_{02}^{11} = \chi_{23}
\end{aligned}$$

$$\begin{aligned}
\nu_{20}^{00}(0,1) &= 1 \\
\nu_{20}^{00}(1,0) &= i \Rightarrow \nu_{20}^{00} = \chi_{10} \\
\nu_{20}^{01}(0,1) &= -1 \\
\nu_{20}^{01}(1,0) &= i \Rightarrow \nu_{20}^{01} = \chi_{12} \\
\nu_{20}^{10}(0,1) &= 1 \\
\nu_{20}^{10}(1,0) &= -i \Rightarrow \nu_{20}^{10} = \chi_{30} \\
\nu_{20}^{11}(0,1) &= -1 \\
\nu_{20}^{11}(1,0) &= -i \Rightarrow \nu_{20}^{11} = \chi_{32}
\end{aligned}$$

$$\begin{aligned}
\nu_{22}^{00}(0,1) &= i \\
\nu_{22}^{00}(1,0) &= i \Rightarrow \nu_{22}^{00} = \chi_{11} \\
\nu_{22}^{01}(0,1) &= -i \\
\nu_{22}^{01}(1,0) &= I \Rightarrow \nu_{23}^{01} = \chi_{13} \\
\nu_{22}^{10}(0,1) &= i \\
\nu_{22}^{10}(1,0) &= -i \Rightarrow \nu_{22}^{10} = \chi_{31} \\
\nu_{22}^{11}(0,1) &= -i \\
\nu_{22}^{11}(1,0) &= -i \Rightarrow \nu_{22}^{11} = \chi_{33}.
\end{aligned}$$

Hence, the output permutation is

$$Q = \text{perm}\{00, 02, 20, 22, 01, 03, 21, 23, 10, 12, 30, 32, 11, 13, 31, 33\}.$$

In fact, $Q = P$. So we have found the factorization

$$F_4 \otimes F_4 = P\big(I_4 \otimes (F_2 \otimes F_2)\big)T'L_4^{16}\big(I_4 \otimes (F_2 \otimes F_2)\big)P.$$

Or, writing $P = Q'L_4^{16}$ and $T = L_4^{16}T'L_4^{16}$,

$$F_4 \otimes F_4 = Q'\big((F_2 \otimes F_2) \otimes I_4\big)T\big(I_4 \otimes (F_2 \otimes F_2)\big)P.$$

6.2.2. *2-D Cooley–Tukey*. Let $A = \mathbf{Z}/m_1 n_1 \mathbf{Z} \times \mathbf{Z}/m_2 n_2 \mathbf{Z}$, and let $B = B_1 \times B_2 = \{(im_1, jm_2) | 0 \le i < n_1, 0 \le j < n_2\}$. Then $B \cong \mathbf{Z}/n_1\mathbf{Z} \times \mathbf{Z}/n_2\mathbf{Z}$, and $C = A/B = C_1 \times C_2 \cong \mathbf{Z}/m_1\mathbf{Z} \times \mathbf{Z}/m_2\mathbf{Z}$. We can choose $\{(i, j) | 0 \le i < m_1,\ 0 \le j < m_2\}$ as coset representatives of A/B.

LEMMA 6.7. *Let $\hat{b} \in \hat{B}$. Then $\hat{b} = \hat{b}_1 \otimes \hat{b}_2$ for $\hat{b}_1 \in \hat{B}_1$ and $\hat{b}_2 \in \hat{B}_2$. Let $\rho_{\hat{b}_1 \otimes \hat{b}_2}$ be ρ restricted to $V_{\hat{b}}$. Then, for $0 \le k < m_1$ and $0 \le l < m_2$,*

$$\rho_{\hat{b}_1 \otimes \hat{b}_2}(k, l) = S_{m_1}\big(\hat{b}_1(-m_1)\big)^k \otimes S_{m_2}\big(\hat{b}_2(-m_2)\big)^l,$$

with respect to the lexicographically ordered basis $\{\delta^b_{(i,j)} | 0 \le i < m_1, 0 \le j < m_2\}$.

Since B is a coherently presentable subgroup we can find $\chi \otimes \psi \in \hat{A}$ such that $\chi|_{B_1} = \hat{b}_1$ and $\psi|_{B_2} = \hat{b}_2$.

LEMMA 6.8. *Let $\chi \otimes \psi \in \hat{A}$ with $\chi|_{B_1} = \hat{b}_1$ and $\psi|_{B_2} = \hat{b}_2$, and let $D_1 = D(\chi(1))$ and $D_2(D(\psi(1))$. Then*

$$\begin{aligned}
&(D_1 \otimes D_2)^{-1}\Big(S_{m_1}\big(\hat{b}_1(-m_1)\big)^k \otimes S_{m_2}\big(\hat{b}_2(-m_2)\big)^l\Big)(D_1 \otimes D_2) \\
&\qquad = \chi_1(k)\chi_2(l)\big(S_{m_1}^k \otimes S_{m_2}^l\big).
\end{aligned}$$

LEMMA 6.9. *Let* $\chi_i \otimes \psi_j$, $0 \le i < n_1$, $0 \le j < n_2$ *be characters in* $\hat{A}$ *that restrict to the characters in* $\hat{B}$. *Then*

$$T = \bigoplus_{i,j} \left(D(\chi_i(1)) \otimes D(\psi_j(1)) \right).$$

Then

$$T\left(\bigoplus_{i,j} \left(S_{m_1}(\chi_i(-m_1))^k \otimes S_{m_2}(\psi_j(-m_2))^l \right) \right) T^{-1}$$

$$= \bigoplus_{i,j} \left(\chi_i(k)\psi_j(l)\left(S_{m_1}^k \otimes S_{m_2}^l \right) \right)$$

Therefore we obtain the following factorization of the multidimensional Fourier transform $F_{m_1 n_1} \otimes F_{m_2 n_2}$.

THEOREM 6.10.

$$F_{m_1 n_1} \otimes F_{m_2 n_2} = Q\left(I_{n_1 n_2} \otimes \left(F_{m_1} \otimes F_{m_2} \right) \right) T\left(F_{n_1} \otimes F_{n_2} \right) \otimes I_{m_1 m_2} \Big) P,$$

where Q *and* P *are permutation matrices*.

6.2.3. *A Non-coherent Case*. We now consider a case when the presentation of $B < A$ is not coherent. From the Introduction, let

$$A = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z} = \{(0,0),(0,1),\dots,(0,7),(1,0),\dots,(1,7)\}$$

and

$$B = \mathbf{Z}/4\mathbf{Z} = \{(0,0),(1,2),(0,4),(1,6)\}.$$

Then B is not coherently presented in A. In fact, there is no presentation in which B is coherently presented. Indeed, there is no basis $\langle a, a' \rangle$ of A, such that there is an r with $\langle ra \rangle$ a basis of B.

Now compute a presentation of $C = A/B$. To do this in general, we would have to compute something like the Smith normal form, but here it is easy to see that $(0,1)$ is of order 4 in A/B and hence $C = \mathbf{Z}/4\mathbf{Z}$.

$$C = \{0', 1', 2', 3'\}$$

and choose $\xi(j') = (0, j)$.

Step 1. $P(\xi)$. With these choices we have

$$\begin{aligned}\{\delta_{00}, \delta_{01}, \dots, \delta_{77}\} = \{&\delta_{00}, \delta_{12}, \delta_{04}, \delta_{16}\} \\ &\vee \{\delta_{01}, \delta_{13}, \delta_{05}, \delta_{17}\} \\ &\vee \{\delta_{02}, \delta_{14}, \delta_{06}, \delta_{10}\} \\ &\vee \{\delta_{03}, \delta_{15}, \delta_{07}, \delta_{11}\}.\end{aligned}$$

With lexicographic order on the indices we have

$$P = \text{perm}\{00, 12, 04, 16, 01, 13, 05, 17, 02, 14, 06, 10, 03, 15, 07, 11\}.$$

Step 2. $F(B) = F_4$. Computing $F(B)$ we have

$$\delta_{00}^{00} = \delta_{00} + \delta_{12} + \delta_{04} + \delta_{16}$$
$$\delta_{00}^{12} = \delta_{00} + i\delta_{12} - \delta_{04} - i\delta_{16}$$
$$\delta_{00}^{04} = \delta_{00} - \delta_{12} + \delta_{04} - \delta_{16}$$
$$\delta_{00}^{16} = \delta_{00} - i\delta_{12} - \delta_{04} + i\delta_{16}$$
$$\delta_{01}^{00} = \delta_{01} + \delta_{13} + \delta_{05} + \delta_{17}$$
$$\delta_{01}^{12} = \delta_{01} + i\delta_{13} - \delta_{05} - i\delta_{17}$$
$$\delta_{01}^{04} = \delta_{01} - \delta_{13} + \delta_{05} - \delta_{17}$$
$$\delta_{01}^{16} = \delta_{01} - i\delta_{13} - \delta_{05} + i\delta_{17}$$
$$\delta_{02}^{01} = \delta_{02} + \delta_{14} + \delta_{06} + \delta_{10}$$
$$\delta_{02}^{12} = \delta_{02} + i\delta_{14} - \delta_{06} - i\delta_{10}$$
$$\delta_{02}^{04} = \delta_{02} - \delta_{14} + \delta_{06} - \delta_{10}$$
$$\delta_{02}^{16} = \delta_{02} - i\delta_{14} - \delta_{06} + i\delta_{10}$$
$$\delta_{03}^{00} = \delta_{03} + \delta_{15} + \delta_{07} + \delta_{11}$$
$$\delta_{03}^{12} = \delta_{03} + i\delta_{15} - \delta_{07} - i\delta_{11}$$
$$\delta_{03}^{04} = \delta_{03} - \delta_{15} + \delta_{07} - \delta_{11}$$
$$\delta_{03}^{16} = \delta_{03} - i\delta_{15} - \delta_{07} + i\delta_{11}$$

For Step 6 we need to note the values of the δ_j^i's at the generators, $(0, 1)$, and $(1, 0)$, of A. All are zero except

$$\delta_{01}^{00}(0, 1) = 1$$
$$\delta_{01}^{12}(0, 1) = 1$$
$$\delta_{01}^{04}(0, 1) = 1$$
$$\delta_{01}^{16}(0, 1) = 1$$

and

$$\delta_{02}^{00}(1,0) = 1$$
$$\delta_{02}^{12}(1,0) = -i$$
$$\delta_{02}^{04}(1,0) = -1$$
$$\delta_{02}^{16}(1,0) = i.$$

Step 3. V_χ. Collect as indicated by the superscripts at stride 4.

$$V_{00} = \langle \delta_{00}^{00}, \delta_{01}^{00}, \delta_{02}^{00}, \delta_{03}^{00} \rangle$$
$$V_{12} = \langle \delta_{00}^{12}, \delta_{01}^{12}, \delta_{02}^{12}, \delta_{03}^{12} \rangle$$
$$V_{04} = \langle \delta_{00}^{04}, \delta_{01}^{04}, \delta_{02}^{04}, \delta_{03}^{04} \rangle$$
$$V_{16} = \langle \delta_{00}^{16}, \delta_{01}^{16}, \delta_{02}^{16}, \delta_{03}^{16} \rangle.$$

Step 4. $T'(\eta)$. We have

$$\rho_{00}(0,1) = S_4,$$
$$\rho_{12}(0,1) = S_4(-1),$$
$$\rho_{04}(0,1) = S_4,$$
$$\rho_{16}(0,1) = S_4(-1).$$

Since $\xi(1') = (0,1)$ there is no need to look at the other generator of A. But it is interesting to observe that $(1,0) \equiv 2(1,0) \bmod B$, or more explicitly, $(1,0) = 2(0,1) + (1,6)$ and we have

$$\rho_{00}(1,0) = \chi_{00}(-(1,6))(\rho_{00}(0,1))^2 = S_4^2 = S_2 \otimes I_2,$$
$$\rho_{12}(1,0) = \chi_{12}(-(1,6))(\rho_{12}(0,1))^2 = iS_4(-1)^2 = iS_2(-1) \otimes I_2,$$
$$\rho_{04}(1,0) = \chi_{04}(-(1,6))(\rho_{04}(0,1))^2 = -S_4^2 = -S_2 \otimes I_2,$$
$$\rho_{16}(1,0) = \chi_{16}(-(1,6))(\rho_{16}(0,1))^2 = -iS_4(-1)^2 = -iS_2(-1) \otimes I_2.$$

Now we compute the twiddle factors. Let $\omega = e^{2\pi i/8}$ and

$$W_4 = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \omega^2 & \\ & & & \omega^3 \end{pmatrix}.$$

Then,

$$T' = I_4 \oplus W_4 \oplus I_4 \oplus W_4.$$

Step 5. $F(C) = F_4$. We have, with $i = \omega^2$ and $-1 = \omega^4$,

$$\nu_0^0 = \delta_{00}^{00} + \delta_{01}^{00} + \delta_{02}^{00} + \delta_{03}^{00}$$
$$\nu_0^1 = \delta_{00}^{00} + \omega^2\delta_{01}^{00} + \omega^4\delta_{02}^{00} + \omega^6\delta_{03}^{00}$$
$$\nu_0^2 = \delta_{00}^{00} + \omega^4\delta_{01}^{00} + \delta_{02}^{00} + \omega^4\delta_{03}^{00}$$
$$\nu_0^3 = \delta_{00}^{00} + \omega^6\delta_{01}^{00} + \omega^4\delta_{02}^{00} + \omega^2\delta_{03}^{00}$$
$$\nu_1^0 = \delta_{00}^{12} + \omega\delta_{01}^{12} + \omega^2\delta_{02}^{12} + \omega^3\delta_{03}^{12}$$
$$\nu_1^1 = \delta_{00}^{12} + \omega^3\delta_{01}^{12} + \omega^6\delta_{02}^{12} + \omega\delta_{03}^{12}$$
$$\nu_1^2 = \delta_{00}^{12} + \omega^5\delta_{01}^{12} + \omega^2\delta_{02}^{12} + \omega^7\delta_{03}^{12}$$
$$\nu_1^3 = \delta_{00}^{12} + \omega^7\delta_{01}^{12} + \omega^6\delta_{02}^{12} + \omega^5\delta_{03}^{12}$$
$$\nu_2^0 = \delta_{00}^{04} + \delta_{01}^{04} + \delta_{02}^{04} + \delta_{03}^{04}$$
$$\nu_2^1 = \delta_{00}^{04} + \omega^2\delta_{01}^{04} + \omega^4\delta_{02}^{04} + \omega^6\delta_{03}^{04}$$
$$\nu_2^2 = \delta_{00}^{04} + \omega^4\delta_{01}^{04} + \delta_{02}^{04} + \omega^4\delta_{03}^{04}$$
$$\nu_2^3 = \delta_{00}^{04} + \omega^6\delta_{01}^{04} + \omega^4\delta_{02}^{04} + \omega^2\delta_{03}^{04}$$
$$\nu_3^0 = \delta_{00}^{16} + \omega\delta_{01}^{16} + \omega^2\delta_{02}^{16} + \omega^3\delta_{03}^{16}$$
$$\nu_3^1 = \delta_{00}^{16} + \omega^3\delta_{01}^{16} + \omega^6\delta_{01}^{16} + \omega\delta_{03}^{16}$$
$$\nu_3^2 = \delta_{00}^{16} + \omega^5\delta_{01}^{16} + \omega^2\delta_{02}^{16} + \omega^7\delta_{03}^{16}$$
$$\nu_3^3 = \delta_{00}^{16} + \omega^7\delta_{01}^{16} + \omega^6\delta_{02}^{16} + \omega^5\delta_{03}^{16}$$

Step 6. Q. To compute the output permutation, we must evaluate the ν_j^i's at the generators of A. Indeed, the characters of A are, in lexicographic order $\chi_{ij} \in \hat{A}$, for $0 \le i < 2$, $0 \le j < 8$ where

$$\chi_{ij}(k, l) = (-1)^{ik}\omega^{jl}.$$

Thus, referring to Steps 2 and 5,

$$\nu_0^0(0,1) = 1$$
$$\nu_0^0(1,0) = 1 \Rightarrow \nu_0^0 = \chi_{00}$$
$$\nu_0^1(0,1) = \omega^2$$
$$\nu_0^1(1,0) = -1 \Rightarrow \nu_0^1 = \chi_{12}$$
$$\nu_0^2(0,1) = \omega^4$$
$$\nu_0^2(1,0) = 1 \Rightarrow \nu_0^2 = \chi_{04}$$
$$\nu_0^3(0,1) = \omega^6$$
$$\nu_0^3(1,0) = -1 \Rightarrow \nu_0^3 = \chi_{16}$$

$$\nu_1^0(0,1) = \omega$$
$$\nu_1^0(1,0) = 1 \Rightarrow \nu_1^0 = \chi_{01}$$
$$\nu_1^1(0,1) = \omega^3$$
$$\nu_1^1(1,0) = -1 \Rightarrow \nu_1^1 = \chi_{13}$$
$$\nu_1^2(0,1) = \omega^5$$
$$\nu_1^2(1,0) = 1 \Rightarrow \nu_1^2 = \chi_{05}$$
$$\nu_1^3(0,1) = \omega^7$$
$$\nu_1^3(1,0) = -1 \Rightarrow \nu_1^3 = \chi_{17}$$

$$\nu_2^0(0,1) = 1$$
$$\nu_2^0(1,0) = -1 \Rightarrow \nu_2^0 = \chi_{10}$$
$$\nu_2^1(0,1) = \omega^2$$
$$\nu_2^1(1,0) = 1 \Rightarrow \nu_2^1 = \chi_{02}$$
$$\nu_2^2(0,1) = \omega^4$$
$$\nu_2^2(1,0) = -1 \Rightarrow \nu_2^2 = \chi_{14}$$
$$\nu_2^3(0,1) = \omega^6$$
$$\nu_2^3(1,0) = 1 \Rightarrow \nu_2^3 = \chi_{06}$$

$$\nu_3^0(0,1) = \omega$$
$$\nu_3^0(1,0) = -1 \Rightarrow \nu_3^0 = \chi_{11}$$
$$\nu_3^1(0,1) = \omega^3$$
$$\nu_3^1(1,0) = 1 \Rightarrow \nu_3^1 = \chi_{03}$$
$$\nu_3^2(0,1) = \omega^5$$
$$\nu_3^2(1,0) = -1 \Rightarrow \nu_3^2 = \chi_{15}$$
$$\nu_3^3(0,1) = \omega^7$$
$$\nu_3^3(1,0) = 1 \Rightarrow \nu_3^3 = \chi_{07}.$$

Hence, the output permutation is

$$Q = \text{perm}\{00, 12, 04, 16, 01, 13, 05, 17, 10, 02, 14, 06, 11, 03, 15, 07\}.$$

And we have found the interesting factorization

$$F_2 \otimes F_8 = Q(I_4 \otimes F_4)T' L_4^{16}(I_4 \otimes F_4)P.$$

Or, writing $Q = Q'L_4^{16}$ and $L_4^{16}T'L_4^{16}$,

$$F_2 \otimes F_8 = Q'(F_4 \otimes I_4)T(I_4 \otimes F_4)P.$$

7. PROGRAMMING CONSIDERATIONS

Our goal as implementers is to construct a program to efficiently evaluate the linear computation

$$y = Fx,$$

where F is generally a 1-D, 2-D, or 3-D Fourier transform. In these cases, if the data is linearly ordered, $F = F_n$, $F_{n_1} \otimes F_{n_2}$, or $F_{n_1} \otimes F_{n_2} \otimes F_{n_3}$. The approach we have suggested is to apply repeatedly the concrete procedure to factor F into small Fourier transforms that we may assume are efficiently implemented. Along the way we will pick up tensor products, diagonal multiplications (twiddle factors), and, most importantly, permutations. The code for each of the resulting factors can be combined to give a program for the computation. The resulting formula can also be algebraically manipulated to produce many different algorithms with different performance characteristics. For a detailed discussion of how this code can be generated see [9].

One of the main features of this paper is to show how, possibly at the cost of non-standard twiddle factors, the class of resulting permutations may be enlarged from the standard approach. The performance bottleneck for implementations of the FFT for large data sets on modern computer architectures is the data flow [3]. These new permutations may enable us to find better implementations of the Fourier transform. This is especially true in the multidimensional case. To see why this is true, consider the choices the programmer has in applying the factorization procedure. In 1-D, the presentation of $B < A$, and hence A/B is essentially unique. This is because of the general fact that a cyclic group has a unique subgroup of a given order. So in 1-D, the only free choice is the cross section ξ. In 2-D, the situation is considerably more complex and fruitful. Given A there are many non-isomorphic B's to choose of a given size. Furthermore, even if we choose isomorphic B's it can happen that the resulting quotients C's are *not* isomorphic. This, together with the choice of cross sections, gives the algorithm designer considerable flexibility in matching an algorithm to a specific machine to obtain a high performance implementation.

We report the results of an experiment we conducted which suggest that these ideas may have a practical value in implementing multidimensional

Fourier transforms. Consider the problem of implementing a square 2-D Fourier transform $F_N \otimes F_N$ with $N = 2^n$. A standard modification of the row–column method uses the factorization

$$y = L_N^{N^2}(I_N \otimes F_N) L_N^{N^2}(I_N \otimes F_N) x$$

where the load-stride at stride N, $L_N^{L^2}$, is transposition. In words, this says: apply F_N to the N rows of the data, transpose the columns into rows, repeat, and transpose back. Presuming that an efficient implementation of F_N is available, this reduces the problem to finding an efficient implementation of transposition.

On machines with a hierarchical or distributed memory the behavior of load-stride permutations for large data sizes varies with the stride. Our test machine was a Sun Sparc 10 model 41 running under Solaris 1.0 with the following memory hierarchy:

16 − KB	on-chip cache
1 − MB	on-board cache
64 − MB	main memory
400 − MB	swapfile

For complex data (8 bytes per point) we obtained wall-clock timings for a straightforward implementation of $y \leftarrow L_s^{2^{24}} x$. These are shown in Table I. Thus, or row–column evaluation of $F_{2^{12}} \otimes F_{2^{12}}$ takes more than 72,000 seconds!

One approach to handling the situation is to use a multipass algorithm to do the transposition [10]. In fact, a simple one would be to do $L_4^{2^{24}}$ six times based on the factorization $L_{st}^{rst} = L_s^{rst} L_t^{rst}$. However, the following observations suggest a faster implementation might be obtained by the methods we have developed in this paper.

TABLE I
Wall Clock Timings of Load–Stride

Stride s	Time (s)
1 (copy)	173
2	197
4	259
16	774
64	2819
4096 (transposition)	$>$ 72000

TABLE II
Wall Clock Timings of 2-D Cooley–Tukey

Factor	Time (secs)
Q	178
$L_{2^{22}}^{2^{24}}$	259
$I_4 \otimes (F_{2^{11}} \otimes F_{2^{11}})$	198
$L_4^{2^{24}} T$	259
$I_{2^{22}} \otimes (F_2 \otimes F_2)$	185
P	399

Applying one of the 2-D Cooley–Tukey factorizations found in Section 6.2.2 to this case we have

$$y = QL_{N/4}^{N^2}\left(I_4 \otimes \left(F_{N/2} \otimes F_{N/2}\right)\right)L_4^{N^2}T\left(I_{N/4} \otimes (F_2 \otimes F_2)\right)P,$$

where P and Q are permutations not too different from $L_4^{N^2}$. Now for $N = 2^{24}$ we have

$$y = QL_{2^{22}}^{2^{24}}\left(I_4 \otimes \left(F_{2^{11}} \otimes F_{2^{11}}\right)\right)L_4^{2^{24}}T\left(I_{2^{22}} \otimes (F_2 \otimes F_2)\right)P.$$

For this case we have obtained timings shown in Table II. Reference to the table shows that a 2-D Cooley–Tukey costs at worst about 1500 s which is much less than a straightforward row–column evaluation at more than 72,000 s.

This experiment was only a test of a quick and dirty code, but we believe it strongly suggests that the methods developed in this paper warrant further practical study. The wide variety of new data flows introduced by the factorization procedure we have developed in this paper may have an implementation advantage over those of traditional multidimensional FFT's on modern computer architectures.

REFERENCES

1. L. Auslander, J. R. Johnson, and R. W. Johnson, Computing finite Fourier transforms exploiting group symmetries. I. The Role of inner products, preprint, 1993.
2. L. Auslander, J. R. Johnson, and R. W. Johnson, Computing finite Fourier transforms exploiting group symmetries. II. Orbital Evaluation, preprint, 1993.
3. D. H. Bailey, FFT's in external or hierarchical memory, *J. Supercomp.* 4 (1990), 23–35.
4. T. Beth, ''Vehrfahren der Schneller Fourier Transform,'' Teubner, Stuttgart, 1984.
5. M. Clausen, Fast Fourier transforms for metabelian groups, *SIAM J. Comput.* 18 No. 3 (1989), 584–593.
6. M. Clausen and U. Baum, ''Fast Fourier Transforms,'' Wissshaftsverlag, Manheim, 1993.

7. P. Diaconis and D. Rockmore. Efficient computation of the Fourier transform on finite groups. *J. Amer. Math. Soc.*, 3(2):297–332, April 1990.
8. J. R. Johnson, R. W. Johnson, D. Rodriguez, and R. Tolimieri. A methodology for designing, modifying, and implementing Fourier Transform algorithms on various architectures. *Circuits Systems Signal Process.*, 9(4):449–500, 1990.
9. R. W. Johnson, C.-H. Huang, and J. R. Johnson. Multilinear algebra and parallel programming. *J. Supercomput.* 5:189–217, 1991.
10. S. D. Kaushik, C.-H. Huang, J. R. Johnson, R. W. Johnson, and P. Sadayappan, Efficient transposition algorithms for large matrices, *In "Proceedings of Supercomputing '93,* Portland, OR, Nov. 21–23, 1993," p. 656–665.
11. D. Masler and D. Rockmore. Adapted diameters and the efficient computation of Fourier transforms on finite groups, preprint, 1994.
12. D. Rockmore. Efficient computation of Fourier inversion for finite groups. *J. Assoc. Comput. Mach.* 41(1):31–66, January 1994.
13. D. Rockmore, Fast Fourier analysis for abelian group extensions. *Adv. Appl. Math.*, 11:164–204, 1990.

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