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Note

On a generalization of the Evans Conjecture

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Abstract

The Evans Conjecture states that a partial Latin square of order n with at most $n-1$ entries can be completed. In this paper we generalize the Evans Conjecture by showing that a partial r-multi Latin square of order n with at most $n-1$ entries can be completed. Using this generalization, we confirm a case of a conjecture of Häggkvist.

Keywords: Complete partial r-multi latin square

1. Introduction

The purpose of this paper is to generalize the following theorem known as the Evans Conjecture [5].

Theorem 1. *If A is a partial Latin square of order n with at most $n-1$ entries, then A can be completed to a Latin square of order n.*

Theorem 1 was independently confirmed by Häggkvist [6] for $n \geqslant 1111$, by Smetaniuk [8] for all n, and by Anderson and Hilton [1] for all n. Since then there have been several attempts to generalize Theorem 1 [1,3].

We begin such an attempt by defining an object that generalizes the Latin square. An r-multi Latin square of order n is an $n \times n$ array of nr symbols so that each entry is an r-set of symbols while each row and column contains the nr symbols. Of course a 1-multi Latin square of order n is a Latin square of order n. The following is a 3-multi Latin square of order 4.

1 2 3	4 5 6	7 8 9	10 11 12
4 6 8	1 9 10	11 12 5	2 3 7
9 10 12	3 7 11	2 4 6	1 5 8
5 7 11	2 8 12	1 3 10	4 6 9

A partial r-multi Latin square of order n is one with at most n^2 entries. As indicated in the definition, we will always assume that a non-empty cell in a partial r-multi Latin square contains r symbols. In addition to this, we will always assume that the nr symbols are $[nr]=\{1, 2, \ldots, nr\}$, unless otherwise stated and that the rows and columns are indexed by $[n]$. The following conjectures attempt to generalize Theorem 1.

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Conjecture 1. If A is a partial r-multi Latin square of order n with at most $n-1$ entries, then A can be completed to an r-multi Latin square of order n.

Conjecture 2. If A is a partial Latin square of order nr with at most $(n-1)$ *disjoint* $r \times r$ squares filled, then A can be completed to a Latin square of order nr.

The first conjecture can be found in [4] where it is shown that if A is a partial r-multi Latin square of order n with at most $n-1$ entries, then A can be completed given that $U \leqslant (n+1)r/2 - pr$ where U is the number of symbols used in A and p is the maximum number of times a symbol appears in A. The second conjecture is due to Häggkvist. In [3], Häggkvist and Denley show that Conjecture 2 holds for $n=3$ and in [2], Denley shows that Conjecture 2 holds given that the filled $r \times r$ squares are Latin squares themselves and that they lie in distinct rows and columns.

In this paper we confirm Conjecture 1 by generalizing the techniques used by Smetaniuk in confirming the Evans Conjecture. We then obtain the following result using this confirmation. Let A be a partial Latin square of order nr with at most $(n-1)$ disjoint $r \times r$ squares in distinct columns. Then there is a permutation of the symbols appearing in a row of a fixed $r \times r$ square so that A can be completed.

2. Completing r-multi partial Latin squares

An operation that has proved to be extremely useful for completing partial Latin squares is interchanging the columns and symbols, or rows and symbols. If one thinks of a Latin square as a set of n^2 triples each representing row, column, and symbol; then interchanging the columns and symbols is nothing more than swapping the second and third elements in each triple. The following two Latin squares demonstrate this.

1	2	3	4
3	1	4	2
2	4	1	3
4	3	2	1

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

We will consider this operation in reference to the following object. Define a (d,k,n)-rectangle to be an $nd \times nk$ array of n symbols so that each symbol appears k times in each row and each symbol appears d times in each column. Immediately one sees that a $(1,1,n)$-rectangle is a Latin square of order n. Let $P(n,r)$ be the set of all r-multi Latin squares of order n and let $Q(d,k,n)$ be the set of all (d,k,n)-rectangles. Furthermore, define ϕ_1 and ϕ_2 so that $\phi_1(A)$ and $\phi_2(A)$ are the result of interchanging columns and symbols and rows and symbols in A, respectively, where A is some rectangular array of symbols.

Lemma 1. *If $A \in Q(1,r,n)$ and $B \in Q(r,1,n)$, then $\phi_1(A), \phi_2(B) \in P(n,r)$.*

Proof. Consider column i of $A \in Q(1,r,n)$. Then the symbol i will appear once in each row and column of $\phi_1(A)$. Consider symbol k in row j of A. Since k appears r times in this row, there will be r symbols appearing in entry (j,k) of $\phi_1(A)$. Hence $\phi_1(A)$ is an r-multi Latin square of order n; that is, $\phi_1(A) \in P(n,r)$. One can easily give a similar argument for $\phi_2(B) \in P(n,r)$. □

Naturally $\phi_1(\phi_1(A))=A$ and $\phi_2(\phi_2(A))=A$. Also, one can easily show that if $C \in P(n,r)$, then $\phi_1(C) \in Q(1,r,n)$ and $\phi_2(C) \in Q(r,1,n)$. Consider the following $(1,3,4)$-rectangle which we will denote as A.

1	1	1	2	2	2	3	3	3	4	4	4
2	4	4	1	3	1	4	1	2	2	3	3
4	3	2	3	4	3	2	4	1	1	2	1
3	2	3	4	1	4	1	2	4	3	1	2

Then the following 3-multi Latin square of order 4 is $\phi_1(A)$.

1 2 3	4 5 6	7 8 9	10 11 12
4 6 8	1 9 10	5 11 12	2 3 7
9 10 12	3 7 11	2 4 6	1 5 8
5 7 11	2 8 12	1 3 10	4 6 9

Also note that if we take the transpose of A above, then we obtain a $(3,1,4)$-rectangle and $\phi_2(A^{\mathrm{T}}) = \phi_1(A)^{\mathrm{T}}$.

Our goal for this section is to confirm Conjecture 1. To do this we prove a generalization of a result of Smetaniuk and, additionally, we will use the following generalization of Ryser's Theorem [7], found in [4], to show a trivial case for completing partial $(1, r, n)$ and $(r, 1, n)$-rectangles.

Theorem 2. *Let A be an $t \times s$ r-multi Latin rectangle. Then A can be completed to an r-multi Latin square L of order n if and only if each symbol in A appears at least $t + s - n$ times.*

Corollary 1. *If A is a partial $(1, r, n)$-rectangle so that the first t rows are filled, then A can be completed.*

Proof. By Lemma 1, $\phi_1(A) \in P(n, r)$ has the first t rows filled. Then by Theorem 2, $\phi_1(A)$ can completed as every symbol appears t times. Therefore A can be completed. $\square$

Corollary 2. *If A is a partial $(r, 1, n)$-rectangle so that the first t columns are filled, then A can be completed.*

We will use the following definitions for the primary impetus of this paper. We define the back diagonal of an $n \times nr$ array to be the set of cells $\{(1, (n-1)r+1), (1, (n-1)r+2), \ldots, (1, nr)\} \cup \{(2, (n-2)r+1), \ldots, (2, (n-1)r)\} \cup \cdots \cup \{(n, 1), \ldots, (n, r)\}$. Analogously, one can define the back diagonal of an $nr \times n$ array. Consider $A \in Q(1, r, n)$. We define a new partial $(1, r, n+1)$-rectangle A_T by retaining the entries from A which lie above the back diagonal of A_T, placing symbol $n + 1$ in the entries which lie down the back diagonal of A_T, and leaving the remaining cells empty.

Lemma 2. *If $A \in Q(1, r, n)$, then A_T can be completed.*

Proof. Let G be a bipartite graph with bipartition (V_1, V_2) where $V_1 = \{\rho_1, \ldots, \rho_n\}$ and $V_2 = \{s_1, \ldots, s_n, s_{n+1}\}$. Let $(\rho_i, s_j) \in E(G)$ whenever symbol j appears in one of the last $(i-1)r$ cells of row i in A. Note that G will contain multiple edges. Additionally, let s_{n+1} be adjacent to each vertex in V_1 with multiplicity r and observe that $d(\rho_i) = ir$.

Let $D(s_j)$ be the set of columns in A_T that do not contain symbol j. An $(n+1)r$-coloring of the edges of G is called good if the coloring is proper and if the edges incident to s_j receive their colors from $D(s_j)$. Our goal is to construct a good $(n+1)r$-coloring of the edges of G so that the r multi edges (ρ_i, s_{n+1}) receive colors $(n+1-i)r+1, \ldots, (n+2-i)r$ for $1 \leqslant i \leqslant n$. By placing color k on (ρ_i, s_j) whenever symbol j appears in cell (i, k) of A, we produce a good coloring of G except that the edges incident to s_{n+1} receive no color. So, to produce a good $(n+1)r$-coloring, we shall rearrange the colors appearing on edges incident to ρ_i by starting with $i = n$.

Color the r multi edges (ρ_n, s_{n+1}) with colors $r+1, \ldots, 2r$. However, there are already r edges colored $r+1, \ldots, 2r$. We will remove the colors from these r edges and replace with the colors $nr+1, \ldots, (n+1)r$. This new coloring is good and the edges incident to ρ_n are colored with $r+1, \ldots, (n+1)r$. Suppose that the r multi edges (ρ_i, s_{n+1}) are colored $(n+1-i)r+1, \ldots, (n+2-i)r$ and that the edges incident to ρ_i are colored with $(n+1-i)r+1, \ldots, (n+1)r$ for $n \geqslant i > l$ while the colors on the edges incident to ρ_i for $i \leqslant l$ have remained unchanged. Consider $i = l$. Color the r multi edges (ρ_l, s_{n+1}) with $(n+1-l)r+1, \ldots, (n+2-l)r$. Now there are already r edges, $(\rho_l, s_{j_1}), \ldots, (\rho_l, s_{j_r})$, colored $(n+1-l)r+1, \ldots, (n+2-l)r$. Recolor these edges with $nr+1, \ldots, (n+1)r$ so that (ρ_l, s_{j_i}) which was colored $(n+1-1)r+i$ now receives color $nr+i$. Then for $1 \leqslant i \leqslant r$, there may already be edges (ρ_{i_1}, s_{j_i}) colored $nr+i$. Recolor these edges with $(n+1-l)r+i$ and observe that by the order of the recoloring, since ρ_{i_i} was incident with an edge colored $nr+i$, it must also have been incident with an edge colored $(n+1-l)r+i$ which we can recolor $nr+i$. By repeating this process, we produce r edge disjoint paths beginning at s_{n+1} and ending at some other vertex of V_2, along which we have alternately recolored the edges with colors $nr+i$ and $(n+1-l)r+i$ for $1 \leqslant i \leqslant r$. We now have produced a good coloring of G in which the r multi edges (ρ_i, s_{n+1}) receive colors $(n+1-i)r+1, \ldots, (n+2-i)r$ and the edges incident to ρ_i receive colors $(n+1-i)r+1, \ldots, (n+1)r$ for $n \geqslant i \geqslant l$. For $i \leqslant l-1$, the colors on the edges incident to ρ_i have remained unchanged; however, we may repeat this procedure to give G a good $(n+1)r$-coloring, ϕ, in which there are no uncolored edges and the r multi edges (ρ_i, s_{n+1}) receive colors $(n+1-i)r+1, \ldots, (n+2-i)r$.Now place symbol j in the empty (i, k) cell of $A_T(n+1, (n+1)r)$ whenever $\phi(\rho_i, s_j) = k$. Thus we have filled the first n rows of A_T and so by Corollary 1, A_T can be completed. $\square$

Using analogous definitions and the same argument above along with Corollary 2, one can similarly show the following lemma.

Lemma 3. *If $A \in Q(r, 1, n)$, then A_T can be completed.*

In addition to this, we will use the following lemma which is again a generalization of a lemma of Smetaniuk [8].

Lemma 4. *Consider a partial $(1, r, n)$-rectangle A with $(n-1)r$ entries so that if symbol j appears in row i of A, it appears r times. Let k be a specified symbol in some row of A. Then the rows and columns of A can be permuted so that the r appearances of this specified k lie on the back diagonal of A, and all other entries lie above it.*

Proof. Clearly the result holds for $n = 2$. Suppose then that the result holds for $n \leqslant l$ and consider $n = l + 1$. Either there is only one row containing the entries, or there is a row which does not contain this specified k but does contain another entry. In the first case we may permute the columns and rows so that k, appearing r times, lies in the last r cells of row 1. For the second case, let the non-empty row, that does not contain the specified k, be row i. Since there are only lr entries, there are r columns which do not contain an entry. Permute the rows and columns so that row 1 and row i are swapped and the last r columns are empty columns. If we remove row 1 and the last r columns, then we have a partial $(1, r, l)$-rectangle A' with at most $(l-1)r$ entries. Thus, by induction, we can permute the rows and columns of A' so that k appears on the back diagonal of A'. Adding back the removed row and the removed r columns completes the proof. □

Again, if A is a partial $(r, 1, n)$-rectangle with analogous conditions of Lemma 4, then using the same argument we can permute the rows and columns to obtain a similar conclusion.

Lemma 5. *Consider a partial $(r, 1, n)$-rectangle A with $(n-1)r$ entries so that if symbol j appears in column i of A, it appears r times. Let k be a specified symbol. Then the rows and columns of A can be permuted so that the r appearances of this specified k lie on the back diagonal of A, and all other entries lie above it.*

Note that in the lemma above, we use the definition of back diagonal for an $nr \times n$ array.

We are now in a position to prove the Evans Conjecture for r-multi Latin squares.

Theorem 3. *Suppose A is a partial r-multi Latin square with at most $(n-1)$ entries, then A can be completed.*

Proof. Without loss of generality we may assume that A has precisely $n-1$ entries. To show that A can be completed, we will induct on n. For $n = 1$, this is trivial. Suppose that the theorem holds for $n \leqslant l - 1$ and consider $n = l$.

We begin by first assuming that there is a column of A, say column k, that contains exactly one entry. Then $\phi_1(A)$ is a partial $(1, r, n)$-rectangle with $(n-1)r$ entries so that if symbol j appears in row i, then it appears r times. Furthermore symbol k appears exactly r times in some row of $\phi_1(A)$ and appears nowhere else. We will show that A can be completed by showing that $\phi_1(A)$ can be completed.

By Lemma 4, we may permute the rows and columns of $\phi_1(A)$ so that the r appearances of k lie on the back diagonal and all other entries lie above it. We will denote this new partial $(1, r, n)$-rectangle by B. Form B' from B by removing an empty row, r empty columns, and k. Then, by induction, B' can be completed on the symbol set $[n]\backslash\{k\}$. We then form B'_T by placing k on the back diagonal and so Lemma 2 guarantees that B'_T can be completed. Undoing the permutations of the rows and columns of the completed B'_T previously mentioned gives a completion of $\phi_1(A)$ and so gives a completion of A.

We now assume that each column of A contains at least pr entries for $p = 0$ or $p \geqslant 2$. If there is a row of A that contains exactly one entry, then we repeat the above argument to show that $\phi_2(A)$ can be completed using Lemmas 5 and 3. Therefore we must suppose that each row and column of A contains at least p entries for $p = 0$ or $p \geqslant 2$. To conclude, the columns and rows of A can be permuted so that the entries of A lie in a $t \times t$ square with $t \leqslant \lfloor \frac{(n-1)}{2} \rfloor$. Therefore by Theorem 2, A can be completed. □

3. Completing partial Latin squares of order nr

Given Theorem 3 and the following definition, we immediately have the following case for Conjecture 2. Let A be a partial Latin square of order nr with disjoint $r \times r$ squares filled. Then we say that A is uniform (or uniformly filled) if when two $r \times r$ squares share a row, then they share r rows, or if they share a column, then they share r columns.

Theorem 4. *If A is a uniform $nr \times nr$ partial Latin square with at most $(n-1)$ disjoint $r \times r$ squares filled as Latin squares, then A can be completed.*

Proof. We begin by assuming that each fixed $r \times r$ square lies in rows $[jr+1, (j+1)r]$ for a distinct j so that $0 \leqslant j \leqslant n-1$, and each fixed $r \times r$ square lies in columns $[kr+1, (k+1)r]$ for a distinct k so that $0 \leqslant k \leqslant n-1$. Form A into a partial r-multi Latin square R of order n by considering only the r symbols that appear in each fixed $r \times r$ square. Then, by Theorem 3, we can complete R. Finally, for each cell of R, we can clearly form a Latin square of order r. □

In addition to this, we will use the following idea to confirm a case of Conjecture 2 when each of the $r \times r$ squares lie in distinct columns (or rows). Again, we assume that each fixed $r \times r$ square lies in columns $[jr+1, (j+1)r]$ for a distinct j for $0 \leqslant j \leqslant n-1$. We begin by dividing A into r partial r-multi Latin squares of order n with at most $n-1$ entries. We do this by joining together rows $i, r+i, 2r+i, \ldots, (n-1)r+i$ of A and by allowing the r cells $(m, (k-1)r+1), \ldots, (m, kr-1), (m, kr)$ to become one cell (m, k) for the ith partial r-multi Latin square of order n where $1 \leqslant i \leqslant r$, $1 \leqslant k \leqslant n$, and $m \in \{i, r+i, 2r+i, \ldots, (n-1)r+i\}$. We then complete these partial r-multi Latin squares and bring them back together dividing each cell back into r cells yielding an $nr \times nr$ array of symbols. Finally, we must then find an appropriate order for the symbols appearing in a cell of one of the r-multi Latin squares so that we may obtain a Latin square of order nr.

Theorem 5. *If A is a partial Latin square of order nr with at most $n-1$ disjoint $r \times r$ squares filled in distinct columns, then there is a permutation of the symbols appearing in each row of each fixed $r \times r$ square allowing for A to be completed.*

Proof. As described above, divide A into r partial r-multi Latin squares of order n, $R_1, \ldots, R_r$, so that R_i has at most $n-1$ entries for $1 \leqslant i \leqslant r$. For notation purposes, we label the rows of R_i, in order, as $\{i, r+i, 2r+i \ldots (n-1)r+i\}$. By Theorem 3, we can complete R_i for each i.

We now form n bipartite graphs $G_k = (V_1, V_2)$ for $1 \leqslant k \leqslant n$, so that $V_1 = \{\rho_1, \ldots, \rho_{nr}\}$, $V_2 = \{s_1, \ldots, s_{nr}\}$, and $(\rho_{i+lr}, s_j) \in E(G_k)$ if symbol j appears in cell $(i+lr, k)$ of R_i for $1 \leqslant i \leqslant r$ and $0 \leqslant l \leqslant n$. Then observe that for each graph G_k, $d(\rho_i) = r$ and $d(s_j) = r$ for $1 \leqslant i \leqslant nr$ and $1 \leqslant j \leqslant nr$.

By König's coloring theorem for bipartite graphs, the edges of G_k can be properly colored using the function $\phi_k : E(G_k) \to \{c_{(k-1)r+1}, \ldots, c_{kr}\}$. Therefore, for each coloring of G_k, we can build a $nr \times r$ array of symbols so that cell $(i, (k-1)r+m)$ receives symbol j if $\phi_k(\rho_i, s_j) = (k-1)r+m$. Each coloring then yields an $nr \times r$ Latin rectangle. Setting these Latin rectangles side by side yields a completion of A where the symbols appearing in a row of a fixed $r \times r$ square may have been permuted. □

Note that in the argument given for Theorem 5, if the $r \times r$ squares are Latin squares, then we do not need for these squares to lie in distinct columns. Additionally, we do not need to allow for a permutation of symbols in the fixed rows of the $r \times r$ squares. Indeed, dividing A as described above yields r partial r-multi Latin squares and G_k is an r-regular graph with the vertices representing the rows containing the fixed $r \times r$ squares removed for each k. Therefore we have the following generalization of Theorem 4.

Theorem 6. *Let A be a partial Latin square of order nr with at most $(n-1)$ disjoint $r \times r$ squares filled as Latin squares so that if two $r \times r$ squares share a column they share r columns. Then A can be completed.*

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