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Numerical Solution for Kawahara Equation by Using Spectral Methods

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Abstract

Some nonlinear wave equations are more difficult to investigate mathematically, as no general analytical method for their solutions exists.The Exponential Time Differencing (ETD) technique requires minimum stages to obtain the requiredaccurateness, which suggests an efficient technique relatingto computational duration thatensures remarkable stability characteristicsupon resolving nonlinear wave equations. This article solves the diagonal example of Kawahara equation via the ETD Runge-Kutta 4 technique. Implementation of this technique is proposed by short Matlab programs.

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Keywords: Exponential techniques; Integration factor techniques; ETD technique; R-K technique, Kawahara equation.

1. Introduction

A number of time-dependent partial differential equations (PDEs) are found to merge nonlinear and linear expressions of low and higher orders respectively. The spatial and temporal high order approximations can be applied suitably to find accurate numerical solutions of such problem.A lucid development of the Exact

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Linear Part (ELP)techniques of any order was given by Cox and Matthews [1]. This refers much to the Exponential Time Differencing (ETD) methods [2-3]. Since then Tokman [4] expressed these formulas which direct to thegroup relating to exponential propagation methods called Exponential Propagation Iterative (EPI) techniques. In order to make better the ETD schemes, Wright [5] deliberated on these schemes and thus reforming the solution in integral form of a nonlinear autonomous system of ODEs to an extension in terms of matrix and vectorfunctions products.

The basic procedure of ETD schemes is to integrate linear terms of the differential equation (DE)exactly, whilesestimating the nonlinear parts via a polynomial to be accurately integrated. Exceptionally a comparable technique is implemented by Lawson [6] and is now applied to the Integrating Factor (IF) techniques. Following in manner of IF techniques [7-9] the two ODEparts are multiplied via a suitable IF, upon which we acquired a DE with, modified variables as such the linear term isexactly resolved.

The ETD schemes are used widespread to unravel stiff systems.Furthermore in [10-11], they contrasted numerous fourth-order techniques which include ETD techniques and relatedconsequences.They found preeminent option with regards to ETD Runge-Kutta 4 (ETDRK4) technique inresolving a range of one-dimensional diffusion-type problems. A wide-ranging utilization of the ETD methods was carried out in accordance with connected work in simulations of stiff problems [12]. In Aziz *et al.* [13-14]the ETDRK4 method was used to solve the diagonal case of Korteweg-de Vries (KdV) equation with Fourier transformation and to implement by the integration factor method.Other papers on this subject include [15-22].

The present article is arranged as ensued: In part 1, we introduce theissue. In part 2, we demonstrate the background of the study which is related to a diagonal example. In part 3, we accomplish animplementation correlated todiagonal case of Kawahara equation, alongside Fast Fourier Transform (FFT). For part 4, a brief conclusion is given.

2. Background of the study

2.1. A diagonal example:Burgers' equation

In this section, we intend to show a diagonal example, which is solved via spectral method [17]. The Burgers' equation is given as

$$u_t - ju_{xx} + uu_x = 0 \qquad x \in [0,1]\,, \quad t \in [0,1] \tag{1}$$

with the initial and Dirichlet boundary conditions imposed by means of

$$u(x,0) = (\sin(2\pi x))^2 (1-x)^{\frac{3}{2}} \tag{2}$$

where $= 500$, $j = 0.0003$ (in lieu of viscous Burgers' equation) and $j = 0$ (in place ofinviscid Burgers' equation), $r = 0.03$.

To solve the equations (1) and (2), wecompose

$$u_t - ju_{xx} + (\tfrac{1}{2}u^2)_x = 0. \tag{3}$$

The use of Fast Fourier Transform (FFT) in (3) gives

$$\hat{u}_t + jk^2\hat{u} + \tfrac{1}{2}ik\widehat{u^2} = 0 \tag{4}$$

where $i = \sqrt{-1}$. Multiplying (4)by e^{jk^2t}, then

$$e^{jk^2t}\hat{u}_t + e^{jk^2t}\varepsilon k^2\hat{u} + \tfrac{1}{2}ik\, e^{jk^2t}\widehat{u^2} = 0\ . \tag{5}$$

Choosing the following substitution

$$\hat{U} = e^{jk^2t}\hat{u} \tag{6}$$

with

$$\hat{U}_t = jk^2 e^{jk^2t}\hat{u} + e^{jk^2t}\hat{u}_t\,, \tag{7}$$

andreplacing (7) in (5), we have

$$\widehat{U}_t + \frac{1}{2} ik\, e^{jk^2t}\widehat{u^2} = 0\ . \tag{8}$$

Performingin Fourier space (using FFT), the numerical discretizing algorithm is achieved via

$$\widehat{U}_t + \frac{i}{2} e^{jk^2t} kF\big((F^{-1}\big(e^{-jk^2t}\widehat{U}\big))^2\big) = 0\ . \tag{9}$$

where$F$ is the Fourier transformed operator . The Matlab program is proposed in [17].

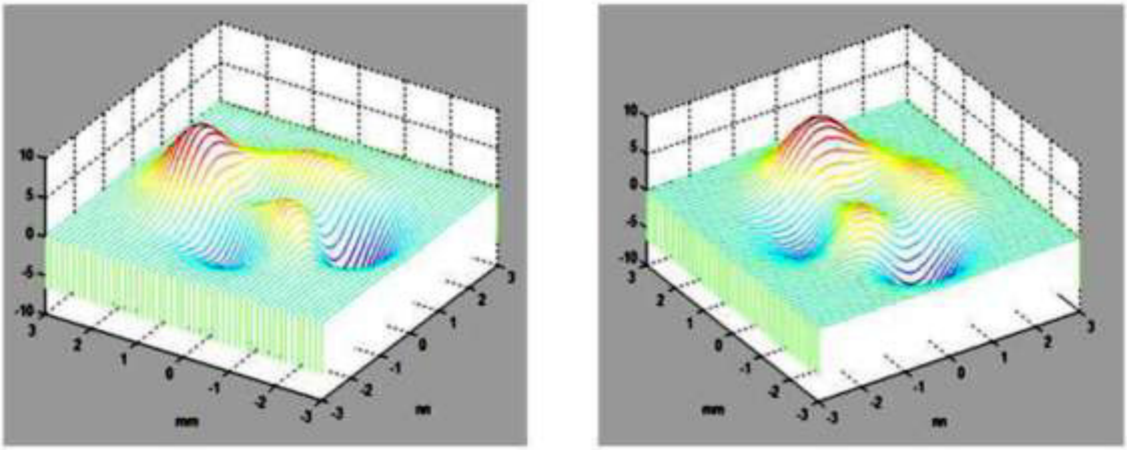

Fig.1. Time development ofinviscid Burgersequation ($j = 0$; Left) and viscous Burgers' equation ($j \neq 0$; Right). Axes are from x = -3 to x = 3, and from t = 0 to t = 150.

3. A diagonal example: Kawahara equation

Let us consider a diagonal example on the Kawahara equation,

$$u_t = -uu_x - u_{xxx} + u_{xxxxx} \qquad x \in [0,32\pi] \tag{10}$$

with a nonlinear hyperbolic term uu_x and two linear dispersive term u_{xxx} and u_{xxxxx} . Furthermore, subscript represents partial differentiation and the initial condition is given as

$$u(x,0) = \frac{105}{109} sech^4\Big(\frac{x}{2\sqrt{13}}\Big)\ . \tag{11}$$

The equation (10) can be written as

$$u_t = -\big(\tfrac{1}{2}u^2\big)_x - u_{xxx} + u_{xxxxx}\ . \tag{12}$$

The equation is then being discretized and the Fourier spectral technique is applied to the spatial part. The Fourier transform is given by

$$\widehat{u_t} = -\frac{ik}{2}\widehat{u^2} + i\,(k^3 + k^5)\,\widehat{u}, \tag{13}$$

where$i = \sqrt{-1}$ and k is the wave number. In the standard form,we have

$$u_t = Lu + N(u,t), \tag{14}$$

where

$$(L\hat{u})(k) = i\,(k^3 + k^5)\hat{u}(k), \tag{15}$$

$$N(\hat{u},t) = N(\hat{u}) = -\frac{ik}{2}\,\big(F\big(\big(F^{-1}(\hat{u})\big)^2\big)\big), \tag{16}$$

and $F$is the discrete Fourier transform.

Working in Fourier space, we consider the ETDRK4 time stepping for solving to $t = 150$,with the ETDRK4 which is given as follows

$$a_n = u_n e^{hL/2} + \left(e^{hL/2} - \mathrm{I}\right) N(u_n, t_n)/L, \tag{17}$$

$$b_n = u_n e^{hL/2} + \left(e^{hL/2} - \mathrm{I}\right) N(a_n, t_n + h/2)/L \tag{18}$$

$$c_n = a_n e^{hL/2} + \left(e^{hL/2} - \mathrm{I}\right) \left(2N(b_n, t_n + h/2) - N(u_n, t_n)\right)/L \tag{19}$$

$$u_{n+1} = a_n e^{hL} + \left\{ \emptyset_1 N(u_n, t_n) + 2\emptyset_2 \left(N(a_n, t_n + h/2) + N(b_n, t_n + h/2)\right) + \emptyset_3 N(c_n, t_n + h) \right\} / (L^3 h^2), \tag{20}$$

where

$$\emptyset_1 = (L^2 h^2 - 3Lh + 4) e^{hL} - Lh - 4\,, \tag{21}$$

$$\emptyset_2 = (Lh - 2) e^{hL} + Lh + 2\,, \tag{22}$$

$$\emptyset_3 = (-Lh + 4) e^{hL} - L^2 h^2 - 3Lh - 4 \tag{23}$$

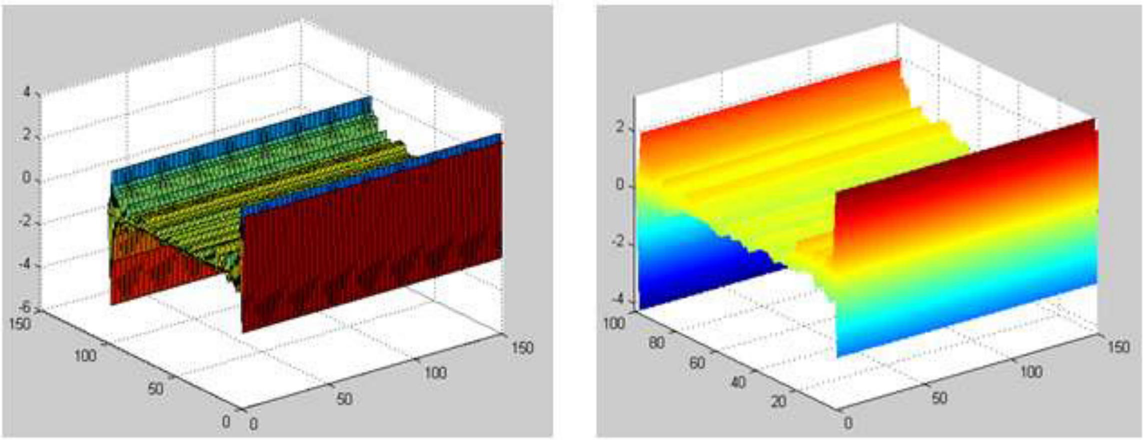

Fig.2. Time development for Kawaharaequation. Axes are from x=0 to $x = 32\pi$ and$t$=0 to t=150.

4. Conclusion

We have presented the solution of Kawahara equation with the initial condition$(x, 0) = \frac{105}{109} sech^4(\frac{x}{2\sqrt{13}})$, $x \in [0,32\pi]$, and applying N = 128 grid points in Fourier spatial discretization. For integrating the system (13), we have used the ETDRK4 method. Figure 2 shows that waves propagating and travelling periodically in time and persisting without change of shape. In spite of the remarkable sensitivity of the equation to perturbations in initial data, we obtained computational time of less than 1 second. The results are created by Matlab code (Appendix A).

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Appendix A.

Matlab codes to solve Kawahara equation and yielding Figure 2.

```
clear
clc
% Spatial grid and initial condition:
N = 128;
x = 32*pi*(1:N)/N;
u=0.9*sech(x*0.13).^4;
v = fft(u);
% Precompute various ETDRK4 scalar quantities:
h = 1/4; % time step
k = [0:N/2-1 0 -N/2+1:-1]; % wave numbers
L =1i*( k.^3 + k.^5); % Fourier multipliers
E = exp(h*L); E2 = exp(h*L/2);
M = 16; % no. of points for complex means
r = exp(1i*pi*((1:M)-.5)/M);
LR1=h*L(:,ones(M,1));
LR2=r(:,ones(M,1));
LR=LR1+LR2;
Q = h*real(mean( (exp(LR/2)-1)./LR ,2));
f1 = h*real(mean((-4-LR+exp(LR).*(4-3*LR+LR.^2))./LR.^3 ,2));
f2 = h*real(mean((2+LR+exp(LR).*(-2+LR))./LR.^3 ,2));
f3 = h*real(mean((-4-3*LR-LR.^2+exp(LR).*(4-LR))./LR.^3 ,2));
uu = u; tt = 0;
tmax = 150; nmax = round(tmax/h);
nplt = floor((tmax/100)/h);
g = -0.5i*k;
 Nv = g.*fft(real(ifft(v)).^2);
 a = E2.*v + Q.*Nv;
 Na = g.*fft(real(ifft(a)).^2);
 b = E2.*v + Q.*Na;
 Nb = g.*fft(real(ifft(b)).^2);
 c = E2.*a + Q.*(2*Nb-Nv);
 Nc = g.*fft(real(ifft(c)).^2);
 v = E.*v + Nv.*f1 + 2*(Na+Nb).*f2 + Nc.*f3;
 for n = 1:nmax
 t = n*h;
 if mod(n,nplt)==0
 u = real(ifft(v));
 uu = [uu,u]; tt = [tt,t];
 end
 end
 nn=length(tt);
 mm=length(x);
 uu2=reshape(uu,mm,nn);
 surf(tt,x,uu2),
 figure
```

```
surf(tt,x,uu2),
shading interp, lighting phong, axis tight
light(color,[1 1 0],position,[-1,2,2])
material([0.30 0.60 0.60 40.00 1.00]);
```