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Rings of real functions in pointfree topology ☆

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ABSTRACT

This paper deals with the algebra $\mathsf{F}(L)$ of real functions on a frame L and its subclasses $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$ of, respectively, lower and upper semicontinuous real functions. It is well known that $\mathsf{F}(L)$ is a lattice-ordered ring; this paper presents explicit formulas for its algebraic operations which allow to conclude about their behaviour in $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$. As applications, idempotent functions are characterized and previous pointfree results about strict insertion of functions are significantly improved: general pointfree formulations that correspond exactly to the classical strict insertion results of Dowker and Michael regarding, respectively, normal countably paracompact spaces and perfectly normal spaces are derived.
The paper ends with a brief discussion concerning the frames in which every arbitrary real function on the α-dissolution of the frame is continuous.

0. Introduction

As is well known, each frame L has associated with it the ring $\mathcal{R}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ of its continuous real functions [2,3]. This is a commutative archimedean (strong) f-ring with unit [2]. By the familiar dual adjunction

$$\mathsf{Top} \underset{\Sigma}{\overset{\mathcal{O}}{\rightleftarrows}} \mathsf{Frm}$$

between the categories of topological spaces and frames there is a bijection

$$\mathsf{Top}(X, \mathbb{R}) \simeq \mathsf{Frm}\big(\mathfrak{L}(\mathbb{R}), \mathcal{O}X\big) \tag{1}$$

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where $\mathcal{O}X$ is the frame of open sets of the topological space X and $\mathbb{R}$ is endowed with its natural topology. Thus the classical ring $C(X)$ [10] is naturally isomorphic to $\mathcal{R}(\mathcal{O}X)$ and the correspondence $L \rightsquigarrow \mathcal{R}(L)$ for frames extends that for spaces.

Now, replace the space X in (1) by a discrete space $(X, \mathcal{P}(X))$. We get

$$\mathbb{R}^X \simeq \mathsf{Top}\big(\big(X, \mathcal{P}(X)\big), \mathbb{R}\big) \simeq \mathsf{Frm}\big(\mathfrak{L}(\mathbb{R}), \mathcal{P}(X)\big).$$

For any L in the category Frm, the role of the lattice $\mathcal{P}(X)$ of all subspaces of X is taken by the lattice $\mathcal{S}(L)$ of all sublocales of L, which justifies to think of the members of

$$\mathsf{Frm}\big(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L)\big) = \mathcal{R}\big(\mathcal{S}(L)\big)$$

as *arbitrary not necessarily continuous* real functions [12] on the frame L. The real functions on a frame L are thus the continuous real functions on the sublocale lattice of L and therefore, from the results of [2], constitute a commutative archimedean (strong) f-ring with unit that we denote by $\mathsf{F}(L)$. It is partially ordered by

$$\begin{aligned} f \leqslant g \;\equiv\; & f(r,-) \leqslant g(r,-) \quad \text{for all } r \in \mathbb{Q} \\ \Leftrightarrow\; & g(-,r) \leqslant f(-,r) \quad \text{for all } r \in \mathbb{Q}. \end{aligned}$$

Since any L is isomorphic to the subframe $\mathfrak{c}L$ of $\mathcal{S}(L)$ of all closed sublocales, the ring $\mathcal{R}(L)$ may be seen as the subring $\mathsf{C}(L)$ of all continuous real functions of $\mathsf{F}(L)$: $f \in \mathsf{F}(L)$ is *continuous* if $f(p,q)$ is a closed sublocale for every p, q, i.e. $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$.

Besides continuity, $\mathsf{F}(L)$ allows to distinguish the two types of semicontinuity: $f \in \mathsf{F}(L)$ is *lower semicontinuous* if $f(r,-)$ is a closed sublocale for every r, and f is *upper semicontinuous* if $f(-,r)$ is a closed sublocale for every r. We shall denote by $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$ respectively the classes of lower and upper semicontinuous functions. Hence, $\mathsf{C}(L) = \mathsf{LSC}(L) \cap \mathsf{USC}(L)$.

The first approach to semicontinuity in pointfree topology was presented in [13]. The approach here considered, summarized above, has wider scope and was introduced recently [12]. The further development of it asks for a better knowledge of the posets $(\mathsf{LSC}(L), \leqslant)$ and $(\mathsf{USC}(L), \leqslant)$ and the behaviour of the lattice-ordered ring operations of $\mathsf{F}(L)$ on them. This is the original motivation for this paper. We present explicit formulas for the algebraic operations of $\mathsf{F}(L)$ that provide, as immediate corollaries, results about their behaviour in $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$. Some of these formulas appear in a similar form in [1, Section 3] but our treatment here, based on the use of scales, simplifies the presentation and proofs. Our results make it possible to improve the study in [11] of strict insertion of frame homomorphisms with general pointfree extensions of the classical strict insertion theorems for normal and countably paracompact spaces (due to Dowker [7]) and perfectly normal spaces (due to Michael [16]).

We begin this paper by reviewing all the required background material (Section 1) and by providing (Section 2) a useful tool for generating the various types of real functions (general, semicontinuous and continuous). Then, we present the new descriptions of the algebraic operations of $\mathsf{F}(L)$ (joins and meets in Section 3, and sums and products in Section 4). Finally, we apply the results of Section 4 to characterize idempotent functions (Section 5) and to obtain the general formulations of the strict insertion theorems (Section 6) and we end with a very short section dealing with the natural question concerning the frames L in which every real function on the α-dissolution of L is continuous. Not surprisingly, this reveals to be related to one of the most important and deep open problems in locale theory.

1. Background and notation

1.1. Frames and locales

In pointfree topology spaces are represented by *generalized lattices of open sets*, called *frames*, defined as complete lattices L in which the distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$ holds for all $a \in L$ and $S \subseteq L$. In particular, a classical space X is represented by its lattice $\mathcal{O}(X)$ of open sets. Continuous maps are represented by *frame homomorphisms*, that is, those maps between frames that preserve arbitrary joins (hence 1, the top) and finite meets (hence 0, the bottom). The category of frames and frame homomorphisms is denoted by Frm. The set of all morphisms from L into M is denoted by $\mathsf{Frm}(L, M)$.

The above representation is contravariant: continuous maps $f : X \to Y$ are represented by frame homomorphisms $h = f^{-1}(\cdot) : \mathcal{O}(Y) \to \mathcal{O}(X)$. This can be easily mended, in order to keep the geometric motivation, by considering, instead of Frm simply its opposite category of *locales* and *localic maps*, and we have "generalized continuous maps" $f : L \to M$ that are precisely frame homomorphisms $h : M \to L$. Since we adopt along the paper the algebraic (frame) approach and reasoning, the reader should keep in mind that the geometric (localic) motivation reads backwards.

Being a Heyting algebra, each frame L has the implication $\to$ satisfying $a \wedge b \leqslant c$ iff $a \leqslant b \to c$. The *pseudocomplement* of an $a \in L$ is $a^* = a \to 0 = \bigvee\{b \in L \mid a \wedge b = 0\}$. Then $(\bigvee A)^* = \bigwedge_{a \in A} a^*$ for all $A \subseteq L$. In particular, $(\cdot)^*$ is order-reversing.

For general notions concerning frames and locales the reader is referred to [14] and [17]. In particular, regarding sublocales, we follow [17].

1.2. The frame of sublocales

A subset S of a locale L is a *sublocale* of L if, whenever $A \subseteq S$, $a \in L$ and $b \in S$, then $\bigwedge A \in S$ and $a \to b \in S$. The set of all sublocales of L forms a co-frame under inclusion, in which arbitrary meets coincide with intersection, $\{1\}$ is the bottom, and L is the top [17].

With the goal of dealing with arbitrary general functions we need to make the co-frame of all sublocales of L into a frame $\mathcal{S}(L)$ by considering the dual ordering: $S_1 \leqslant S_2$ iff $S_2 \subseteq S_1$. Thus, given $\{S_i \in \mathcal{S}(L) \mid i \in I\}$, we have $\bigvee_{i\in I} S_i = \bigcap_{i\in I} S_i$ and $\bigwedge_{i\in I} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in I} S_i\}$. Also, $\{1\}$ is the top and L is the bottom in $\mathcal{S}(L)$ that we simply denote by 1 and 0, respectively.

For any $a \in L$, the sets $\mathfrak{c}(a) = {\uparrow} a$ and $\mathfrak{o}(a) = \{a \to b \mid b \in L\}$ are the *closed* and *open* sublocales of L, respectively. Their main properties are subsumed in the following:

Proposition 1.1. *For any $a, b \in L$ and $A \subseteq L$:*

(a) $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \wedge \mathfrak{c}(b)$,
(b) $\mathfrak{c}(\bigvee A) = \bigvee_{a\in A} \mathfrak{c}(a)$,
(c) $\mathfrak{o}(a) \geqslant \mathfrak{c}(b)$ *if and only if* $a \wedge b = 0$,
(d) $\mathfrak{o}(a) \leqslant \mathfrak{c}(b)$ *if and only if* $a \vee b = 1$,
(e) $\mathfrak{c}(a) = \mathfrak{o}(b)$ *if and only if a and b are complements of each other,*
(f) $\mathfrak{c}(a) \vee \mathfrak{o}(a) = 1$ *and* $\mathfrak{c}(a) \wedge \mathfrak{o}(a) = 0$.

Thus $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathcal{S}(L)$. This implies that L is Boolean whenever all sublocales of L are clopen. Note also that the map $a \mapsto \mathfrak{c}(a)$ is a frame embedding $L \hookrightarrow \mathcal{S}(L)$, i.e. L and the subframe $\mathfrak{c}L$ of $\mathcal{S}(L)$ consisting of all closed sublocales, are isomorphic.

1.3. Frames of reals

There are various equivalent ways of introducing the *frame of reals* $\mathfrak{L}(\mathbb{R})$ (see e.g. [14] and [2,6]). In [2,6], $\mathfrak{L}(\mathbb{R})$ is the frame given by the generators (p, q) for $p, q \in \mathbb{Q}$ and the defining relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leqslant r < q \leqslant s$,
(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,
(R4) $\bigvee_{p,q\in\mathbb{Q}} (p, q) = 1$.

Here it will be useful to adopt the equivalent description of $\mathfrak{L}(\mathbb{R})$ introduced in [15] (see also [13]) and to take the elements $(r, -) = \bigvee_{s\in\mathbb{Q}} (r, s)$ and $(-, s) = \bigvee_{r\in\mathbb{Q}} (r, s)$ as primitive notions. Specifically, the *frame of reals* $\mathfrak{L}(\mathbb{R})$ is equivalently given by the generators $(r, -)$ and $(-, r)$ for $r \in \mathbb{Q}$ subject to the defining relations

(r1) $(r, -) \wedge (-, s) = 0$ whenever $r \geqslant s$,
(r2) $(r, -) \vee (-, s) = 1$ whenever $r < s$,
(r3) $(r, -) = \bigvee_{s>r} (s, -)$, for every $r \in \mathbb{Q}$,
(r4) $(-, r) = \bigvee_{s<r} (-, s)$, for every $r \in \mathbb{Q}$,
(r5) $\bigvee_{r\in\mathbb{Q}} (r, -) = 1$,
(r6) $\bigvee_{r\in\mathbb{Q}} (-, r) = 1$.

With $(p, q) = (p, -) \wedge (-, q)$ one goes back to (R1)–(R4).

1.4. Rings of real functions

For any frame L, the algebra $\mathcal{R}(L)$ of continuous real functions on L has as its elements the frame homomorphisms $f : \mathfrak{L}(\mathbb{R}) \to L$. The operations are determined by the operations of $\mathbb{Q}$ as lattice-ordered ring as follows (see [2] for more details):

(1) For $\diamond = +, \cdot, \wedge, \vee$:

$$(f \diamond g)(p, q) = \bigvee\bigl\{f(r, s) \wedge g(t, u) \mid \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\bigr\}$$

where $\langle \cdot, \cdot\rangle$ stands for open interval in $\mathbb{Q}$ and the inclusion on the right means that $x \diamond y \in \langle p, q\rangle$ whenever $x \in \langle r, s\rangle$ and $y \in \langle t, u\rangle$.

(2) $(-f)(p, q) = f(-q, -p)$.

(3) For each $r \in \mathbb{Q}$, a nullary operation $\mathbf{r}$ defined by

$$\mathbf{r}(p,q) = \begin{cases} 1 & \text{if } p < r < q \\ 0 & \text{otherwise.} \end{cases}$$

(4) For each $0 < \lambda \in \mathbb{Q}$, $(\lambda \cdot f)(p,q) = f(\frac{p}{\lambda}, \frac{q}{\lambda})$.

Indeed, these stipulations define maps from $\mathbb{Q} \times \mathbb{Q}$ to L and turn the defining relations (R1)–(R4) of $\mathfrak{L}(\mathbb{R})$ into identities in L and consequently determine frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$. The result that $\mathcal{R}(L)$ is an f-ring follows from the fact that any identity in these operations which is satisfied by $\mathbb{Q}$ also holds in $\mathcal{R}(L)$.

Given a frame L, we denote $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L)) = \mathcal{R}(\mathcal{S}(L))$ by $\mathsf{F}(L)$. In particular, each $\mathsf{F}(L)$ is an f-ring with operations defined by the formulas above. In Sections 3 and 4 we will provide explicit formulas for describing them.

An $f \in \mathsf{F}(L)$ is called an *arbitrary real function* [12] on L. Further f is:

(1) *lower semicontinuous* if $f(p, -)$ is a closed sublocale for every $p \in \mathbb{Q}$;
(2) *upper semicontinuous* if $f(-, q)$ is a closed sublocale for every $q \in \mathbb{Q}$.

The classes of lower and upper semicontinuous functions on L will be denoted by $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$ respectively.

Since any L is isomorphic to the subframe $\mathfrak{c}L$ of $\mathcal{S}(L)$ of all closed sublocales, the ring $\mathcal{R}(L)$ may be seen as the subring $\mathsf{C}(L)$ of all continuous real functions of $\mathsf{F}(L)$: $f \in \mathsf{F}(L)$ is *continuous* if $f(p,q)$ is a closed sublocale for every p, q.

Remark 1.2. (1) Each bijective and increasing map $\varphi : \mathbb{Q} \to \mathbb{Q}$ determines a bijection $\varphi(\cdot) : \mathsf{F}(L) \to \mathsf{F}(L)$ defined by

$$(\varphi f)(r, -) = f\big(\varphi(r), -\big) \quad \text{and} \quad (\varphi f)(-, r) = f\big(-, \varphi(r)\big) \quad \text{for every } r \in \mathbb{Q}.$$

When restricted to $\mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$) it becomes a bijection from $\mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$) onto $\mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$). Moreover, $\varphi(\cdot)$ is an order isomorphism.

(2) On the other hand, each bijective and decreasing map $\varphi : \mathbb{Q} \to \mathbb{Q}$ also determines a bijection $\varphi(\cdot) : \mathsf{F}(L) \to \mathsf{F}(L)$ defined by:

$$(\varphi f)(r, -) = f\big(-, \varphi(r)\big) \quad \text{and} \quad (\varphi f)(-, r) = f\big(\varphi(r), -\big) \quad \text{for every } r \in \mathbb{Q}.$$

Now, when restricted to $\mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$) it becomes a bijection from $\mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$) onto $\mathsf{USC}(L)$ (resp. $\mathsf{LSC}(L)$), showing that the posets $(\mathsf{LSC}(L), \leqslant)$ and $(\mathsf{USC}(L), \leqslant)$ are isomorphic. In this case $\varphi(\cdot)$ is order-reversing and one has

$$\varphi(f \wedge g) = \varphi f \vee \varphi g \quad \text{for each } f, g \in \mathsf{F}(L).$$

In particular, when $\varphi(r) = -r$ for each $r \in \mathbb{Q}$ we shall denote this bijection by $-(\cdot)$ (it evidently coincides with the $-(\cdot)$ of Section 1.4(2)).

2. Scales in $\mathcal{S}(L)$

In order to define a real function $f \in \mathsf{F}(L)$ it suffices to consider a map from the set of generators $\{(r, -), (-, r) \mid r \in \mathbb{Q}\}$ to $\mathcal{S}(L)$ that turns the defining relations (r1)–(r6) of $\mathfrak{L}(\mathbb{R})$ into identities in $\mathcal{S}(L)$. This can be easily done with scales (descending trails in [2]): a family $(S_p \mid p \in \mathbb{Q})$ of sublocales of L is a *scale* if

(S1) $S_p \vee S_q{}^* = 1$ whenever $p < q$, and
(S2) $\bigvee_{p \in \mathbb{Q}} S_p = 1 = \bigvee_{p \in \mathbb{Q}} S_p{}^*$.

Note that the terminology *scale* used here differs from its use in [14] where it refers to maps into L from the unit interval of $\mathbb{Q}$ and not all of $\mathbb{Q}$.

Remark 2.1. By condition (S1) a scale is necessarily an antitone family. Further, if a family $\mathcal{C}$ consists of complemented sublocales, then $\mathcal{C}$ satisfies (S1) if and only if it is antitone. Indeed, if $\mathcal{C}$ is antitone and each sublocale S_p has a complement $\neg S_p$, then $S_p \vee S_q{}^* = S_p \vee \neg S_q \geqslant S_p \vee \neg S_p = 1$ whenever $p < q$.

The following lemma, essentially proved in [12], will play a key role in the rest of the paper.

Lemma 2.2. *Let* $(S_r \mid r \in \mathbb{Q})$ *be a scale and let*

$$f(p, -) = \bigvee_{r > p} S_r \quad \text{and} \quad f(-, q) = \bigvee_{r < q} S_r{}^*, \quad p, q \in \mathbb{Q}.$$

Then:

(a) *The above two formulas determine an* $f \in \mathsf{F}(L)$.
(b) *If any* S_r *is closed then* $f \in \mathsf{LSC}(L)$.
(c) *If any* S_r *is open then* $f \in \mathsf{USC}(L)$.
(d) *If any* S_r *is clopen then* $f \in \mathsf{C}(L)$. $\square$

Examples 2.3. As basic examples of real functions we list:

(1) **Constant functions:** For each $r \in \mathbb{Q}$ let $(S_t^r \mid t \in \mathbb{Q})$ be defined by $S_t^r = 1$ if $t < r$ and $S_t^r = 0$ if $t \geqslant r$. Clearly, this is a scale. The corresponding function in $\mathsf{C}(L)$ provided by Lemma 2.2 is given for each $p, q \in \mathbb{Q}$ by

$$\mathbf{r}(p,-) = \begin{cases} 1 & \text{if } p < r \\ 0 & \text{if } p \geqslant r \end{cases} \quad \text{and} \quad \mathbf{r}(-,q) = \begin{cases} 0 & \text{if } q \leqslant r \\ 1 & \text{if } q > r \end{cases}$$

and coincides with the $\mathbf{r}$ of 1.4(3).

(2) **Characteristic functions:** Let S be a complemented sublocale of L. Then $(S_r \mid r \in \mathbb{Q})$ defined by $S_r = 1$ if $r < 0$, $S_r = \neg S$ if $0 \leqslant r < 1$ and $S_r = 0$ if $r \geqslant 1$, is a scale. We shall denote by χ_S the corresponding real function in $\mathsf{F}(L)$ and refer to it as the *characteristic function* of S. It is defined for each $p, q \in \mathbb{Q}$ by

$$\chi_S(p,-) = \begin{cases} 1 & \text{if } p < 0 \\ \neg S & \text{if } 0 \leqslant p < 1 \\ 0 & \text{if } p \geqslant 1 \end{cases} \quad \text{and} \quad \chi_S(-,q) = \begin{cases} 0 & \text{if } q \leqslant 0 \\ S & \text{if } 0 < q \leqslant 1 \\ 1 & \text{if } q > 1. \end{cases}$$

3. The posets $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$

The aim of the following two sections is to provide alternative descriptions to [2] of the lattice-ordered ring operations of $\mathsf{F}(L)$, by using scales. We shall use these alternative descriptions to study the behaviour of the operations in $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$. In this section we start with the lattice operations.

3.1. Finite joins and meets

Given $f, g \in \mathsf{F}(L)$, if we define $S_p = f(p,-) \vee g(p,-)$ for each $p \in \mathbb{Q}$ then, for each $p < q$,

$$\begin{aligned} S_p \vee {S_q}^* &= f(p,-) \vee g(p,-) \vee \big(f(q,-)^* \wedge g(q,-)^*\big) \\ &= \big(f(p,-) \vee g(p,-) \vee f(q,-)^*\big) \wedge \big(f(p,-) \vee g(p,-) \vee g(q,-)^*\big) = 1. \end{aligned}$$

Consequently,

$$\mathcal{C}_{f\vee g} = \big(f(p,-) \vee g(p,-) \mid p \in \mathbb{Q}\big)$$

satisfies condition (S1) of a scale. Moreover

$$\bigvee_{p\in\mathbb{Q}} S_p = \bigvee_{p\in\mathbb{Q}} \big(f(p,-) \vee g(p,-)\big) = \Big(\bigvee_{p\in\mathbb{Q}} f(p,-)\Big) \vee \Big(\bigvee_{p\in\mathbb{Q}} g(p,-)\Big) = 1 \quad \text{and}$$
$$\bigvee_{p\in\mathbb{Q}} {S_p}^* = \bigvee_{p\in\mathbb{Q}} \big(f(p,-)^* \wedge g(p,-)^*\big) \geqslant \bigvee_{p\in\mathbb{Q}} \big(f(-,p) \wedge g(-,p)\big) \geqslant \bigvee_{r,s\in\mathbb{Q}} \big(f(-,r) \wedge g(-,s)\big)$$

(since for any $r, s \in \mathbb{Q}$, $p = r \vee s \in \mathbb{Q}$ and $f(-,r) \wedge g(-,s) \leqslant f(-,p) \wedge g(-,p)$), from which it follows that

$$\bigvee_{p\in\mathbb{Q}} {S_p}^* = \Big(\bigvee_{p\in\mathbb{Q}} f(-,p)\Big) \wedge \Big(\bigvee_{p\in\mathbb{Q}} g(-,p)\Big) = 1.$$

Hence $\mathcal{C}_{f\vee g}$ is a scale. It is straightforward to check that the real function generated by $\mathcal{C}_{f\vee g}$ is precisely the supremum $f \vee g$ in $\mathsf{F}(L)$.

Note also that, for each $p, q \in \mathbb{Q}$,

$$(f \vee g)(p,-) = \bigvee_{r>p} \big(f(r,-) \vee g(r,-)\big) = f(p,-) \vee g(p,-) \quad \text{and}$$
$$(f \vee g)(-,q) = \bigvee_{r<q} \big(f(r,-) \vee g(r,-)\big)^* = f(-,q) \wedge g(-,q).$$

(For the latter identity, if $r < q$ then $(f(r,-) \vee g(r,-))^* = f(r,-)^* \wedge g(r,-)^* \leqslant f(-,q) \wedge g(-,q)$; conversely, $f(-,q) \wedge g(-,q) = \bigvee_{r_1,r_2<q} (f(-,r_1) \wedge g(-,r_2)) \leqslant \bigvee_{r<q} (f(r,-)^* \wedge g(r-)^*)$.)

Then, immediately, if $f, g \in \mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$) we have also $f \vee g \in \mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$).

Concerning meets, since $f \leqslant g$ iff $-g \leqslant -f$ for every $f, g \in \mathsf{F}(L)$, the infimum $f \wedge g$ of $f, g \in \mathsf{F}(L)$ exists and is given by $f \wedge g = -(-f \vee -g)$. Equivalently, $f \wedge g$ is the real function defined by the scale

$$\mathcal{C}_{f\wedge g} = \big(f(p,-) \wedge g(p,-) \mid p \in \mathbb{Q}\big).$$

Note also that, for each $p, q \in \mathbb{Q}$,

$$(f \wedge g)(p,-) = (-f \vee -g)(-,-p) = -f(-,-p) \wedge -g(-,-p) = f(p,-) \wedge g(p,-) \quad \text{and}$$
$$(f \wedge g)(-,q) = (-f \vee -g)(-q,-) = -f(-q,-) \vee -g(-q,-) = f(-,q) \vee g(-,q).$$

Therefore, if $f, g \in \mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$) then $f \wedge g \in \mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$).

In summary, we have:

Proposition 3.1. *The poset $\mathsf{F}(L)$ has binary joins and meets; $\mathsf{LSC}(L)$, $\mathsf{USC}(L)$ and $\mathsf{C}(L)$ are closed under these joins and meets.* $\square$

Remark 3.2. The lattice operations defined above on $\mathsf{F}(L)$, when applied to elements of the form (p, q), coincide with those of [2] (see Section 1.4). In fact, let $f, g \in \mathsf{F}(L)$ and $p, q \in \mathbb{Q}$.

(1) Regarding joins we have

$$\begin{aligned}(f \vee g)(p,q) &= (f \vee g)(p,-) \wedge (f \vee g)(-,q) \\ &= \big(f(p,-) \vee g(p,-)\big) \wedge \big(f(-,q) \wedge g(-,q)\big) \\ &= \big(f(p,q) \wedge g(-,q)\big) \vee \big(f(-,q) \wedge g(p,q)\big) \\ &= \Big(\bigvee_{s<q} f(p,q) \wedge g(s,q)\Big) \vee \Big(\bigvee_{r<q} f(r,q) \wedge g(p,q)\Big),\end{aligned}$$

and the latter is equal to $\bigvee\{f(r,s) \wedge g(t,u) \mid \langle r,s\rangle \vee \langle t,u\rangle = \langle r \vee t, s \vee u\rangle \subseteq \langle p,q\rangle\}$:

If $s < q$, then $\langle p,q\rangle \vee \langle s,q\rangle = \{x \vee y \mid x \in \langle p,q\rangle, y \in \langle s,q\rangle\} = \langle p \vee s, q\rangle \subseteq \langle p,q\rangle$. If $r < q$, then $\langle r,q\rangle \vee \langle p,q\rangle = \{x \vee y \mid x \in \langle r,q\rangle, y \in \langle p,q\rangle\} = \langle r \vee p, q\rangle \subseteq \langle p,q\rangle$. Hence the inequality $\leqslant$ follows. Conversely, let r, s, t and u such that $\langle r,s\rangle \vee \langle t,u\rangle \subseteq \langle p,q\rangle$, i.e. such that $p \leqslant r \vee t$ and $s \vee u \leqslant q$. We distinguish several cases: if $p \leqslant r$ and $t \geqslant q$, then $f(r,s) \wedge g(t,u) \leqslant f(p,q) \wedge g(t,q) = 0$; if $p \leqslant r$ and $t < q$, then $f(r,s) \wedge g(t,u) \leqslant f(p,q) \wedge g(t,q) \leqslant \bigvee_{s<q} f(p,q) \wedge g(s,q)$; if $p \leqslant t$ and $r \geqslant q$, then $f(r,s) \wedge g(t,u) \leqslant f(r,q) \wedge g(p,q) = 0$; finally, if $p \leqslant t$ and $r < q$, then $f(r,s) \wedge g(t,u) \leqslant f(r,q) \wedge g(p,q) \leqslant \bigvee_{r<q} f(r,q) \wedge g(p,q)$.

(2) Concerning meets, it follows immediately from the bijection $-(\cdot)$:

$$\begin{aligned}(f \wedge g)(p,q) &= (-f \vee -g)(-q,-p) \\ &= \bigvee\big\{-f(r,s) \wedge -g(t,u) \mid \langle r,s\rangle \vee \langle t,u\rangle = \langle r \vee t, s \vee u\rangle \subseteq \langle -q,-p\rangle\big\} \\ &= \bigvee\big\{f(-s,-r) \wedge g(-u,-t) \mid \langle r,s\rangle \vee \langle t,u\rangle = \langle r \vee t, s \vee u\rangle \subseteq \langle -q,-p\rangle\big\} \\ &= \bigvee\big\{f\big(r',s'\big) \wedge g\big(t',u'\big) \bigm| \big\langle r',s'\big\rangle \wedge \big\langle t',u'\big\rangle = \big\langle r' \vee t', s' \wedge u'\big\rangle \subseteq \langle p,q\rangle\big\}.\end{aligned}$$

3.2. Arbitrary joins and meets

We now turn to the question about arbitrary joins and meets in $\mathsf{F}(L)$, $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$.

Lemma 3.3. *Let $\varnothing \neq \mathcal{F} \subseteq \mathsf{F}(L)$. If $\bigvee_{f\in\mathcal{F}} f(p,-)$ is a complemented sublocale for every $p \in \mathbb{Q}$ and $\bigvee_{p\in\mathbb{Q}} \bigwedge_{f\in\mathcal{F}} f(-,p) = 1$, then $\bigvee\mathcal{F}$ exists in $\mathsf{F}(L)$.*

Proof. Let $S_p = \bigvee_{f\in\mathcal{F}} f(p,-)$ for each $p \in \mathbb{Q}$ and $\mathcal{C}_{\bigvee\mathcal{F}} = (S_p \mid p \in \mathbb{Q})$. Since each S_p is complemented and $\mathcal{C}_{\bigvee\mathcal{F}}$ is antitone, it follows from Remark 2.1 that $\mathcal{C}_{\bigvee\mathcal{F}}$ satisfies condition (S1) of a scale. Moreover

$$\bigvee_{p\in\mathbb{Q}} S_p = \bigvee_{p\in\mathbb{Q}} \bigvee_{f\in\mathcal{F}} f(p,-) = \bigvee_{f\in\mathcal{F}} \bigvee_{p\in\mathbb{Q}} f(p,-) = 1 \quad \text{and}$$
$$\bigvee_{p\in\mathbb{Q}} {S_p}^* = \bigvee_{p\in\mathbb{Q}} \Big(\bigvee_{f\in\mathcal{F}} f(p,-)\Big)^* = \bigvee_{p\in\mathbb{Q}} \bigwedge_{f\in\mathcal{F}} f(p,-)^* = \bigvee_{p\in\mathbb{Q}} \bigwedge_{f\in\mathcal{F}} f(-,p) = 1.$$

Consequently, $\mathcal{C}_{\bigvee\mathcal{F}}$ is a scale.

The real function generated by $\mathcal{C}_{\bigvee\mathcal{F}}$ is precisely the supremum $\bigvee\mathcal{F}$ of $\mathcal{F}$ in $\mathsf{F}(L)$ and is given for each $p, q \in \mathbb{Q}$ by

$$\Big(\bigvee\mathcal{F}\Big)(p,-) = \bigvee_{f\in\mathcal{F}} f(p,-) \quad \text{and} \quad \Big(\bigvee\mathcal{F}\Big)(-,q) = \bigvee_{r<q} \bigwedge_{f\in\mathcal{F}} f(r,-)^* = \bigvee_{r<q} \bigwedge_{f\in\mathcal{F}} f(-,r).$$

(For the latter identity let $r < s < q$. Then

$$\bigvee_{s<q} \bigwedge_{f\in\mathcal{F}} f(-,s) \geqslant \bigwedge_{f\in\mathcal{F}} f(-,s) \geqslant \bigwedge_{f\in\mathcal{F}} f(r,-)^*.$$

The other inequality follows immediately since $f(-,r) \leqslant f(r,-)^*$.) $\square$

Now we can prove the following completeness result:

Corollary 3.4. *Let $\varnothing \neq \mathcal{F} \subseteq \mathsf{LSC}(L)$ and suppose there is a $g \in \mathsf{F}(L)$ such that $f \leqslant g$ for every $f \in \mathcal{F}$. Then $\bigvee\mathcal{F}$ exists and belongs to $\mathsf{LSC}(L)$. (Equivalently, $\bigvee\mathcal{F}$ exists and belongs to $\mathsf{LSC}(L)$ if and only if $\bigvee\mathcal{F}$ exists in $\mathsf{F}(L)$.)*

Dually, let $\varnothing \neq \mathcal{F} \subseteq \mathsf{USC}(L)$ and suppose there is a $g \in \mathsf{F}(L)$ such that $g \leqslant f$ for every $f \in \mathcal{F}$. Then $\bigwedge\mathcal{F}$ exists and belongs to $\mathsf{USC}(L)$. (Equivalently, $\bigwedge\mathcal{F}$ exists and belongs to $\mathsf{USC}(L)$ if and only if $\bigwedge\mathcal{F}$ exists in $\mathsf{F}(L)$.)

Proof. Let $\varnothing \neq \mathcal{F} \subseteq \mathsf{LSC}(L)$ and $g \in \mathsf{F}(L)$ such that $f \leqslant g$ for every $f \in \mathcal{F}$. Since $\bigvee_{f\in\mathcal{F}} f(p,-)$ is a closed (hence complemented) sublocale and $\bigvee_{p\in\mathbb{Q}} \bigwedge_{f\in\mathcal{F}} f(-,p) \geqslant \bigvee_{p\in\mathbb{Q}} g(-,p) = 1$, the result follows immediately from Lemma 3.3. The second assertion can be proved by a similar argument. $\square$

Finally, in the case of continuous real functions, we have the following:

Corollary 3.5. *Let $\varnothing \neq \mathcal{F} \subseteq \mathsf{C}(L)$. If there is a $g \in \mathsf{F}(L)$ such that $f \leqslant g$ for every $f \in \mathcal{F}$, then $\bigvee\mathcal{F}$ exists and belongs to $\mathsf{LSC}(L)$. Dually, if there is a $g \in \mathsf{F}(L)$ such that $g \leqslant f$ for every $f \in \mathcal{F}$, then $\bigwedge\mathcal{F}$ exists and belongs to $\mathsf{USC}(L)$.* $\square$

3.3. Order-completeness

As is well known (see e.g. [17]) the frame $\mathcal{S}(L)$ is always completely regular and zero-dimensional. Therefore, by the identity $\mathsf{F}(L) = \mathcal{R}(\mathcal{S}(L))$, $\mathsf{F}(L)$ is an *l*-ring of continuous functions of a completely regular and zero-dimensional frame. This means that any result concerning $\mathcal{R}(L)$ for completely regular and zero-dimensional frames L is in particular true for $\mathsf{F}(L)$. In a sense, for a given L, the study of $\mathsf{F}(L)$ is more general than that of $\mathcal{R}(L)$ (since $\mathcal{R}(L) \simeq \mathsf{C}(L) \subseteq \mathsf{F}(L)$), but on the other hand the study of all $\mathsf{F}(L)$ is just a particular case of the study of all $\mathcal{R}(L)$ (for those L which are completely regular and zero-dimensional).

Recall from [5] that an *l*-ring is called *order complete* if every non-void subset S which is bounded above has a join $\bigvee S$; similarly, it is called *σ-complete* if $\bigvee S$ exists for any countable subset of this type. In Section 2 of [5], the authors prove a series of results for a completely regular L. Now we have:

Proposition 3.6. (Cf. [5, Proposition 1].) *$\mathsf{F}(L)$ is order complete iff $\mathcal{S}(L)$ is extremally disconnected.*

Since $\mathcal{S}(L)$ is zero-dimensional, this means that $\mathsf{F}(L)$ is not, in general, order complete: it is order complete precisely when every sublocale of L is complemented (since in any extremally disconnected the second De Morgan law $(\bigwedge_{i\in I} x_i)^* = \bigvee_{i\in I} x_i^*$ holds, every element of a zero-dimensional and extremally disconnected frame is evidently complemented). Then, by [19, Proposition 26], we may conclude that $\mathsf{F}(L)$ is order complete if and only if the lattice of complemented sublocales of L is closed under arbitrary joins in $\mathcal{S}(L)$.

Given a frame L, let $B(L)$ denote the *Boolean part* of L, that is, the Boolean algebra of complemented elements of L, and let βL denote the Stone–Čech compactification of L. Again by [5] we have the following:

Corollary 3.7. (Cf. [5, Corollaries 1 and 2].) *The following assertions are equivalent for any frame L:*

(a) $\mathsf{F}(L)$ *is order complete.*
(b) $\mathcal{S}(L)$ *is extremally disconnected.*
(c) $B(\mathcal{S}(L))$ *is complete.*
(d) $\beta\mathcal{S}(L)$ *is extremally disconnected.*

Note that the equivalence (b) $\Leftrightarrow$ (c) is a particular case of result III.3.5 of [14]: a zero-dimensional frame L is extremally disconnected iff BL is complete.

There is also a corresponding result for σ-completeness:

Corollary 3.8. (Cf. [5, Proposition 2 and Corollary 3].) *The following assertions are equivalent for any frame L:*

(a) $\mathsf{F}(L)$ *is σ-complete.*
(b) $\mathcal{S}(L)$ *is basically disconnected* (*i.e.* $\mathrm{coz}(f)^* \vee \mathrm{coz}(f)^{**} = 1$ *for every cozero element* $\mathrm{coz}(f) = f(-,0) \vee f(0,-)$ *of* $\mathcal{S}(L)$).

(c) *$\beta\mathcal{S}(L)$ is basically disconnected.*

Finally, by [5, Remark 3] we know that

$\mathsf{F}(L)$ *is regular iff every* $\operatorname{coz}(f)$ *is complemented.*

Thus, immediately,

$\mathsf{F}(L)$ *is order complete* $\Rightarrow$ $\mathsf{F}(L)$ *is regular* $\Rightarrow$ $\mathsf{F}(L)$ *is σ-complete.*

4. Algebraic operations in $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$

We now pursue with the operations of scalar product, sum and product.

4.1. Product with a scalar

Given $0 < \lambda \in \mathbb{Q}$ and $f \in \mathsf{F}(L)$, if we define $S_p = f(\frac{p}{\lambda}, -)$ for each $p \in \mathbb{Q}$ then we have that for each $p < q$

$$S_p \vee {S_q}^* = f\left(\frac{p}{\lambda}, -\right) \vee f\left(\frac{q}{\lambda}, -\right)^* \geqslant f\left(\frac{p}{\lambda}, -\right) \vee f\left(-, \frac{q}{\lambda}\right) = 1,$$

$\bigvee_{p\in\mathbb{Q}} S_p = \bigvee_{p\in\mathbb{Q}} f(\frac{p}{\lambda}, -) = 1$ and $\bigvee_{p\in\mathbb{Q}} S_p^* \geqslant \bigvee_{p\in\mathbb{Q}} f(-, \frac{p}{\lambda}) = 1$. Consequently,

$$\mathcal{C}_{\lambda\cdot f} = \left(f\left(\frac{p}{\lambda}, -\right) \;\middle|\; p \in \mathbb{Q}\right)$$

is a scale. The real function generated by $\mathcal{C}_{\lambda\cdot f}$ which we denote by $\lambda \cdot f$ is defined for each $p, q \in \mathbb{Q}$ as

$$(\lambda \cdot f)(p, -) = f\left(\frac{p}{\lambda}, -\right) \quad \text{and} \quad (\lambda \cdot f)(-, q) = f\left(-, \frac{q}{\lambda}\right).$$

It coincides again with the corresponding operation in $\mathcal{R}(\mathcal{S}(L))$ (Section 1.4):

$$(\lambda \cdot f)(p, q) = (\lambda \cdot f)(p, -) \wedge (\lambda \cdot f)(-, q) = f\left(\frac{p}{\lambda}, -\right) \wedge f\left(-, \frac{q}{\lambda}\right) = f\left(\frac{p}{\lambda}, \frac{q}{\lambda}\right).$$

Let $f \in \mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$) and $0 < \lambda \in \mathbb{Q}$. It follows immediately that $\lambda \cdot f \in \mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$).

4.2. Sum

We first note the following:

Lemma 4.1. *Let $f, g \in \mathsf{F}(L)$. For each $p \in \mathbb{Q}$ define*

$$S_p^{f+g} = \bigvee_{r\in\mathbb{Q}} \big(f(r, -) \wedge g(p - r, -)\big) \quad \textit{and} \quad T_p^{f+g} = \bigvee_{s\in\mathbb{Q}} \big(f(-, s) \wedge g(-, p - s)\big).$$

(a) *If $p \geqslant q \in \mathbb{Q}$ then $S_p^{f+g} \wedge T_q^{f+g} = 0$.*
(b) *If $p < q \in \mathbb{Q}$ then*

$$S_p^{f+g} \wedge T_q^{f+g} = \bigvee\big\{f(r, s) \wedge g(t, u) \;\big|\; \langle r, s\rangle + \langle t, u\rangle \subseteq \langle p, q\rangle\big\} \quad \textit{and} \quad S_p^{f+g} \vee T_q^{f+g} = 1.$$

Proof. (a) Let $p, q, r, s \in \mathbb{Q}$ with $p \geqslant q$. Then either $s \leqslant r$ or $q - s < p - r$ and so either $f(r, -) \wedge f(-, s) = 0$ or $g(p - r, -) \wedge g(-, q - s) = 0$. Hence $S_p^{f+g} \wedge T_q^{f+g} = 0$.

(b) Let $p, q, r, s \in \mathbb{Q}$ with $p < q$. Since $\langle r, s\rangle + \langle t, u\rangle = \langle r + t, s + u\rangle$, it follows that $\langle r, s\rangle + \langle t, u\rangle \subseteq \langle p, q\rangle$ if and only if $p \leqslant r + t$ and $q \geqslant s + u$, that is, if and only if $p - r \leqslant t$ and $q - s \geqslant u$. Consequently

$$\begin{aligned} S_p^{f+g} \wedge T_q^{f+g} &= \bigvee_{r,s\in\mathbb{Q}} \big(f(r, -) \wedge g(p - r, -) \wedge f(-, s) \wedge g(-, q - s)\big) \\ &= \bigvee_{r,s\in\mathbb{Q}} \big(f(r, s) \wedge g(p - r, q - s)\big) \\ &= \bigvee\big\{f(r, s) \wedge g(t, u) \;\big|\; \langle r, s\rangle + \langle t, u\rangle \subseteq \langle p, q\rangle\big\}. \end{aligned}$$

Regarding the second assertion, let $p < q \in \mathbb{Q}$ and $t = \frac{q-p}{2} > 0$. Then $\bigvee_{r\in\mathbb{Q}} f(r, r+t) = \bigvee_{s\in\mathbb{Q}} g(s, s+t) = 1$. Let $r, s \in \mathbb{Q}$. If $r + s > p$ then $f(r, r+t) \wedge g(s, s+t) \leqslant f(r, -) \wedge g(p-r, -) \leqslant S_p^{f+g}$. Otherwise, if $r + s \leqslant p$ then $s + t \leqslant q - r - t$ and so $f(r, r+t) \wedge g(s, s+t) \leqslant f(-, r+t) \wedge g(-, q-(r+t)) \leqslant T_q^{f+g}$. Hence

$$1 = \bigvee_{r,s\in\mathbb{Q}} \big(f(r, r+t) \wedge g(s, s+t)\big) \leqslant S_p^{f+g} \vee T_q^{f+g}. \qquad \square$$

Proposition 4.2. *For any $f, g \in \mathsf{F}(L)$ the family $(S_p^{f+g} \mid p \in \mathbb{Q})$ is a scale.*

Proof. Let $p < q \in \mathbb{Q}$. Take $r \in \mathbb{Q}$ such that $p < r < q$. It follows immediately from Lemma 4.1(a) and (b) that $S_p^{f+g} \vee (S_q^{f+g})^* \geqslant S_p^{f+g} \vee T_r^{f+g} = 1$. On the other hand $\bigvee_{p\in\mathbb{Q}} S_p^{f+g} = \bigvee_{p,r\in\mathbb{Q}} (f(r, -) \wedge g(p-r, -)) = \bigvee_{r\in\mathbb{Q}} (f(r, -) \wedge \bigvee_{p\in\mathbb{Q}} g(p-r, -)) = \bigvee_{r\in\mathbb{Q}} f(r, -) = 1$ and $\bigvee_{p\in\mathbb{Q}} (S_p^{f+g})^* \geqslant \bigvee_{p\in\mathbb{Q}} T_p^{f+g} = \bigvee_{p,s\in\mathbb{Q}} (f(-, s) \wedge g(-, p-s)) = \bigvee_{s\in\mathbb{Q}} (f(-, s) \wedge \bigvee_{p\in\mathbb{Q}} g(-, p-s)) = \bigvee_{s\in\mathbb{Q}} f(-, s) = 1$. $\square$

We shall write $f + g$ (the *sum* of f and g) to denote the real function generated by $(S_p^{f+g} \mid p \in \mathbb{Q})$. It coincides with the sum operation in $\mathcal{R}(\mathcal{S}(L))$ (Section 1.4):

Corollary 4.3. *Let $f, g \in \mathsf{F}(L)$. Then*:

(a) $(f+g)(p, -) = \bigvee_{r\in\mathbb{Q}} (f(r, -) \wedge g(p-r, -))$ *for every* $p \in \mathbb{Q}$.
(b) $(f+g)(-, q) = \bigvee_{s\in\mathbb{Q}} (f(-, s) \wedge g(-, q-s))$ *for every* $q \in \mathbb{Q}$.
(c) $(f+g)(p, q) = \bigvee\{f(r, s) \wedge g(t, u) \mid \langle r, s\rangle + \langle t, u\rangle \subseteq \langle p, q\rangle\}$ *for every* $p, q \in \mathbb{Q}$.

Proof. (a) By Lemma 2.2,

$$(f+g)(p, -) = \bigvee_{t>p} S_t^{f+g} = \bigvee_{t>p} \bigvee_{r\in\mathbb{Q}} \big(f(r, -) \wedge g(t-r, -)\big) = \bigvee_{r\in\mathbb{Q}} \big(f(r, -) \wedge g(p-r, -)\big).$$

(b) By Lemma 2.2, $(f+g)(-, q) = \bigvee_{r<q} (S_r^{f+g})^*$ and therefore $(f+g)(-, q) \leqslant T_q^{f+g}$ (since by Lemma 4.1(b), $S_r^{f+g} \vee T_q^{f+g} = 1$ for $r < q$). On the other hand, $T_q^{f+g} = \bigvee_{s\in\mathbb{Q}} \bigvee_{r<q} (f(-, s) \wedge g(-, r-s)) = \bigvee_{r<q} T_r^{f+g} \leqslant \bigvee_{r<q} (S_r^{f+g})^*$. Hence $(f+g)(-, q) = T_q^{f+g} = \bigvee_{s\in\mathbb{Q}} (f(-, s) \wedge g(-, q-s))$.

(c) It follows immediately from Lemma 4.1(b). $\square$

Hence we have:

Corollary 4.4. *Let $f, g \in \mathsf{F}(L)$.*

(a) *If $f, g \in \mathsf{LSC}(L)$ then $f + g \in \mathsf{LSC}(L)$.*
(b) *If $f, g \in \mathsf{USC}(L)$ then $f + g \in \mathsf{USC}(L)$.*
(c) *If $f, g \in \mathsf{C}(L)$ then $f + g \in \mathsf{C}(L)$.* $\square$

Given $f, g \in \mathsf{F}(L)$, since $f - g = f + (-g)$ we also have:

Corollary 4.5. *Let $f, g \in \mathsf{F}(L)$.*

(a) $(f-g)(p, -) = \bigvee_{r\in\mathbb{Q}} f(r, -) \wedge g(-, r-p)$ *for every* $p \in \mathbb{Q}$.
(b) $(f-g)(-, q) = \bigvee_{s\in\mathbb{Q}} f(-, s) \wedge g(s-q, -)$ *for every* $q \in \mathbb{Q}$.
(c) *If $f \in \mathsf{LSC}(L)$ and $g \in \mathsf{USC}(L)$ then $f - g \in \mathsf{LSC}(L)$.*
(d) *If $f \in \mathsf{USC}(L)$ and $g \in \mathsf{LSC}(L)$ then $f - g \in \mathsf{USC}(L)$.*
(e) *If $f, g \in \mathsf{C}(L)$ then $f - g \in \mathsf{C}(L)$.* $\square$

4.3. Product

We now turn to the product, starting with the case $f, g \geqslant \mathbf{0}$:

Lemma 4.6. *Let $\mathbf{0} \leqslant f, g \in \mathsf{F}(L)$. For each $p \in \mathbb{Q}$ define*

$$S_p^{f\cdot g} = \begin{cases} \bigvee_{r>0}(f(r,-) \wedge g(\frac{p}{r},-)) & \text{if } p \geqslant 0 \\ 1 & \text{if } p < 0 \end{cases} \quad \text{and}$$

$$T_q^{f\cdot g} = \begin{cases} \bigvee_{s>0}(f(-,s) \wedge g(-,\frac{q}{s})) & \text{if } q > 0 \\ 0 & \text{if } q \leqslant 0. \end{cases}$$

(a) *If $p \geqslant q \in \mathbb{Q}$ then $S_p^{f\cdot g} \wedge T_q^{f\cdot g} = 0$.*
(b) *If $p < q \in \mathbb{Q}$ then $S_p^{f\cdot g} \wedge T_q^{f\cdot g} = \bigvee\{f(r,s) \wedge g(t,u) \mid \langle r,s\rangle \cdot \langle t,u\rangle \subseteq \langle p,q\rangle\}$ and $S_p^{f\cdot g} \vee T_q^{f\cdot g} = 1$.*

Proof. (a) Let $p,q,r,s \in \mathbb{Q}$ with $p \geqslant q > 0$ (the case $q \leqslant 0$ is trivial) and $r,s > 0$. Then either $s \leqslant r$ or $\frac{q}{s} \leqslant \frac{p}{r}$ and so either $f(r,-) \wedge f(-,s) = 0$ or $g(\frac{p}{r},-) \wedge g(-,\frac{q}{s}) = 0$. Hence $S_p^{f+g} \wedge T_q^{f+g} = 0$.

(b) Let $p,q \in \mathbb{Q}$ with $0 \leqslant p < q$ (the case $p < 0$ is similar). Then

$$\begin{aligned} S_p^{f\cdot g} \wedge T_q^{f\cdot g} &= \bigvee_{r,s>0}\left(f(r,-) \wedge g\left(\frac{p}{r},-\right) \wedge f(-,s) \wedge g\left(-,\frac{q}{s}\right)\right) \\ &= \bigvee\left\{f(r,s) \wedge g\left(\frac{p}{r},\frac{q}{s}\right) \;\middle|\; 0 < r < s,\, 0 \leqslant \frac{p}{r} < \frac{q}{s}\right\} \\ &\leqslant \bigvee\bigl\{f(r,s) \wedge g(t,u) \bigm| \langle r,s\rangle \cdot \langle t,u\rangle \subseteq \langle p,q\rangle\bigr\} \end{aligned}$$

since $\langle r,s\rangle \cdot \langle \frac{p}{r},\frac{q}{s}\rangle = \langle p,q\rangle$ for $0 < r < s$ and $0 \leqslant \frac{p}{r} < \frac{q}{s}$. Conversely, if $\langle r,s\rangle \cdot \langle t,u\rangle \subseteq \langle p,q\rangle$ then either $s,u < 0$ or $r,t > 0$. If $s,u < 0$, then $f(r,s) \wedge g(t,u) = 0$; on the other hand, if $r,t > 0$ we have that $\langle r,s\rangle \cdot \langle t,u\rangle = \langle rt,su\rangle \subseteq \langle p,q\rangle$ and so $p \leqslant rt$ and $q \geqslant su$. Hence $f(r,s) \wedge g(t,u) \leqslant f(r,s) \wedge g(\frac{p}{r},\frac{q}{s}) \leqslant \bigvee_{0<r,s}(f(r,s) \wedge g(\frac{p}{r},\frac{q}{s})) = S_p^{f\cdot g} \wedge T_q^{f\cdot g}$.

Regarding the second assertion, let $0 \leqslant p < q \in \mathbb{Q}$ (the case $p < 0$ is trivial) and $t \in \mathbb{Q}$ such that $1 < t^2 \leqslant \frac{q}{p}$. We have that $\bigvee_{r>0} f(r,rt) = f(0,-)$ and $\bigvee_{s>0} g(s,st) = g(0,-)$. Let $0 < r,s \in \mathbb{Q}$. If $rs > p$ then $f(r,rt) \wedge g(s,st) \leqslant f(r,-) \wedge g(\frac{p}{r},-) \leqslant S_p^{f\cdot g}$. Otherwise, if $rs \leqslant p$ then $st \leqslant \frac{q}{rt}$ and so $f(r,rt) \wedge g(s,st) \leqslant f(-,rt) \wedge g(-,\frac{q}{rt}) \leqslant T_q^{f\cdot g}$. Hence

$$f(0,-) \wedge g(0,-) = \bigvee_{r,s>0}\bigl(f(r,rt) \wedge g(s,st)\bigr) \leqslant S_p^{f\cdot g} \vee T_q^{f\cdot g}.$$

On the other hand, $T_q^{f\cdot g} \vee f(0,-) = \bigvee_{s>0}((f(-,s) \vee f(0,-)) \wedge (g(-,\frac{q}{s}) \vee f(0,-))) = \bigvee_{s>0}(g(-,\frac{q}{s}) \vee f(0,-)) = 1$ and, similarly, $T_q^{f\cdot g} \vee g(0,-) = 1$, hence

$$1 = \bigl(T_q^{f\cdot g} \vee f(0,-)\bigr) \wedge \bigl(T_q^{f\cdot g} \vee g(0,-)\bigr) = T_q^{f\cdot g} \vee \bigl(f(0,-) \wedge g(0,-)\bigr) \leqslant S_p^{f\cdot g} \vee T_q^{f\cdot g}. \qquad \square$$

Proposition 4.7. *For any $\mathbf{0} \leqslant f, g \in \mathsf{F}(L)$ the family $(S_p^{f\cdot g} \mid p \in \mathbb{Q})$ is a scale.*

Proof. Let $p < q \in \mathbb{Q}$. Take $r \in \mathbb{Q}$ such that $p < r < q$. It follows immediately from Lemma 4.6(a) and (b) that $S_p^{f\cdot g} \vee (S_q^{f\cdot g})^* \geqslant S_p^{f\cdot g} \vee T_r^{f\cdot g} = 1$. On the other hand, $\bigvee_{p\in\mathbb{Q}} S_p^{f\cdot g} = 1$ and $\bigvee_{p\in\mathbb{Q}} (S_p^{f\cdot g})^* \geqslant \bigvee_{p\in\mathbb{Q}} T_p^{f\cdot g} = \bigvee_{p,s>0}(f(-,s) \wedge g(-,\frac{p}{s})) = \bigvee_{s>0}(f(-,s) \wedge \bigvee_{p>0} g(-,\frac{p}{s})) = \bigvee_{s>0} f(-,s) = 1$. $\square$

Let $\mathbf{0} \leqslant f, g \in \mathsf{F}(L)$. We shall write $f \cdot g$ (the *product* of f and g) to denote the real function generated by the scale $(S_p^{f\cdot g} \mid p \in \mathbb{Q})$. It coincides with the product operation in $\mathcal{R}(\mathcal{S}(L))$ (Section 1.4):

Corollary 4.8. *Let $\mathbf{0} \leqslant f, g \in \mathsf{F}(L)$. Then*:

(a) $(f\cdot g)(p,-) = \begin{cases} \bigvee_{r>0}(f(r,-) \wedge g(\frac{p}{r},-)) & \text{if } p \geqslant 0 \\ 1 & \text{if } p < 0. \end{cases}$

(b) $(f\cdot g)(-,q) = \begin{cases} \bigvee_{s>0}(f(-,s) \wedge g(-,\frac{q}{s})) & \text{if } q > 0 \\ 0 & \text{if } q \leqslant 0. \end{cases}$

(c) $(f\cdot g)(p,q) = \bigvee\{f(r,s) \wedge g(t,u) \mid \langle r,s\rangle \cdot \langle t,u\rangle \subseteq \langle p,q\rangle\}$ *for every $p,q \in \mathbb{Q}$.*

Proof. (a) If $p < 0$ then $(f\cdot g)(p,-) = \bigvee_{r>p} S_r^{f\cdot g} = 1$. On the other hand, if $p \geqslant 0$ then $(f\cdot g)(p,-) = \bigvee_{t>p} S_t^{f\cdot g} = \bigvee_{t>p}\bigvee_{r>0}(f(r,-) \wedge g(\frac{t}{r},-)) = \bigvee_{r>0}(f(r,-) \wedge g(\frac{p}{r},-))$.

(b) By Lemma 2.2, $(f\cdot g)(-,q)=\bigvee_{p<q}(S_p^{f\cdot g})^*\leqslant T_q^{f\cdot g}$. On the other hand, let $q>0$. It follows then, using Lemma 4.6(b), that $\bigvee_{s>0}(f(-,s)\wedge g(-,\frac{q}{s}))=\bigvee_{s>0}\bigvee_{0<p<q}(f(-,s)\wedge g(-,\frac{p}{s}))=\bigvee_{0<p<q}T_p^{f\cdot g}\leqslant\bigvee_{0<p<q}(S_p^{f\cdot g})^*=(f\cdot g)(-,q)$.

(c) It follows immediately from Lemma 4.6(b). □

Hence we have:

Corollary 4.9. *Let* $\mathbf{0}\leqslant f,g\in\mathsf{F}(L)$.

(a) *If* $f,g\in\mathsf{LSC}(L)$ *then* $f\cdot g\in\mathsf{LSC}(L)$.
(b) *If* $f,g\in\mathsf{USC}(L)$ *then* $f\cdot g\in\mathsf{USC}(L)$.
(c) *If* $f,g\in\mathsf{C}(L)$ *then* $f\cdot g\in\mathsf{C}(L)$. □

In order to extend this result to the product of two arbitrary f and g let

$$f^+=f\vee\mathbf{0}\quad\text{and}\quad f^-=(-f)\vee\mathbf{0}$$

for any $f\in\mathsf{F}(L)$. Note that $f=f^+-f^-$. Since $\mathcal{R}(\mathcal{S}(L))$ is an ℓ-ring, from general properties of ℓ-rings we have that

$$f\cdot g=\left(f^+\cdot g^+\right)-\left(f^+\cdot g^-\right)-\left(f^-\cdot g^+\right)+\left(f^-\cdot g^-\right).$$

In particular, if $f,g\leqslant\mathbf{0}$, then $f\cdot g=f^-\cdot g^-=(-f)\cdot(-g)$. Hence:

Corollary 4.10. *Let* $f,g\in\mathsf{F}(L)$.

(a) *If* $f,g\in\mathsf{LSC}(L)$ *and* $f,g\leqslant\mathbf{0}$ *then* $f\cdot g\in\mathsf{USC}(L)$.
(b) *If* $f,g\in\mathsf{USC}(L)$ *and* $f,g\leqslant\mathbf{0}$ *then* $f\cdot g\in\mathsf{LSC}(L)$.
(c) *If* $f,g\in\mathsf{C}(L)$ *then* $f\cdot g\in\mathsf{C}(L)$. □

Remark 4.11. Replacing the frame $\mathfrak{L}(\mathbb{R})$ of reals by the frame $\mathfrak{L}(\overline{\mathbb{R}})$ of *extended reals* (defined by dropping conditions (r5) and (r6) in 1.3) we may deal with *rings of extended real functions*. Their study, more difficult, is left for a subsequent paper [4].

5. An application to idempotent functions

An $f\in\mathsf{F}(L)$ is *idempotent* if $f\cdot f=f$. Obvious examples of idempotents in $\mathsf{F}(L)$ are the characteristic functions χ_S (for complemented sublocales S of L).

By using the descriptions of the algebraic operations of $\mathsf{F}(L)$ obtained in Section 4, the following properties are now easy to check.

Properties 5.1. *The following hold for any* $f,g\in\mathsf{F}(L)$:

(a) $(f\cdot g)(0,-)=(f(0,-)\wedge g(0,-))\vee(f(-,0)\wedge g(-,0))$.
(b) $(f\cdot g)(-,0)=(f(0,-)\wedge g(-,0))\vee(f(-,0)\wedge g(0,-))$.
(c) $(\mathbf{1}-f)(0,-)=f(-,1)$ *and* $(\mathbf{1}-f)(-,0)=f(1,-)$. □

With them at hand we can easily prove the following result that strengthens Lemma 2.5 of [8].

Proposition 5.2. *An* $f\in\mathsf{F}(L)$ *is idempotent if and only if* $f(0,1)=f(-,0)=f(1,-)=0$.

Proof. Clearly $f\cdot f=f$ if and only if $f\cdot(\mathbf{1}-f)=\mathbf{0}$ if and only if $(f\cdot(\mathbf{1}-f))(0,-)=0=(f\cdot(\mathbf{1}-f))(-,0)$. But by the preceding properties we have

$$\begin{aligned}\big(f\cdot(\mathbf{1}-f)\big)(0,-)&=\big(f(0,-)\wedge(\mathbf{1}-f)(0,-)\big)\vee\big(f(-,0)\wedge(\mathbf{1}-f)(-,0)\big)\\&=\big(f(0,-)\wedge f(-,1)\big)\vee\big(f(-,0)\wedge f(1,-)\big)=f(0,1)\end{aligned}$$

and

$$\begin{aligned}\big(f\cdot(\mathbf{1}-f)\big)(-,0)&=\big(f(0,-)\wedge(\mathbf{1}-f)(-,0)\big)\vee\big(f(-,0)\wedge(\mathbf{1}-f)(0,-)\big)\\&=\big(f(0,-)\wedge f(1,-)\big)\vee\big(f(-,0)\wedge f(-,1)\big)\\&=f(1,-)\vee f(-,0).\qquad\square\end{aligned}$$

Corollary 5.3. *Let L be a frame. Then*:

(a) *An $f \in \mathsf{F}(L)$ is idempotent iff $f = \chi_S$ for some complemented sublocale S of L.*
(b) *An $f \in \mathsf{C}(L)$ is idempotent iff $f = \chi_{\mathfrak{c}(a)}$ for some complemented element a of L.*

Proof. (a) We only need to prove necessity. Let $f \in \mathsf{F}(L)$ be idempotent and $S = f(-,1)$. Since $f(-,1) \vee f(0,-) = 1$ and $f(-,1) \wedge f(0,-) = f(0,1) = 0$, it follows that S is a complemented sublocale of L with complement $f(0,-)$. It is easy to check now that $f = \chi_S$.

(b) This is obvious since we have that $f \in \mathsf{C}(L)$ if and only if $f \in \mathsf{F}(L)$ and $f(p,q)$ is a closed sublocale of L for each $p, q \in \mathbb{Q}$. It follows that f must be of the form χ_S with both S and $\neg S$ being closed sublocales of L. $\square$

We can now conclude from Proposition 2.2 of [8]) that:

(1) There exists a Boolean isomorphism between idempotent real functions on L and the complemented sublocales of L.
(2) There exists a Boolean isomorphism between idempotent continuous real functions on L and the complemented elements of L.

6. Applications to strict insertion

The results in the preceding section allow now to improve the study in the previous paper [11] with the pointfree assertions corresponding exactly to the following classical insertion theorems of Dowker [7] and Michael [16] regarding, respectively, normal countably paracompact spaces and perfectly normal spaces:

(Dowker) *A topological space X is normal and countably paracompact if and only if, given $f, g : X \to \mathbb{R}$ such that $f < g$, f is upper semicontinuous and g is lower semicontinuous, there is a continuous $h : X \to \mathbb{R}$ such that $f < h < g$.*

(Michael) *A topological space X is perfectly normal if and only if, given $f, g : X \to \mathbb{R}$ such that $f \leqslant g$, f is upper semicontinuous and g is lower semicontinuous, there is a continuous $h : X \to \mathbb{R}$ such that $f \leqslant h \leqslant g$ and $f(x) < h(x) < g(x)$ whenever $f(x) < g(x)$.*

To begin with, we recall from [12] the fundamental pointfree Katětov–Tong insertion theorem:

(Pointfree Katětov–Tong) *A frame L is normal if and only if, given $f \in \mathsf{USC}(L)$ and $g \in \mathsf{LSC}(L)$ with $f \leqslant g$, there exists an $h \in \mathsf{C}(L)$ such that $f \leqslant h \leqslant g$.*

Now for each $f, g \in \mathsf{F}(L)$ define

$$\iota(f,g) = \bigvee_{p \in \mathbb{Q}} \big(f(-,p) \wedge g(p,-)\big) \in \mathcal{S}(L).$$

One writes $f < g$ whenever $\iota(f,g) = 1$ [11]. Note that the relation $<$ is indeed stronger than $\leqslant$: if $f < g$ then, for every $r \in \mathbb{Q}$, $f(r,-) = f(r,-) \wedge \bigvee_{p \in \mathbb{Q}}(f(-,p) \wedge g(p,-)) \leqslant \bigvee_{p \geqslant r}(f(-,p) \wedge g(p,-)) \leqslant g(r,-) \wedge \bigvee_{p \in \mathbb{Q}}(f(-,p) \wedge g(p,-)) = g(r,-)$.
Moreover:

Lemma 6.1. *For any $r \in \mathbb{Q}$ and any $f, g, f_i, g_i \in \mathsf{F}(L)$ $(i = 1, 2)$ we have*:

(a) *$\iota(\mathbf{r}, f) = f(r,-)$; in particular, $\mathbf{r} < f$ iff $f(r,-) = 1$.*
(b) *$\iota(f, \mathbf{r}) = f(-,r)$; in particular, $f < \mathbf{r}$ iff $f(-,r) = 1$.*
(c) *$\iota(f,g) = \iota(\mathbf{0}, g - f)$; in particular, $f < g$ iff $\mathbf{0} < g - f$.*
(d) *$\iota(\lambda \cdot f, \lambda \cdot g) = \iota(f,g)$; in particular, $f < g$ iff $\lambda \cdot f < \lambda \cdot g$ for every $0 < \lambda \in \mathbb{Q}$.*
(e) *$\iota(f_1, g_1) \leqslant \iota(f_2, g_2)$ whenever $f_2 \leqslant f_1$ and $g_1 \leqslant g_2$.*

Proof. (a) $\iota(\mathbf{r}, f) = \bigvee_{p \in \mathbb{Q}}(\mathbf{r}(-,p) \wedge f(p,-)) = \bigvee_{p > r} f(p,-) = f(r,-)$.

(b) may be proved in a similar way.

(c) $\iota(\mathbf{0}, g - f) = \bigvee_{p \in \mathbb{Q}}(\mathbf{0}(-,p) \wedge (g-f)(p,-)) = \bigvee_{p>0} \bigvee_{r \in \mathbb{Q}}(g(r,-) \wedge f(-,r-p)) = \bigvee_{r \in \mathbb{Q}}(g(r,-) \wedge (\bigvee_{p>0} f(-,r-p))) = \bigvee_{r \in \mathbb{Q}}(g(r,-) \wedge f(-,r)) = \iota(f,g)$.

(d) and (e) are clear. $\square$

We shall also need the following:

Remark 6.2. (Cf. Remark 1.2.) Each bijective and increasing map φ from $\{q \in \mathbb{Q} \mid 0 \leqslant q < 1\}$ into $\{q \in \mathbb{Q} \mid 0 \leqslant q\}$ determines a bijection $\varphi(\cdot)$ from the set of all $f \in \mathsf{F}(L)$ such that $\mathbf{0} \leqslant f$ into the set of all $f \in \mathsf{F}(L)$ such that $\mathbf{0} \leqslant \mathsf{F}(L) < \mathbf{1}$, defined by:

$$(\varphi f)(r,-) = \begin{cases} 1 & \text{if } r < 0 \\ f(\varphi(r),-) & \text{if } 0 \leqslant r < 1 \\ 0 & \text{if } r \geqslant 1, \end{cases} \quad \text{and} \quad (\varphi f)(-,r) = \begin{cases} 0 & \text{if } r < 0 \\ f(-,\varphi(r)) & \text{if } 0 \leqslant r < 1 \\ 1 & \text{if } r \geqslant 1. \end{cases}$$

Indeed,

$$\begin{aligned} \mathbf{0} \leqslant f \quad &\Leftrightarrow \quad f(-,0) = 0 \\ &\Leftrightarrow \quad (\varphi f)(-,0) = 0 \quad \text{and} \quad \iota(\varphi f, \mathbf{1}) = \varphi f(-,1) = 1 \\ &\Leftrightarrow \quad \mathbf{0} \leqslant \varphi f \quad \text{and} \quad \varphi f < \mathbf{1}. \end{aligned}$$

Also, $\iota(\mathbf{0}, f) = \iota(\mathbf{0}, \varphi f)$ and so $\mathbf{0} < f$ iff $\mathbf{0} < \varphi f$. Finally, $f \in \mathsf{LSC}(L)$ iff $\varphi f \in \mathsf{LSC}(L)$, and $f \in \mathsf{USC}(L)$ iff $\varphi f \in \mathsf{USC}(L)$. We shall denote the inverse of $\varphi(\cdot)$ by $\varphi^{-1}(\cdot)$.

The following result was proved in [11] and shown to be a (pointfree) generalization of Dowker's Theorem above.

Proposition 6.3. *The following are equivalent for a normal frame L:*

(a) *L is countably paracompact.*
(b) *For each $g \in \mathsf{LSC}(L)$ with $\mathbf{0} < g \leqslant \mathbf{1}$, there exists an $h \in \mathsf{C}(L)$ such that $\mathbf{0} < h \leqslant g$.*

We can now extend it in the following way:

Theorem 6.4 *(Pointfree Dowker insertion theorem). A frame L is normal and countably paracompact if and only if, given $f \in \mathsf{USC}(L)$ and $g \in \mathsf{LSC}(L)$ with $f < g$, there exists an $h \in \mathsf{C}(L)$ such that $f < h < g$.*

Proof. Assume L is a normal and countably paracompact frame and consider $f \in \mathsf{USC}(L)$ and $g \in \mathsf{LSC}(L)$ with $f < g$. By Corollary 4.5(c) and Lemma 6.1(c), $\mathbf{0} < g - f \in \mathsf{LSC}(L)$. Let φ be a bijective and increasing map from $\{q \in \mathbb{Q} \mid 0 \leqslant q < 1\}$ into $\{q \in \mathbb{Q} \mid 0 \leqslant q\}$. By Remark 6.2 we have that $\mathbf{0} < \varphi(g-f) \leqslant \mathbf{1}$ and $\varphi(g-f) \in \mathsf{LSC}(L)$. Therefore by Proposition 6.3 there exists a continuous $k > \mathbf{0}$ such that $\mathbf{0} < k \leqslant \varphi(g-f)$ and so $\mathbf{0} < \varphi^{-1}(k) \leqslant g - f$. Then $f + \frac{\varphi^{-1}(k)}{2} \leqslant g - \frac{\varphi^{-1}(k)}{2}$ and by Katětov–Tong insertion there is a continuous h such that

$$f + \frac{\varphi^{-1}(k)}{2} \leqslant h \leqslant g - \frac{\varphi^{-1}(k)}{2}.$$

This is the required continuous h since $k > \mathbf{0}$ implies $g - h \geqslant \frac{\varphi^{-1}(k)}{2} > \mathbf{0}$ and $h - f \geqslant \frac{\varphi^{-1}(k)}{2} > \mathbf{0}$ and hence $f < h < g$ (by Lemma 6.1(c) and (d) and Remark 6.2).

Conversely, it suffices to show that L is normal (and to use then Proposition 6.3). Let $a \vee b = 1$ in L. We need to prove that there exist $u, v \in L$ satisfying $u \wedge v = 0$ and $a \vee u = b \vee v = 1$. Consider $f = \chi_{\mathfrak{c}(a)}$ and $g = \chi_{\mathfrak{o}(b)} + \mathbf{1}$. We know that f is upper semicontinuous and g is lower semicontinuous. Further,

$$\begin{aligned} \iota(f,g) &\geqslant \left(\chi_{\mathfrak{c}(a)}\left(-,\frac{1}{2}\right) \wedge (\chi_{\mathfrak{o}(b)} + \mathbf{1})\left(\frac{1}{2},-\right)\right) \vee \left(\chi_{\mathfrak{c}(a)}\left(-,\frac{3}{2}\right) \wedge (\chi_{\mathfrak{o}(b)} + \mathbf{1})\left(\frac{3}{2},-\right)\right) \\ &= \big(\mathfrak{c}(a) \wedge 1\big) \vee \big(1 \wedge \mathfrak{c}(b)\big) = 1, \end{aligned}$$

that is, $f < g$. Hence, by hypothesis, there is a continuous h satisfying $f < h < g$. In particular, $h(1,-) = \mathfrak{c}(u)$ and $h(-,1) = \mathfrak{c}(v)$ for some $u, v \in L$. Clearly, $u \wedge v = 0$. Moreover, from $f < h$ it follows that

$$1 = \bigvee_{p \in \mathbb{Q}} \big(f(-,p) \wedge h(p,-)\big) = \bigvee_{0 < p \leqslant 1} \big(\mathfrak{c}(a) \wedge h(p,-)\big) \vee \bigvee_{p > 1} h(p,-) \leqslant \mathfrak{c}(a) \vee \mathfrak{c}(u),$$

which shows that $a \vee u = 1$. Similarly, $h < g$ implies $b \vee v = 1$. Hence L is normal. $\square$

Remark 6.5. Theorem 6.4 applied to $L = \mathcal{O}X$, for a normal and countably paracompact space X, yields the result of Dowker quoted earlier in a very straightforward way:

Let $f, g : X \to \mathbb{R}$ such that $f < g$, f is upper semicontinuous and g is lower semicontinuous. To begin with, observe that $g : X \to \mathbb{R}$ induces a lower semicontinuous $\widetilde{g} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(\mathcal{O}X)$ via the scale $(\mathfrak{c}(g^{-1}(]p,+\infty[)) \mid p \in \mathbb{Q})$ (see [12, Section 6] for the details). By Lemma 2.2,

$$\widetilde{g}(p,-)=\bigvee_{r>p}\mathfrak{c}\big(g^{-1}\big(]r,+\infty[\big)\big)=\mathfrak{c}\big(g^{-1}\big(]p,+\infty[\big)\big)\quad\text{for each } p\in\mathbb{Q}.$$

Similarly, $f : X \to \mathbb{R}$ induces an upper semicontinuous $\widetilde{f} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(\mathcal{O}X)$ (via the scale $(\mathfrak{o}(f^{-1}(]-\infty,q[)) \mid q \in \mathbb{Q}))$ satisfying

$$\widetilde{f}(-,q)=\bigvee_{s<q}\mathfrak{c}\big(f^{-1}\big(]-\infty,s[\big)\big)=\mathfrak{c}\big(f^{-1}\big(]-\infty,q[\big)\big)\quad\text{for each } q\in\mathbb{Q}.$$

Further

$$\begin{aligned}\widetilde{f}<\widetilde{g}\quad&\Leftrightarrow\quad \iota(\widetilde{f},\widetilde{g})=1\quad\Leftrightarrow\quad \bigvee_{p\in\mathbb{Q}}\big(\mathfrak{c}\big(f^{-1}\big(]-\infty,p[\big)\big)\wedge\mathfrak{c}\big(g^{-1}\big(]p,+\infty[\big)\big)\big)=1\\ &\Leftrightarrow\quad \mathfrak{c}\Big(\bigcup\nolimits_{p\in\mathbb{Q}}\big(f^{-1}\big(]-\infty,q[\big)\cap g^{-1}\big(]p,+\infty[\big)\big)\Big)=1\\ &\Leftrightarrow\quad \bigcup\nolimits_{p\in\mathbb{Q}}\big(f^{-1}\big(]-\infty,q[\big)\cap g^{-1}\big(]p,+\infty[\big)\big)=X\\ &\Leftrightarrow\quad f(x)<g(x)\quad\text{for every } x\in X.\end{aligned}$$

Therefore by Theorem 6.4 there is a continuous $\widetilde{h}$ such that $\widetilde{f} < \widetilde{h} < \widetilde{g}$. It is now a straightforward exercise to conclude that the $h : X \to \mathbb{R}$ defined by $h(x) \in]p,q[$ iff $x \in \widetilde{h}(p,q)$ (for any $p,q \in \mathbb{Q}$) is a continuous map satisfying $f < h < g$.

The following result was proved in [11].

Proposition 6.6. *A frame L is perfectly normal if and only if it is normal and given $g \in \mathsf{LSC}(L)$ with $\mathbf{0} \leqslant g \leqslant \mathbf{1}$ there exists an $h \in \mathsf{C}(L)$ such that $\mathbf{0} \leqslant h \leqslant g$ and $\iota(\mathbf{0},h) = h(0,-) = g(0,-) = \iota(\mathbf{0},g)$.*

We can also extend it as follows:

Theorem 6.7 *(Pointfree Michael insertion theorem). A frame L is perfectly normal if and only if, given $f \in \mathsf{USC}(L)$ and $g \in \mathsf{LSC}(L)$ with $f \leqslant g$, there exists an $h \in \mathsf{C}(L)$ such that $f \leqslant h \leqslant g$ and $\iota(f,h) = \iota(h,g) = \iota(f,g)$.*

Proof. We only need to prove necessity. Assume L is a perfectly normal frame and consider $f \in \mathsf{USC}(L)$ and $g \in \mathsf{LSC}(L)$ wit $f \leqslant g$. By Corollary 4.5(c) and Lemma 6.1(c), $\mathbf{0} \leqslant g - f \in \mathsf{LSC}(L)$. Let φ be a bijective and increasing map from $\{q \in \mathbb{Q} \mid 0 \leqslant q < 1\}$ into $\{q \in \mathbb{Q} \mid 0 \leqslant q\}$. By Remark 6.2 we have that $\mathbf{0} \leqslant \varphi(g-f) \leqslant \mathbf{1}$ and $\varphi(g-f) \in \mathsf{LSC}(L)$. Therefore by Proposition 6.6 there exists a continuous k such that $\mathbf{0} \leqslant k \leqslant \varphi(g-f)$ and $\iota(\mathbf{0},k) = \iota(\mathbf{0},\varphi(g-f))$. It follows that $\mathbf{0} \leqslant \varphi^{-1}(k) \leqslant g-f$ and so $f + \frac{\varphi^{-1}(k)}{2} \leqslant g - \frac{\varphi^{-1}(k)}{2}$. Then by the pointfree Katětov–Tong insertion result (quoted at the beginning of the present section) there is a continuous h such that

$$f+\frac{\varphi^{-1}(k)}{2}\leqslant h\leqslant g-\frac{\varphi^{-1}(k)}{2}.$$

This is the required continuous h since, by Lemma 6.1 and Remark 6.2, $\iota(\mathbf{0},k) = \iota(\mathbf{0},\frac{k}{2}) = \iota(\mathbf{0},\frac{\varphi^{-1}(k)}{2}) \leqslant \iota(\mathbf{0},h-f) = \iota(f,h) \leqslant \iota(f,g) = \iota(\mathbf{0},g-f) = \iota(\mathbf{0},\varphi(g-f)) = \iota(\mathbf{0},k)$, hence $\iota(f,h) = \iota(f,g)$. Similarly, $\iota(\mathbf{0},k) \leqslant \iota(\mathbf{0},g-h) = \iota(h,g) \leqslant \iota(f,g) = \iota(\mathbf{0},k)$ and thus $\iota(h,g) = \iota(f,g)$. $\square$

Just as in Remark 6.5, it can be shown that Theorem 6.7 applied to $\mathcal{O}X$ for a perfectly normal space X, yields the original result of Michael for spaces.

7. When is every real function continuous?

Since the sublocale lattice $\mathcal{S}(L)$ of a frame L is also a frame, the second sublocale lattice $\mathcal{S}^2(L)$ and an embedding $\mathcal{S}(L) \hookrightarrow \mathcal{S}^2(L)$ exist. In fact, for each frame L there is a tower

$$L\hookrightarrow\mathcal{S}(L)\hookrightarrow\mathcal{S}^2(L)\hookrightarrow\mathcal{S}^3(L)\hookrightarrow\cdots\tag{2}$$

of sublocale lattices $S^\alpha(L)$ ([14,20]) over all ordinals α. Each $S^\alpha(L)$ is the *α-dissolution* of L and a frame L is called *α-soluble* [18] if its α-dissolution is Boolean.

Not much is known about the tower (2). One of the few known facts is that it can continue into the transfinite and, in some cases, may never stop. It certainly stops when a Boolean frame is reached (because a frame is 0-soluble iff it is Boolean [14]). Furthermore, a frame is 1-soluble iff it is scattered (equivalently, if all its (Boolean) sublocales are complemented) and it is 2-soluble iff each sublocale $S \neq 0$ of L has a nonzero complemented Boolean sublocale [18].

Applying the functor $\mathcal{R}$ to (2) we get the tower

$$\mathcal{R}(L) \hookrightarrow \mathsf{F}(L) = \mathcal{R}\big(\mathcal{S}(L)\big) \hookrightarrow \mathsf{F}\big(\mathcal{S}(L)\big) = \mathcal{R}\big(\mathcal{S}^2(L)\big) \hookrightarrow \mathsf{F}\big(\mathcal{S}^2(L)\big) = \mathcal{R}\big(\mathcal{S}^3(L)\big) \hookrightarrow \cdots$$

and it seems then natural to ask, for each ordinal α, which frames L satisfy the identity $\mathsf{F}(\mathcal{S}^\alpha(L)) = \mathcal{R}(\mathcal{S}^\alpha(L))$.

For each ordinal α, the α-soluble frames are precisely the frames L for which

$$\mathsf{F}\big(\mathcal{S}^\alpha(L)\big) = \mathcal{R}\big(\mathcal{S}^\alpha(L)\big).$$

Indeed: if $\mathcal{S}^\alpha(L)$ is Boolean then $\mathcal{S}^{\alpha+1}(L) = \mathcal{S}^\alpha(L)$ and so $\mathsf{F}(\mathcal{S}^\alpha(L)) = \mathcal{R}(\mathcal{S}^{\alpha+1}(L)) = \mathcal{R}(\mathcal{S}^\alpha(L))$; conversely, if $\mathsf{F}(\mathcal{S}^\alpha(L)) = \mathcal{R}(\mathcal{S}^\alpha(L))$, then for each complemented sublocale S of $\mathcal{S}^\alpha(L)$ the characteristic function χ_S belongs to $\mathsf{F}(\mathcal{S}^\alpha(L)) = \mathcal{R}(\mathcal{S}^\alpha(L))$, from which it follows that S is a clopen sublocale; but by zero-dimensionality any sublocale is a join of complemented sublocales thus any sublocale of $\mathcal{S}^\alpha(L)$ is clopen and, consequently, $\mathcal{S}^\alpha(L)$ is Boolean.

In particular, Boolean frames are precisely the frames L where $\mathsf{F}(L) = \mathcal{R}(L)$, that is, where *every real function on L is continuous*. In this case the insertion theorems of the preceding section and of the papers [12] and [9] trivialize (and L is immediately extremally disconnected, monotonically normal, perfectly normal, completely normal, etc.).

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