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Homothetic interval orders☆

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Abstract

We give a characterization of the non-empty binary relations $\succ$ on a $\mathbb{N}^*$-set A such that there exist two morphisms of $\mathbb{N}^*$-sets $u_1, u_2 : A \to \mathbb{R}_+$ verifying $u_1 \leqslant u_2$ and $x \succ y \Leftrightarrow u_1(x) > u_2(y)$. They are called *homothetic interval orders*. If $\succ$ is a homothetic interval order, we also give a representation of $\succ$ in terms of one morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ and a map $\sigma : u^{-1}(\mathbb{R}_+^*) \times A \to \mathbb{R}_+^*$ such that $x \succ y \Leftrightarrow \sigma(x, y)u(x) > u(y)$. The pairs (u_1, u_2) and (u, σ) are "uniquely" determined by $\succ$, which allows us to recover one from each other. We prove that $\succ$ is a semiorder (resp. a weak order) if and only if σ is a constant map (resp. $\sigma = 1$). If moreover A is endowed with a structure of commutative semigroup, we give a characterization of the homothetic interval orders $\succ$ represented by a pair (u, σ) so that u is a morphism of semigroups.

Keywords: $\mathbb{N}^*$-set; Semigroup; Weak order; Semiorder; Interval order; Intransitive indifference; Independence; Homothetic structure; Representation

0. Introduction

Let us start with an example, which has been our main source of inspiration for this work. Consider a two-armed-balance, the two arms of which not necessarily being of the same length; such a balance is said to be *biased*. Let P_1 and P_2 denote its two pans. If the arms are not of the same length, we assume that P_1 is located at the end of the shorter arm. Suppose we are also given a set A of objects to put on P_1 and P_2. We define a binary relation $\succ$ on A as follows: $x \succ y$ if the balance tilts towards x when we put x on P_1 and y on P_2. This relation is always asymmetric and transitive, but it is negatively transitive if and only if the two arms are of the same length. However, we can observe it is always *strongly transitive*: $x \succ y \succsim z \succ t \Rightarrow x \succ t$ where $y \succsim z \Leftrightarrow z \nsucc y$. In particular, $\succ$ is an *interval order* (cf. Section 1). Furthermore, suppose that A is endowed with a structure of $\mathbb{N}^*$-*set*; i.e. suppose there exists a map $\mathbb{N}^* \times A \to A$, $(m, x) \mapsto mx$ such that $m(nx) = (mn)x$ and $1x = x$. Then the relation $\succ$ verifies the following property of *homothetic independence*: $x \succ y \Leftrightarrow (mx \succ my,\ \forall m \in \mathbb{N}^*)$. We can continue to identify the properties satisfied by $\succ$. That naturally brings us to introduce the notion of *homothetic structure* (cf. Section 2). A homothetic structure is by definition a $\mathbb{N}^*$-set A

☆ We are very grateful to Duncan Luce for his comments on the paper. As he pointed out to us, the notion of "homothetic interval order" is not standard. Roughly speaking, it is a kind of structure of extensive multiples [11] such that the underlying order is an interval order (and not necessarily a weak order).

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endowed with a binary relation $\succ$ verifying five properties of compatibility, the most striking two being the homothetic independence introduced before and the following property (Archimedean condition): if $x \succ y$, then $\exists m \in \mathbb{N}^*$ such that $mx \succ (m+1)y$. A homothetic structure $(A, \succ)$ is called a *homothetic interval order* if the relation $\succ$ is asymmetric and strongly transitive. The main goal of this paper is to give a characterization of the *homothetic interval orders* via their representations in $\mathbb{R}_+$.

So let $(A, \succ)$ be a non-empty homothetic interval order. Let $\sim$ denotes the *indifference relation* on A defined by $x \sim y \Leftrightarrow x \not\succ y \not\succ x$. If $(A, \succ)$ is obtained from a biased balance as above, then we know there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ (the mass) and a real number $\alpha \in]0, 1]$ (the ratio of the shortest arm to the longest one) such that $x \succ y \Leftrightarrow \alpha u(x) > u(y)$. This is the kind of result we are looking for here.

Let us begin with the simplest case: $\succ$ is a *homothetic weak order*, that is the relation $\succ$ is negatively transitive; or (equivalently) the indifference $\sim$ is an equivalence relation. Then we prove (Proposition 4.1) that there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$, unique up to multiplication by a positive scalar, such that $x \succ y \Leftrightarrow u(x) > u(y)$. Let us point out that no countable hypothesis on the quotient-set $A/\sim$ is needed here.

Now let us return to the general case. So as to simplify this introduction, we assume that $\forall (x, y) \in A \times A$, the set $\mathscr{P}_{x,y} = \{mn^{-1} : (m, n) \in \mathbb{N}^* \times \mathbb{N}^*, mx \succ ny\}$ is *non-empty*. Hence we can put $s_{x,y} = \inf_{\mathbb{R}} \mathscr{P}_{x,y} \in \mathbb{R}_+$. This invariant is one the most important tool of this work; we prove in particular that $x \succ y \Leftrightarrow s_{x,y} < 1$. Let $\mathscr{E}(A)$ be the set of pairs (u, σ) made up of a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+^*$ and a map $\sigma : A/\mathbb{N}^* \times A/\mathbb{N}^* \to \mathbb{R}_+^*$ such that $\sigma(x, y)\sigma(z, t) = \sigma(x, t)\sigma(z, y)$ and $\sigma(x, x) \leqslant 1$. The main result of this paper (Propositions 6.1 and 7.2) is stated as follows.

Main result: The four following conditions are equivalent:

(1) there exists a pair $(u, \sigma) \in \mathscr{E}(A)$ such that $x \succ y \Leftrightarrow \sigma(x, y)u(x) > u(y)$;
(2) there exist a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+^*$ and a map $\gamma : A/\mathbb{N}^* \to]0, 1]$ such that $x \succ y \Leftrightarrow \gamma(x)u(x) > \gamma(y)^{-1}u(y)$;
(3) there exist two morphisms of $\mathbb{N}^*$-sets $u_1, u_2 : A \to \mathbb{R}_+^*$ such that $u_1 \leqslant u_2$ and $x \succ y \Leftrightarrow u_1(x) > u_2(y)$;
(4) $\succ$ is a homothetic interval order.

Moreover, if $\succ$ is a homothetic interval order, then the pair (u, γ) of (2) is *unique* up to multiplication of u by a positive scalar; and the pair (u_1, u_2) of (3) is *unique* up to multiplication by a positive scalar (i.e., up to replacing it by $(\lambda u_1, \lambda u_2)$ for a constant $\lambda > 0$).

The link between the two characterizations (2) and (3) is precisely described (Corollary 7.4): if (u, γ) is a pair verifying (2), then the pair $(u_1, u_2) = (\gamma u, \gamma^{-1} u)$ clearly verifies (3). Conversely, if (u_1, u_2) is a pair verifying (3), then the pair $(u, \gamma) = ((u_1 u_2)^{1/2}, (u_1 \bar{u}_2)^{1/2})$ verifies (2); where $\bar{u}_2 : A \to \mathbb{R}_+^*$ denotes the map defined by $\bar{u}_2(x) = u_2(x)^{-1}$.

For $i = 0, 1, 2$, we define as follows a binary relation $\succ_i$ on A:

- $x \succ_0 y \Leftrightarrow s_{x,y} < s_{y,x}$,
- $x \succ_1 y \Leftrightarrow (mx \succsim z \succ my,\ \exists (z, m) \in A \times \mathbb{N}^*)$,
- $x \succ_2 y \Leftrightarrow (mx \succ z \succsim my,\ \exists (z, m) \in A \times \mathbb{N}^*)$.

Suppose $\succ$ is a homothetic interval order. Then we prove that for $i = 0, 1, 2$, $\succ_i$ is a *homothetic weak order*; i.e., a homothetic structure which is a weak order. Moreover, for any (i.e., for one) pair (u, γ) verifying (2), u represents $\succ_0$; and for any (i.e., for one) pair (u_1, u_2) verifying (3), u_i represents $\succ_i$ $(i = 1, 2)$. Let $\gamma_\succ : A/\mathbb{N}^* \to]0, 1]$ denote the map defined by $\gamma_\succ = \gamma$ for any (i.e., for one) pair (u, γ) verifying (2). We prove (Proposition 7.5) that the following conditions are equivalent:

- $\gamma_\succ$ is a constant map;
- $\succ_1 = \succ_2$;
- $\succ$ is a semiorder.

We are also interested in the case of a commutative semigroup A (Sections 5 and 8). A binary relation $\succ$ on A is said to be $\circ$-*independent* if $x \succ y \Leftrightarrow (x \circ z \succ y \circ z,\ \forall z \in A)$. We introduce a weaker notion of compatibility between $\circ$ and $\succ$ called $\circ$-*pseudoindependence* (cf. Section 5). We prove in particular (Corollary 8.3) that if $(A, \circ)$ is a commutative semigroup endowed with a non-empty homothetic interval order $\succ$, then the weak order $\succ_0$ is $\circ$-independent if and only if $\succ$ is a $\circ$-pseudoindependent semiorder; we also remark (Proposition 8.2) that $\succ$ is $\circ$-pseudoindependent if and only if for $i = 1, 2$, $\succ_i$ is $\circ$-independent. Note that it was already known that the Archimedean condition—called *super-Archimedean* in [4]—is equivalent to the additive representability of a weakly ordered positive semigroup (see [4,6]). More generally, let us point out that some of our axioms are very similar to other conditions that have been already introduced elsewhere to ensure the existence of an additive utility representation.

Let us make a few remarks about the nature of the results explained above. Characterization (3) with the help of two maps u_1 and u_2, is the usual way to represent interval orders ([8] Theorem 2.7); in fact, the homothetic weak orders $\succ_1$ and $\succ_2$ are simple variants of the weak orders associated with $\succ$ by Fishburn ([8, Theorem 2.6]. Novelty resides in that the pair of morphisms (u_1, u_2) is *unique* up to mutiplication by a positive constant. The advantage provided by the characterization (2) is to put in a prominent position the twisting factor $\gamma_\succ : A/\mathbb{N}^* \to]0, 1]$, conveying explicitly the guiding line of our thinking: to consider a homothetic interval order $\succ$ as a deformation of its associated homothetic weak order $\succ_0$. This characterization leads us to contemplate a classification of homothetic interval orders in terms of their invariant $\gamma_\succ$, a task left to a future work. Let us mention that this paper is a generalization of [12], in which we deal with the particular case of a $\mathbb{N}^*$-set A so that $\forall(x, y) \in A^2$, $\exists(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $mx = ny$.

Intransitive indifference in preference theory reflects the "vagueness" of individual decisions. In the past 50 years, numerous authors as Luce [14], Scott and Suppes [17], Fishburn [8–10], have suggested that a preference relation on a set A could be represented by a pair (u, v) of real functions on A. Such a pair (u, v) provides, for each $x \in A$, a non-empty interval $[u(x), v(x)]$ which can be interpreted as the lower and upper bounds of the utility perceived of the object x. Hence the name "interval order" which goes back to Fishburn [10], and generalizes the concept of semiorder introduced by Luce [14]. The representation theorems—that is, lists of conditions ensuring the existence of the pair (u, v)—given by the previous authors and many others (e.g. [18,2,3,7,5,15,16,1]) are quite similar to the ones contained here. But all of those theorems need an "extra hypothesis" of topological nature on the set A (e.g., a finite or countable condition; a connected topological structure; a mixture space structure), whereas our extra hypothesis—A is a $\mathbb{N}^*$-set—is of algebraic nature. This naturally leads us to develop a purely algebraic treatment of the problem. In fact, the only topological property we use—the density of $\mathbb{Q}$ in $\mathbb{R}$—concerns the space of the representation. As a direct consequence of the construction, we obtain the unicity of the pair (u, v). Moreover, our method actually extends to the study of any *positive homothetic order* (i.e., a binary operation on a $\mathbb{N}^*$-set A verifying the conditions ($_h$I) and ($_h$P) of Section 2) as we will show in [13].

Notations, writing conventions: The symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ denote respectively the field of real numbers, the field of rational numbers, and the ring of integers. For every part $X \subset \mathbb{R}$ and every $r \in \mathbb{R}$, we put $X_{>r} = \{x \in X : x > r\}$ and $X_{\geqslant r} = \{x \in X : x \geqslant r\}$. Let $\mathbb{R}_+ = \mathbb{R}_{\geqslant 0}$, $\mathbb{R}_+^* = \mathbb{R}_{>0}$, $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$ and $\mathbb{N}^* = \mathbb{Z}_{>0}$; and for every part $X \subset \mathbb{R}_+$, let $X^* = X \cap \mathbb{R}_+^*$.

Let $\mathbb{R}_+^\infty = \mathbb{R}_+ \coprod \{\infty\}$ where ∞ denotes an arbitrary element not belonging to $\mathbb{R}$. The standard strict order $>$ on $\mathbb{R}_+$ extends naturally to a strict order on $\mathbb{R}_+^\infty$, still denoted $>$: for $x \in \mathbb{R}_+$, we put $\infty > x$, $x \not> \infty$ and $\infty \not> \infty$. And for $x, y \in \mathbb{R}_+^\infty$, we put $x \geqslant y \Leftrightarrow y \not> x$. For every part $X \subset \mathbb{R}_+^\infty$, we put

$$\inf_{\mathbb{R}_+^\infty} X = \begin{cases} \inf_{\mathbb{R}_+}(X \cap \mathbb{R}_+) & \text{if } X \cap \mathbb{R}_+ \neq \emptyset, \\ \infty & \text{if not.} \end{cases}$$

Let (writing conventions) $\infty^{-1} = 0$, $0^{-1} = \infty$ and $\emptyset^{-1} = \emptyset$. And for all non-empty parts $X \subset \mathbb{R}_+^\infty$ and $Y, Z \subset \mathbb{R}_+$, we put $X^{-1} = \{q^{-1} : q \in X\} \subset \mathbb{R}_+^\infty$ and $YZ = \{yz : y \in Y, z \in Z\} \subset \mathbb{R}_+$.

At last, if A is a set, for $n \in \mathbb{Z}_{\geqslant 1}$, we put $A^n = A \times \cdots \times A$ (n times).

1. Let A be a set endowed with a binary relation $\succ$. Let $\sim$ and $\succsim$ denote the binary relations on A defined as follows:

- $x \sim y \Leftrightarrow x \not\succ y \not\succ x$,
- $x \succsim y \Leftrightarrow (x \succ y \text{ or } x \sim y)$.

The relation $\succ$ is said to be:

(A) *asymmetric* if $\forall(x, y) \in A^2$, we have $x \succ y \Rightarrow y \not\succ x$;
(T) *transitive* if $\forall(x, y, z) \in A^3$, we have $x \succ y \succ z \Rightarrow x \succ z$;

(ST) *strongly transitive* if it satisfies (A) and $\forall(x, y, z, t) \in A^4$, we have $x \succ y \succsim z \succ t \Rightarrow x \succ t$;
(NT) *negatively transitive* if it satisfies (A) and the relation $\succsim$ is transitive;
(S) *strict* if $\forall(x, y) \in A^2$, we have $x \succsim y \succsim x \Rightarrow x = y$.

The relation $\succ$ satisfies (A) if and only if $\forall(x, y) \in A^2$, we have $x \not\succ y \Leftrightarrow y \succsim x$. Then we deduce that if $\succ$ satisfies (A), then it satisfies (NT) if and only if the two following equivalent properties are true ($x, y, z \in A$):

- $\forall(x, y, z) \in A^3$, we have $x \succ y \succsim z \Rightarrow x \succ z$;
- $\forall(x, y, z) \in A^3$, we have $x \succsim y \succ z \Rightarrow x \succ z$.

Moreover, we have the (well-known) implications:

$$(NT) \Rightarrow (ST) \Rightarrow (T) \,\&(A).$$

1.1. Remarks. Suppose the relation $\succ$ satisfies (A). Then we have:

- $\succ$ satisfies (ST) if and only if $\forall(x, y, z, t) \in A^4$, we have $(x \succ y \text{ and } z \succ t) \Rightarrow (x \succ t \text{ or } z \succ y)$;
- $\succ$ satisfies (NT) if and only if $\forall(x, y, z, t) \in A^4$, we have $x \succsim y \succ z \succsim t \Rightarrow x \succ t$;
- $\succ$ satisfies (S) if and only if $\forall(x, y) \in A^2$, we have $x \neq y \Rightarrow (x \succ y \text{ or } y \succ x)$;
- if $\succ$ satisfies (T), then it satisfies (NT) if and only if $\sim$ is an equivalence relation.

Using the terminology of Fishburn [8], we will say that the relation $\succ$ is a:

- *interval order* if it satisfies (ST);
- *semiorder* if it is an interval order and $\forall(x, y, z, t) \in A^4$, we have $x \succ y \succ z \Rightarrow (t \succ z \text{ or } x \succ t)$;
- *weak order* if it satisfies (NT);
- *strict order* if it satisfies (NT) and (S).

It is easy to check that the definition of interval order given above coincides with the one of [8]. Thus, we have the implications:

strict order $\Rightarrow$ weak order $\Rightarrow$ semiorder $\Rightarrow$ interval order.

1.2. Definition. Let A be a set endowed with a *non-empty* binary relation $\succ$ (i.e., satisfying: $\exists(x, y) \in A^2$ such that $x \succ y$; in particular, A is non-empty), and let u be a map $A \to \mathbb{R}_+$. We say that u represents $\succ$ if $\forall(x, y) \in A^2$, we have $x \succ y \Leftrightarrow u(x) > u(y)$.

2. In this section, we define the notion of a G-structure on a G-set A. In the next sections, we will use this notion only for $G = \mathbb{N}^*$, but for convenience writing, and because our results naturally extend to other settings (e.g. $G = \mathbb{R}_+^*$), we introduce it for any commutative monoid G.

So let G be a commutative monoid (written multiplicatively); i.e., a set endowed with a map $G \times G \to G$, $(g, g') \mapsto gg'$ and an element $1 = 1_G \in G$, such that $\forall(g, g', g'') \in G^3$, we have $(gg')g'' = g(g'g'')$, $gg' = g'g$ and $1g = g$. We call *G-set* a set A endowed with a map $G \times A \to A$, $(g, x) \mapsto gx$ such that $\forall(g, g', x) \in G^2 \times A$, we have $g(g'x) = (gg')x$ and $1x = x$. If A is a G-set, we denote A/G the quotient-set of A by the equivalence relation $\sim_G$ on A defined by

- $x \sim_G y$ if and only if $\exists(g, g') \in G^2$ such that $gx = g'y$.

Let G be a commutative monoid, and let A be a G-set endowed with a binary relation $\succ$. The relation $\succ$ is said to be:

($_G$I) *G-independent* if $\forall(x, y, g) \in A^2 \times G$, we have $x \succ y \Leftrightarrow gx \succ gy$;
($_G$SS) *G-strongly separable* if $\forall(x, y, z, t) \in A^4$ such that $x \succ y$ and $z \succ t$, $\exists(g, g', g'') \in G^3$ such that $gx \succ g'z \succsim g''z \succ gy$;
($_G$C) *G-coherent* if $\forall(x, y, z) \in A^3$ such that $x \succ y \succsim z$, $\exists(g, g') \in G^2$ such that $gx \succ g'z$.

From Section 1, we know that if the relation $\succ$ satisfies (NT), then it satisfies ($_G$C). Suppose moreover that G is endowed with a *weak order* $>$. Then the relation $\succ$ is said to be:

($_G$A) *G-Archimedean* if $\forall (x, y) \in A^2$ such that $x \succ y$, $\exists (g, g') \in G^2$ such that $g' > g$ and $gx \succ g'y$;
($_G$P) *G-positive* if $\forall (x, y, g, g') \in A^2 \times G^2$ such that $g > g'$, we have $x \succ y \Rightarrow gx \succ g'y$.

2.1. Remark. Let G be a commutative monoid endowed with a weak order $>$, and let A be a G-set endowed with a binary relation $\succ$. Let ($_G$NI) (resp. ($_G$NP)) denote the property obtained by replacing the symbol $\succ$ by the symbol $\succsim$ in ($_G$I) (resp. in ($_G$P). It is easy to prove that if $\succ$ satisfies (A), ($_G$I), ($_G$A) and ($_G$P), then $\succsim$ satisfies ($_G$NI) and ($_G$NP). □

2.2. Definition. Let G be a commutative semigroup endowed with a weak order $>$. A binary relation $\succ$ on a G-set A is called a:

- *G-structure* if it satisfies ($_G$I), ($_G$SS), ($_G$C), ($_G$A) and ($_G$P);
- *G-strict order* if it is a G-structure and a strict order;
- *G-weak order* if it is a G-structure and a weak order;
- *G-semiorder* if it is a G-structure and a semiorder.
- *G-interval order* if it is a G-structure and an interval order.

The set $\mathbb{N}^*$ is a monoid for the multiplication, and the standard strict order $>$ on $\mathbb{R}_+$ induces by restriction a strict order on $\mathbb{N}^*$. To ease the notation, we will replace the index $\mathbb{N}^*$ in ($_{\mathbb{N}^*}$I), ($_{\mathbb{N}^*}$SS) (etc.), by an index "h" for *homothetic*; and we will call *homothetic structure* (resp. *homothetic strict order*, etc.) a $\mathbb{N}^*$-structure (resp. a $\mathbb{N}^*$-strict order, etc.). In this paper, we intend to give a characterization—by means of their representations in $\mathbb{R}_+$—of the $\mathbb{N}^*$-sets endowed with a non-empty homothetic interval order. We will also give a characterization of the $\mathbb{N}^*$-sets endowed with a non-empty homothetic semiorder (resp. a non-empty homothetic weak order, a non-empty homothetic strict order).

3. Let A be a $\mathbb{N}^*$-set endowed with a binary relation $\succ$. For $x, y \in A$, we denote $\mathscr{P}_{x,y} = \mathscr{P}^{\succ}_{x,y}$ and $\mathscr{Q}_{x,y} = \mathscr{Q}^{\succ}_{x,y}$ the subsets of $\mathbb{Q}_{>0}$ defined by

$$\mathscr{P}_{x,y} = \{mn^{-1} : (m, n) \in (\mathbb{N}^*)^2, mx \succ ny\},$$

$$\mathscr{Q}_{x,y} = \{mn^{-1} : (m, n) \in (\mathbb{N}^*)^2, mx \succsim ny\};$$

and we put $s_{x,y} = \inf_{\mathbb{R}_+^\infty} \mathscr{P}_{x,y}$ and $r_{x,y} = \inf_{\mathbb{R}_+^\infty} \mathscr{Q}_{x,y}$. If $\succ$ satisfies (A), then $\forall (x, y) \in A^2$, we have the partitions of $\mathbb{Q}_{>0}$:

$$\mathbb{Q}_{>0} = \mathscr{P}_{x,y} \coprod \mathscr{Q}^{-1}_{y,x} = \mathscr{P}^{-1}_{y,x} \coprod \mathscr{Q}_{x,y}. \tag{3.1}$$

3.2. Lemma. *Let A be $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$ satisfying* ($_h$A) *and* ($_h$P). *Then $\forall (x, y) \in A^2$, we have $\mathscr{P}_{x,y} = \mathbb{Q}_{>s_{x,y}}$.*

Proof. Let $x, y \in A$, and put $s = s_{x,y}$. If $\mathscr{P}_{x,y} = \emptyset$, then there is nothing to prove. Thus, we may (and do) assume that $\mathscr{P}_{x,y} \neq \emptyset$. From ($_h$P), if $q \in \mathscr{P}_{x,y}$, then $\mathbb{Q}_{\geqslant q} \subset \mathscr{P}_{x,y}$. If $q \in \mathbb{Q}_{>s}$, then by definition of s, $\exists q' \in \mathscr{P}_{x,y}$ such that $s \leqslant q' < q$. Thus, we have $\mathbb{Q}_{>s} \subset \mathscr{P}_{x,y}$. From ($_h$A), we have $s \in \mathbb{Q}_{>0} \Rightarrow s \notin \mathscr{P}_{x,y}$. From which we deduce that $\mathscr{P}_{x,y} = \mathbb{Q}_{>s}$. □

If A is a $\mathbb{N}^*$-set endowed with a binary relation $\succ$, we denote $A^* = A^*_\succ$ and $A^{**} = A^{**}_\succ$ the subsets of A defined as follows:

$$A^* = \{x \in A : \mathscr{P}_{x,y} \neq \emptyset,\ \exists y \in A\},$$

$$A^{**} = \{x \in A : \mathscr{P}_{x,y} \neq \emptyset,\ \forall y \in A\}.$$

3.3. Remarks. Suppose that the relation $\succ$ satisfies ($_\mathrm{h}$I). Then A^* is a sub-$\mathbb{N}^*$-set of A, and we have:

- $\succ$ satisfies ($_\mathrm{h}$SS) if and only if $\forall(x, y, z) \in A^2 \times A^*$ such that $x \succ y$, $\exists(p, m, n) \in (\mathbb{N}^*)^3$ such that $px \succ mz \succsim nz \succ py$;
- if $\succ$ satisfies ($_\mathrm{h}$SS), then $\succ$ satisfies ($_\mathrm{h}$C) if and only if $A^{**} = A^*$. $\square$

3.4. Lemma. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty interval order $\succ$ satisfying* ($_\mathrm{h}$I), ($_\mathrm{h}$SS) *and* ($_\mathrm{h}$C), *and let $(x, a) \in (A^*)^2$. Then $\forall y \in A$, we have $\mathscr{P}_{x,y} = \mathscr{P}_{x,a}\mathscr{Q}_{a,a}\mathscr{P}_{a,y}$.*

Proof. Because $A^{**} = A^*$, we have $\mathscr{P}_{x,a} \neq \emptyset$ and $\mathscr{P}_{a,y} \neq \emptyset$. From (FT) and ($_\mathrm{h}$I), we have $\mathscr{P}_{x,a}\mathscr{Q}_{a,a}\mathscr{P}_{a,y} \subset \mathscr{P}_{x,y}$. And from ($_\mathrm{h}$SS) and ($_\mathrm{h}$I), we have $\mathscr{P}_{x,y} \subset \mathscr{P}_{x,a}\mathscr{Q}_{a,a}\mathscr{P}_{a,y}$. $\square$

4. The following proposition characterizes the $\mathbb{N}^*$-sets endowed with a *homothetic weak order* (resp. a *homothetic strict order*).

4.1. Proposition. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. The two following conditions are equivalent*:

(1) *there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ which represents $\succ$*;
(2) *$\succ$ is a* homothetic weak order.

Moreover, if $\succ$ is a homothetic weak order, then the morphism u of (1) *is* unique *up to multiplication by a positive scalar. And $\succ$ is a* homothetic strict order *if and only if there exists an* injective *morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ which represents $\succ$.*

Proof. Suppose that there exists a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ which represents $\succ$. Clearly, we have $u^{-1}(u(A)^*) = A^*$, and the relation $\succsim$ is given by: $x \succsim y \Leftrightarrow u(x) \geqslant u(y)$. Then it is easy to check (and left to the reader) that $\succ$ is a homothetic weak order.

Conversely, suppose $\succ$ is a homothetic weak order. Let $(x, y) \in A^2$. From ($_\mathrm{h}$I) and 3.2, we have $x \succ y \Leftrightarrow s_{x,y} < 1$. And from (3.1) and 3.2, we have $\mathscr{Q}_{y,x} = \mathbb{Q}_{\geqslant r_{y,x}}$ with $r_{y,x} = s_{x,y}^{-1}$.

Let us prove that $\mathscr{P}_{x,x} \neq \emptyset \Leftrightarrow s_{x,x} = 1$. The implication $\Leftarrow$ is clear. Conversely, if $s_{x,x} \neq 1$, then $r_{x,x} < 1$. Hence $\exists(m, n) \in (\mathbb{N}^*)^2$ such that $m < n$ and $mx \succsim nx$. From ($_\mathrm{h}$NI) and ($_\mathrm{h}$NP) (cf. Remark 2.1), we have $m^2x \succsim mnx \succsim n^2x$, from which we obtain (using (NT)) $m^2x \succsim n^2x$. Therefore, $\forall k \in \mathbb{N}^*$, we have $m^kx \succsim n^kx$. Because $\lim_{k\to+\infty}(m/n)^k = 0$, we obtain $r_{x,x} = 0$; i.e., $\mathscr{P}_{x,x} = \emptyset$.

Because the relation $\succ$ is non-empty, we have $A^* \neq \emptyset$. Choose an element $a \in A^*$. We have $\mathscr{P}_{a,a} \neq \emptyset$; i.e., $s_{a,a} = 1$.

Suppose that $x \succ y$. From 3.3, we have $\mathscr{P}_{x,a} \neq \emptyset$, hence $r_{a,x} \in \mathbb{R}_{>0}$. Let us prove that $s_{a,x} = r_{a,x}$. From 3.4, we have $\mathscr{P}_{x,y} = \mathscr{P}_{x,a}\mathscr{Q}_{a,a}\mathscr{P}_{a,y} = \mathscr{P}_{x,a}\mathscr{P}_{a,y}$, which implies the equality $s_{x,y} = s_{x,a}s_{a,y} = r_{a,x}^{-1}s_{a,y}$. Hence we have $s_{a,y} < r_{a,x}$ because $s_{x,y} < 1$. Seeing that $r_{a,x} \in \mathbb{R}_+$, we have $\mathscr{Q}_{a,x} \neq \emptyset$. Let $(m, n) \in (\mathbb{N}^*)^2$ such that $ma \succsim nx$. Because $s_{a,a} = 1 = s_{x,x}$, from ($_\mathrm{h}$P) and ($_\mathrm{h}$NI), $\forall p \in \mathbb{N}^*\backslash\{1\}$, we have $(p+1)ma \succ pma \succsim pnx \succ (p-1)nx$; therefore, (using (ST)), we have $(p+1)ma \succ (p-1)nx$. Tending towards the limit, we obtain the inclusion $\mathbb{Q}_{>m/n} \subset \mathscr{P}_{a,x}$. So we have $r_{a,x} \geqslant s_{a,x}$, which is an equality because $\mathscr{P}_{a,x} \subset \mathscr{Q}_{a,x}$. Finally, we obtain $s_{a,x} > s_{a,y}$.

We no longer suppose that $x \succ y$.

Let us prove that $r_{a,x} \in \mathbb{R}_+$ by reducing it to the absurdity: suppose $r_{a,x} = \infty$; i.e., suppose $\mathscr{P}_{x,a} = \mathbb{Q}_{>0}$. Then ($_\mathrm{h}$I) we have $x \succ a$; therefore ($_\mathrm{h}$SS), $\exists(p, m, n) \in (\mathbb{N}^*)^3$ such that $pa \succ mx \geqslant nx \succ pb$. In particular, $p/m \in \mathscr{P}_{a,x}$; contradiction. Hence $r_{a,x} \in \mathbb{R}_+$.

Let $u = u_a : A \to \mathbb{R}_+$ be the map defined by $u(x) = r_{a,x}$. From ($_\mathrm{h}$NI), $\forall(z, t, m) \in A^2 \times \mathbb{N}^*$, we have $\mathscr{Q}_{z,mt} = m\mathscr{Q}_{z,t}$. Hence u is a morphism of $\mathbb{N}^{xc*}$-sets. Let us prove that $x \succ y \Leftrightarrow u(x) > u(y)$. We have seen that if $x \succ y$, then $r_{a,x} = s_{a,x} > s_{a,y}$. But we have the inclusion $\mathscr{P}_{a,y} \subset \mathscr{Q}_{a,y}$, from which we deduce the implication: $x \succ y \Rightarrow u(x) > u(y)$. Conversely, suppose $u(x) > u(y)$. Then $\exists(m, n) \in (\mathbb{N}^*)^2$ such that $ma \succsim ny$ and $ma \not\succsim nx$. But $ma \not\succsim nx \Leftrightarrow nx \succ ma$, from which we obtain $nx \succ ma \succsim ny$. From (NT) we have $nx \succ ny$; hence ($_\mathrm{h}$I) we have $x \succ y$. We thus proved that u represents $\succ$. And clearly, $\succ$ satisfies (S) if and only if u is injective.

We still have to prove the uniqueness property. Let $v : A \to \mathbb{R}_+$ be another morphism of $\mathbb{N}^*$-sets such that $\forall (x, y) \in A^2$, we have $x \succ y \Leftrightarrow v(x) > v(y)$. Because $u^{-1}(u(A)^*) = A^* = v^{-1}(v(A)^*)$, $\forall x \in A$, we have $u(x) \neq 0 \Leftrightarrow v(x) \neq 0$. Let $\lambda : A \to \mathbb{R}_{>0}$ be the map defined by

$$\lambda(x) = \begin{cases} u(x)^{-1}v(x) & \text{if } u(x) \neq 0 \\ u(a)^{-1}v(a) & \text{if not.} \end{cases}$$

Because u and v are morphisms of $\mathbb{N}^*$-sets, λ factorizes through the quotient-set $A/\mathbb{N}^*$. Suppose $\exists x \in A$ such that $\lambda(x) \neq \lambda(a)$. Put $\alpha = \lambda(a)\lambda(x)^{-1}$. First of all suppose $\alpha < 1$. Then $\exists q \in \mathbb{Q}_{>0}$ such that $\alpha u(a)u(x)^{-1} < q < u(a)u(x)^{-1}$. In other words, we have $v(a) < qv(x)$ and $qu(x) < u(a)$, contradiction. Now if $\alpha > 0$, then $\exists q' \in \mathbb{Q}_{>0}$ such that $u(a)u(x)^{-1} < q' < \alpha u(a)u(x)^{-1}$; i.e., $u(a) < q'u(x)$ and $q'v(x) < v(a)$, contradiction. Hence $\alpha = 1$, and λ is a constant map. This completes the proof of the proposition. □

4.2. Corollary. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic weak order $\succ$, and let $a \in A^*$. Then the map $A \to \mathbb{R}_+$, $x \mapsto r_{a,x}$ is a morphism of $\mathbb{N}^*$-sets which represents $\succ$.*

5. Let $(A, \circ)$ be a *commutative semigroup*; i.e., a set A endowed with a map $A \times A \to A$, $(x, y) \mapsto x \circ y$ such that $\forall (x, y, z) \in A^3$, we have

- $x \circ (y \circ z) = (x \circ y) \circ z$ (associativity),
- $x \circ y = y \circ x$ (commutativity).

Let us remark that A is a fortiori a $\mathbb{N}^*$-set, for the operation of $\mathbb{N}^*$ on A defined by the map $\mathbb{N}^* \times A \to A$, $(m, x) \mapsto mx = x \circ \cdots \circ x$ (m times). For all parts $X, Y \subset A$, we put $X \circ Y = \{x \circ y : x \in X, y \in Y\} \subset A$

A binary relation $\succ$ on A is said to be:

($_\circ$I) $\circ$-*independent* if $\forall (x, y, z) \in A^3$, we have $x \succ y \Leftrightarrow x \circ z \succ y \circ z$;
($_\circ$PI) $\circ$-*pseudoindependent* if $A^* \circ (A \backslash A^*) \subset A^*$ and $\forall (x, y, z, t) \in A^4$, we have

$$\begin{cases} (x \succ y, z \succ t) \Rightarrow x \circ z \succ y \circ t, \\ (x \succsim y, z \succsim t) \Rightarrow x \circ z \succsim y \circ t. \end{cases}$$

5.1 Proposition. [variant of 4.1] *Let $(A, \circ)$ be a commutative semigroup endowed with a non-empty binary relation $\succ$. The three following conditions are equivalent*:

(1) *there exists a morphism of semigroups $u : A \to \mathbb{R}_+$ which represents $\succ$*;
(2) $\succ$ *is a* $\circ$-independent homothetic weak order;
(3) $\succ$ *is a* $\circ$-pseudoindependent homothetic weak order.

Moreover, if $\succ$ is a homothetic weak order, then the morphism u of (1) *is* unique *up to multiplication by a positive scalar.*

Proof. The implication (1) $\Rightarrow$ (2) is clear.

To prove the implication (2) $\Rightarrow$ (3), suppose that $\succ$ is a $\circ$-independent homothetic weak order. Let $(x, y) \in A^* \times (A \backslash A^*)$ be such that $x \circ y \in A \backslash A^*$. Thus, we have $x \succ x \circ y$. From ($_\circ$I), we have $x \circ y \succ (x \circ y) \circ y = x \circ (2y)$ and $y \succ 2y$, hence $y \in A^*$; contradiction. Therefore, $A^* \circ (A \backslash A^*) \subset A^*$. Then using (T) and (NT), we easily deduce that the relation $\succ$ is $\circ$-pseudoindependent. So we have (2) $\Rightarrow$ (3).

To prove the implication (3) $\Rightarrow$ (1), suppose that $\succ$ is a $\circ$-pseudoindependent homothetic weak order. Choose an element $a \in A^*$, and let $u = u_a : A \to \mathbb{R}_+$ be the morphism of $\mathbb{N}^*$-sets defined by $u(x) = r_{a,x}$. From 4.3, u represents $\succ$. Let $(x, y) \in A^2$. If $(m, n, m', n') \in (\mathbb{N}^*)^4$ satisfies $ma \succsim nx$ and $m'a \succsim n'y$, then from ($_\circ$PI), we have $(nm' + n'm)a \succsim nn'(x \circ y)$. Therefore, we have $r_{a,x\circ y} \leqslant m/n + m'/n'$. From which we deduce that $r_{a,x\circ y} \leqslant r_{a,x} + r_{a,y}$; i.e., that $u(x \circ y) \leqslant u(x) + u(y)$.

First of all suppose that $(x, y) \in (A^*)^2$. If $(m, n, m', n') \in (\mathbb{N}^*)^4$ is such that $mx \succ na$ et $m'y \succ n'a$, then from ($_\circ$PI), we have $mm'(x \circ y) \succ (m'n + mn')a$. Hence we have $s_{x\circ y,a} \leqslant mm'/m'n + mn' = (n/m + n'/m')^{-1}$. From this we deduce that $r_{a,x\circ y} = s^{-1}_{x\circ y,a} \geqslant s^{-1}_{x,a} + s^{-1}_{y,a} = r_{a,x} + r_{a,y}$; i.e., that $u(x \circ y) \geqslant u(x) + u(y)$. Hence we have $u(x \circ y) = u(x) + u(y)$.

Now suppose that $(x, y) \in (A\backslash A^*)^2$. Then the inequality $u(x \circ y) \leqslant u(x) + u(y) = 0$ implies $u(x \circ y) = 0$. So we have $u(x \circ y) = 0 = u(x) + u(y)$.

Last of all suppose $(x, y) \in A^* \times (A\backslash A^*)$. Assume that $u(x \circ y) < u(x) + u(y)$. Because $u(y) = 0$, we have $x \succ x \circ y$. Hence ($_h$P), $\exists(m, n) \in (\mathbb{N}^*)^2$ such that $m > n$ and $nx \succ m(x \circ y) = nx \circ z$ with $z = (m - n)x \circ my$. But $(m - n)x \in A^*$ and $my \in A\backslash A^*$. Thus, from ($_\circ$PI), we have $z \in A^*$. Because $(nx, z) \in (A^*)^2$, we have (cf. above) $u(nx \circ z) = u(nx) + u(z)$. And because $nx \succ nx \circ z$, we also have $u(nx) > u(nx \circ z)$; contradiction. Hence we have $u(x \circ y) = u(x) + u(y)$.

Because $x \circ y = y \circ x$, the case $(x, y) \in (A\backslash A^*) \times A^*$ is already done.

So we proved that u is a morphism of semigroups. This completes the proof of the implication (3) ⇒ (1).

At last, the uniqueness property is a consequence of 4.1. □

6. Let E be a set, and let $E' \subset E$ be a subset. Let $\mathscr{G}(E' \times E)$ denote the set of maps $f : E' \times E \to \mathbb{R}^*_+$ such that $\forall(x', y', x, y) \in (E')^2 \times E^2$, we have $f(x', x') \leqslant 1$ and $f(x', x)f(y', y) = f(x', y)f(y', x)$. And let $\mathscr{G}_0(E' \times E)$ denote the subset of $\mathscr{G}(E' \times E)$ made up of maps f such that $\forall(x', y') \in (E')^2$, we have $f(x', y') = f(y', x')$. Remark that if $f \in \mathscr{G}_0(E' \times E)$, then $\forall(x', y') \in (E')^2$, we have $f(x, y) = f(x, x)^{1/2} f(y, y)^{1/2} \leqslant 1$.

Let A be a $\mathbb{N}^*$-set endowed with a binary relation $\succ$ satisfying ($_h$I). Put $\bar{A} = A/\mathbb{N}^*$ and let $\bar{A}^* = \bar{A}^*_\succ$ denote the subset of $\bar{A}$ defined by $\bar{A}^* = A^*_\succsim/\mathbb{N}^*$. We denote $\mathscr{E}(A, \succ)$ the set of pairs (u, σ) made up of a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ and a map $\sigma \in \mathscr{G}(\bar{A}^* \times \bar{A})$; i.e., a map $\sigma \in \mathscr{G}(A^* \times A)$ such that $\forall(x, y, m, n) \in A^* \times A \times (\mathbb{N}^*)^2$, we have $\sigma(mx, ny) = \sigma(x, y)$. We denote $\mathscr{E}_0(A, \succ) \subset \mathscr{E}(A, \succ)$ the subset made up of pairs (u, σ) such that $\sigma \in \mathscr{G}_0(\bar{A}^* \times \bar{A})$. At last, for$(u, \sigma) \in \mathscr{E}(A, \succ)$, we denote σ^* the restriction $\sigma|_{\bar{A}^* \times \bar{A}^*}$.

The following proposition characterizes the homothetic interval orders.

6.1. Proposition. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. The two following conditions are equivalent*:

(1) *there exists a pair $(u, \sigma) \in \mathscr{E}(A, \succ)$ such that $\forall(x, y) \in A^2$, we have $x \succ y \Leftrightarrow \sigma(x, y)u(x) > u(y)$;*
(2) *$\succ$ is a* homothetic interval order.

Moreover, if $\succ$ is a homothetic interval order, then there exists a pair $(u, \sigma) \in \mathscr{E}_0(A, \succ)$ verifying (1); *and if $(u_1, \sigma_1), (u_2, \sigma_2) \in \mathscr{E}_0(A, \succ)$ are two pairs verifying* (1), *then $\sigma^*_2 = \sigma^*_1$ and there exists a (unique) constant $\lambda > 0$ such that $u_2 = \lambda u_1$.*

Proof. Suppose that there exists a pair $(u, \sigma) \in \mathscr{E}(A, \succ)$ verifying (1). Clearly, we have $u^{-1}(u(A)^*) = A^*$. For $x \in A$, put $\overline{x} = u(x)$. Let $(x, y) \in A^2$ such that $x \succ y$, and suppose that $y \succ x$. Then we have $\sigma(y, x)\sigma(x, y)\overline{x} > \sigma(y, x)\overline{y} > \overline{x}$. And because $\sigma \in \mathscr{G}(A^* \times A)$, we also have $\sigma(y, x)\sigma(x, y) = \sigma(y, y)\sigma(x, x) \leqslant 1$, which contradicts the inequality $\sigma(y, x)\sigma(x, y)\overline{x} > \overline{x}$. Therefore, $\succ$ satisfies (A).

Because $\succ$ satisfies (A), for $(x, y) \in A \times A^*$, we have $x \succsim y \Leftrightarrow \overline{x} \geqslant \sigma(y, x)\overline{y}$. Let $(x, y, z, t) \in A^4$ such that $x \succ y \succsim z \succ t$. Thus, we have

$$\begin{cases} \sigma(x, y)\overline{x} > \overline{y} \geqslant \sigma(z, y)\overline{z}, \\ \sigma(z, t)\overline{z} > \overline{t}, \end{cases}$$

hence $(\sigma(x, y)\sigma(z, t)/\sigma(z, y))\overline{x} > \overline{t}$. But $\sigma(x, y)\sigma(z, t) = \sigma(x, t)\sigma(z, y)$, hence $\sigma(x, t)\overline{x} > \overline{t}$; i.e., $x \succ t$. Therefore, $\succ$ satisfies (ST); so it is an interval order.

It remains to prove that $\succ$ is a homothetic structure. The conditions ($_h$I), ($_h$A) and ($_h$P) are clearly satisfied. Let $(x, y, z) \in A^3$ such that $x \succ y \succsim z$. We have $\sigma(x, y)\overline{x} > \overline{y}$, hence $\overline{x} > 0$ and $\exists m \in \mathbb{N}^*$ such that $m\sigma(x, z)\overline{x} > \overline{z}$; i.e., such that $mx \succ z$. Therefore, $\succ$ satisfies ($_h$C). Concerning the condition ($_h$SS), let $(x, y, z, t) \in A^4$ such that $x \succ y$ and $z \succ t$. We have $\sigma(x, y)\overline{x} > \overline{y}$ and $r = \sigma(z, y)\overline{z} > 0$. Hence $\exists(p, m, n) \in (\mathbb{N}^*)^3$ such that

$$\sigma(x, y)\overline{x} > \frac{m}{p\sigma(z, z)} r \geqslant \frac{n}{p} r > \overline{y}.$$

Because $\sigma(x, y)(\sigma(z, z)/\sigma(z, y)) = \sigma(x, z)$, multiplying by $p(\sigma(z, z)/\sigma(z, y))$, we obtain

$$p\sigma(x, z)\overline{x} > m\overline{z} \geqslant n\sigma(z, z)\overline{z} > p\,\frac{\sigma(z, z)}{\sigma(z, y)}\,\overline{y};$$

i.e., $px \succ mz \succsim nz \succ py$. Therefore, $\succ$ satisfies ($_{\rm h}$SS).

Conversely, suppose that $\succ$ is a homothetic interval order. Then $\forall (x, y) \in A^2$, we have (cf. the proof of 4.1) $x \succ y \Leftrightarrow s_{x,y} < 1$ and $\mathscr{Q}_{y,x} = \mathbb{Q}_{\geqslant r_{y,x}}$ with $r_{y,x} = s_{x,y}^{-1}$.

Let $>$ denote the binary relation on A defined by $x > y \Leftrightarrow s_{x,y} < s_{y,x}$; i.e., by $x > y \Leftrightarrow \mathscr{P}_{x,y} \supsetneqq \mathscr{P}_{y,x}$. In particular, we have $x > y \Rightarrow x \in A^*$. Clearly, $>$ satisfies (A). Let $(x, y, z) \in A^3$ such that $x > y > z$. If $z \in A\backslash A^*$, then $\emptyset = \mathscr{P}_{z,x} \subsetneqq \mathscr{P}_{x,z}$. And if $z \in A^*$, then from (3.4), we have $\mathscr{P}_{z,x} = \mathscr{P}_{z,y}\mathscr{Q}_{y,y}\mathscr{P}_{y,z} \subsetneqq \mathscr{P}_{x,z}$. Therefore, $>$ satisfies (T).

Let $\approx$ denote the binary relation on A defined by $x \approx y \Leftrightarrow x \not> y \not> x$. Thus, we have

$$x \approx y \Leftrightarrow s_{x,y} = s_{y,x} \Leftrightarrow \mathscr{P}_{x,y} = \mathscr{P}_{y,x}.$$

We clearly have $x \approx y \Leftrightarrow y \approx x$. Let us prove that $\approx$ is transitive. Let $(x, y, z) \in A^3$ such that $x \approx y \approx z$. Because $\mathscr{P}_{x,y} = \mathscr{P}_{y,x}$, we have $(x, y) \in (A^*)^2 \cup (A\backslash A^*)^2$. If $(x, y) \in (A^*)^2$, then from 3.4, we have $\mathscr{P}_{x,z} = \mathscr{P}_{x,y}\mathscr{Q}_{y,y}\mathscr{P}_{y,z} = \mathscr{P}_{y,x}\mathscr{Q}_{y,y}\mathscr{P}_{z,y} = \mathscr{P}_{z,y}$; i.e., $x \approx z$. Suppose that $(x, y) \in (A\backslash A^*)^2$. Because $A^{**} = A^*$, we have $\mathscr{P}_{x,z} = \mathscr{P}_{y,z} = \emptyset = \mathscr{P}_{z,y}$; i.e., $z \in A\backslash A^*$, which implies $\mathscr{P}_{z,x} = \emptyset$. Hence $x \approx z$.

Because $\approx$ is transitive, it is an equivalence relation. Hence $>$ is a weak order. Let us remark that $\forall (x, y) \in A^2$, we have $x \succ y \Rightarrow x > y$, therefore $x \geqslant y \Rightarrow x \succsim y$.

Let us prove that $>$ is a homothetic structure. For $(x, y, m, n) \in A^2 \times (\mathbb{N}^*)^2$, we have $\mathscr{P}_{mx,ny} = (n/m)\mathscr{P}_{x,y}$. From which we deduce that $>$ satisfies ($_{\rm h}$I), ($_{\rm h}$A) and ($_{\rm h}$P). Because $>$ satisfies (NT), $>$ satisfies ($_{\rm h}$C). Concerning the condition ($_{\rm h}$SS), let $(x, y, z, t) \in A^4$ such that $x > y$ and $z > t$. Because $(x, z) \in (A^*)^2$, we have 3.4 $\mathscr{P}_{x,y} = \mathscr{P}_{x,z}\mathscr{Q}_{z,z}\mathscr{P}_{z,y}$. And if $y \in A^*$, we also have $\mathscr{P}_{y,x} = \mathscr{P}_{y,z}\mathscr{Q}_{z,z}\mathscr{P}_{z,x}$. Because $s_{x,y} < s_{y,x}$ with $s_{y,x} = \infty$ if $y \in A\backslash A^*$, $\exists (p, m, n) \in (\mathbb{N}^*)^3$ such that $n < m$, $(m/p)^2 s_{x,z} < s_{z,x}$ and $(p/n)^2 s_{z,y} < s_{y,z}$; i.e., such that $px > mz \geqslant nz > py$. Thus, $>$ satisfies ($_{\rm h}$SS), and $>$ is a homothetic structure.

Because $>$ is a homothetic weak order, from 4.1, there exists a morphism of $\mathbb{N}^*$-set $u : A \to \mathbb{R}_+$ such that $\forall (x, y) \in A^2$, we have $x > y \Leftrightarrow u(x) > u(y)$. For $x \in A$, we have $u(x) = 0$ if and only if $\forall y \in A$, we have $r_{y,x} = 0$; i.e., if and only if $x \in A\backslash A^*$. Thus, we have $u^{-1}(u(A)^*) = A^*$. Let $\sigma^* : A^* \times A^* \to \mathbb{R}_+^*$ be the map defined by $\sigma^*(x, y) = r_{y,x}u(x)^{-1}u(y)$. We extend σ^* to $A^* \times A$ in the following way: let us choose an element $a \in A^*$, and for $(x, y) \in A^* \times (A\backslash A^*)$, put $\sigma(x, y) = \sigma^*(x, a)$. For $(x, y, m, n) \in (A^*)^2 \times (\mathbb{N}^*)^2$, we have $r_{my,nx} = (n/m)\, r_{y,x}$. Therefore, σ factorizes through $\bar{A}^* \times \bar{A}$. For $(x, y, z, t) \in (A^*)^4$, we have $\sigma^*(x, x) = r_{x,x} \leqslant 1$ and $\mathscr{P}_{x,y} = \mathscr{P}_{x,t}\mathscr{Q}_{t,t}\mathscr{P}_{t,y}$, from which we deduce that $s_{x,y} = s_{x,t}r_{t,t}s_{t,y}$ and (switching to the inverse) that $r_{y,x} = r_{t,x}s_{t,t}r_{y,t}$; hence $r_{y,x}r_{t,z} = r_{t,x}(r_{t,z}s_{t,t}r_{y,t}) = r_{t,x}r_{y,z}$ and $\sigma(x, y)\sigma(z, t) = \sigma(x, t)\sigma(z, y)$. From the definition of σ, this last equality remains true for $(y, t) \in A^2$. Hence $(\sigma, u) \in \mathscr{E}(A, \succ)$, and by construction $\forall (x, y) \in A^2$, we have $x \succ y \Leftrightarrow \sigma(x, y)u(x) > u(y)$.

It remains to prove the last two assertions of the proposition. For $(x, y) \in (A^*)^2$, we have $r_{y,x} = \sigma(x, y)u(x)u(y)^{-1}$, hence

$$\begin{aligned} u(x) > u(y) &\Leftrightarrow \sigma(x, y)u(x)u(y)^{-1} > \sigma(y, x)u(y)u(x)^{-1} \\ &\Leftrightarrow \sigma(x, y)^{1/2}u(x) > \sigma(y, x)^{1/2}u(y); \end{aligned}$$

which is possible only if $\sigma(x, y) = \sigma(y, x)$. Hence $(u, \sigma) \in \mathscr{E}_0(A, \succ)$. Concerning the uniqueness property, for $i = 1, 2$, let $(u_i, \sigma_i) \in \mathscr{E}_0(A, \succ)$ such that $\forall (x, y) \in A^2$, we have $x \succ y \Leftrightarrow \sigma_i(x, y)u_i(x) > u_i(y)$. Let recall that $u_1^{-1}(u_1(A)^*) = A^* = u_2^{-1}(u_2(A)^*)$. For $x \in A$, write $u_2(x) = \lambda_x u_1(x)$ with $\lambda_x > 0$ and $\lambda_x = 1$ if $u_1(x) = 0$. Remark that the map $x \mapsto \lambda_x$ factorizes through $\bar{A}$. For $(x, y) \in (A^*)^2$, we have (easy checking left to the reader) $\sigma_2(x, y) = \lambda_x^{-1}\lambda_y\sigma_1(x, y)$, therefore,

$$\begin{aligned} &\sigma_2(x, y) = \sigma_2(y, x) \\ &\Leftrightarrow \lambda_x^{-1}\lambda_y\sigma_1(x, y) = \lambda_y^{-1}\lambda_x\sigma_1(y, x) \\ &\Leftrightarrow \lambda_y^2 = \lambda_x^2; \end{aligned}$$

i.e., $\lambda_x = \lambda_y$. So $x \mapsto \lambda_x$ is a constant map on A^*. This completes the proof of the proposition. $\square$

6.2. Remark. Let A be $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. If $(u, \sigma) \in \mathscr{E}(A, \succ)$ is a pair verifying 6.1-(1), then we have $u^{-1}(u(A)^*) = A^*$; and the relation $\succ$ is completely determined by the pair $(u|_{A^*}, \sigma^*)$. But for $\sigma \in \mathscr{G}_0(A^* \times A)$ and $(x, y) \in (A^*)^2$, we have $\sigma(x, y) = \gamma(x)\gamma(y)$ with $\gamma(x) = \sigma(x, x)^{1/2}$. Therefore, the condition (1) of 6.1 is equivalent to the following condition (1′):

(1′) there exists a morphism of $\mathbb{N}^*$-sets $u^* : A^* \to \mathbb{R}_+$ and a map $\gamma : \bar{A}^* \to]0, 1]$, such that $\forall (x, y) \in (A^*)^2$, we have $x \succ y \Leftrightarrow \gamma(x)u(x) > \gamma(y)^{-1}u(y)$.

Moreover, if $\succ$ is a homothetic interval order, then the pair (u^*, γ) of (1′) is unique up to multiplication of u^* by a positive scalar.

6.3. Corollary/Definition. Let A be a $\mathbb{N}^*$-set endowed with a non-empty interval homothetic order $\succ$, and let $(u, \sigma) \in \mathscr{E}_0(A, \succ)$ be a pair verifying 6.1-(1). Then u represents the homothetic weak order $\succ_0$ (denoted $>$ in the proof of 6.1) on A defined by $x \succ_0 y \Leftrightarrow r_{y,x} > r_{x,y}$; and $\forall (x, y) \in (A^*)^2$, we have $\sigma^*(x, y) = r_{y,x}u(y)u(x)^{-1}$. The invariant $\sigma^* \in \mathscr{G}_0(\bar{A}^* \times \bar{A}^*)$ does not depend on u; we denote it $\sigma^*_\succ$. At last, let $\gamma^*_\succ : \bar{A}^* \to \mathbb{R}^*_+$ denote the map defined by $\gamma^*_\succ(x) = \sigma^*_\succ(x, x)^{1/2}$; so we have $\sigma^*_\succ(x, y) = \gamma^*_\succ(x)\gamma^*_\succ(y)$.

6.4. Corollary. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$, and let $u : A \to \mathbb{R}_+$ be a morphism of $\mathbb{N}^*$-sets which represents $\succ_0$. Then $\forall (x, y) \in (A^*)^2$, we have $u(x)u(y)^{-1} = (r_{y,x}s_{y,x})^{1/2}$.*

Proof. For $(x, y) \in (A^*)^2$, we have $\sigma^*_\succ(x, y) = r_{y,x}u(y)u(x)^{-1}$ and $\sigma(x, y) = \sigma(y, x)$; from which we deduce that $u(x)u(y)^{-1} = (r_{y,x}r^{-1}_{x,y})^{1/2} = (r_{y,x}s_{y,x})^{1/2}$. □

6.5. Remark. Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$, and let $u : A \to \mathbb{R}^*_+$ be a morphism of $\mathbb{N}^*$-sets which represents $\succ_0$. One may wonder if the map $A \times A \to \mathbb{R}^\infty_+$, $(x, y) \mapsto r_{y,x} = s^{-1}_{x,y}$ factorizes through the product-map $u \times u$; i.e., if $\forall (x, y, x', y') \in A^4$ such that $u(x) = u(x')$ and $u(y) = u(y')$, we have $r_{x,y} = r_{x',y'}$. In general, the answer is negative: cf. Example 7.6.

Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$, and let $u : A \to \mathbb{R}_+$ be a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ which represents $\succ_0$. Choose an element $a \in A^*$ and let $\sigma^a_\succ : A^* \times A \to \mathbb{R}^*_+$ denote the map extending $\sigma^*_\succ$ defined by $\sigma^a_\succ(x, y) = \sigma^*_\succ(x, a)$ for $(x, y) \in A^* \times (A \backslash A^*)$. Then $(u, \sigma^a_\succ) \in \mathscr{E}_0(A, \succ)$ and $\forall (x, y) \in A^2$, we have $x \succ y \Leftrightarrow \sigma^a_\succ(x, y)u(x) > u(y)$. The map $\sigma^a_\succ$ is *split*: there exist two maps $\sigma_1 : \bar{A}^* \to \mathbb{R}^*_+$ and $\sigma_2 : \bar{A} \to \mathbb{R}^*_+$ such that $\sigma^a_\succ = \sigma_1 \times \overline{\sigma}_2$ with $\overline{\sigma}_2(x) = \sigma_2(x)^{-1}$ $(x \in A)$. In fact, for $(x, y) \in (A^*)^2$, put $\sigma_1(x) = s_{a,a}r_{a,x}u(x)^{-1}$ and $\sigma^*_2(y) = s_{a,y}u(y)^{-1}$; because $r_{y,x} = r_{y,a}s_{a,a}r_{a,x}$ 3.4, we have $\sigma_1(x)\sigma^*_2(y)^{-1} = \sigma^*_\succ(x, y)$. Let $\sigma_2 : A \to \mathbb{R}^*_+$ be the map extending σ^*_2 defined by $\sigma_2(y) = \sigma_2(a)$ for $y \in A \backslash A^*$. The maps $\sigma_1 : A^* \to \mathbb{R}^*_+$ and $\sigma_2 : A \to \mathbb{R}^*_+$ defined in this way factorize through $\bar{A}^*$ and $\bar{A}$, respectively. And by construction, we have $\sigma^a_\succ = \sigma_1 \times \overline{\sigma}_2$. In other words, $\forall (x, y) \in A^2$, we have $x \succ y \Leftrightarrow u_1(x) > u_2(y)$ with $u_i(x) = \sigma_i(x)u$. For $i = 1, 2$, the map $u_i : A \to \mathbb{R}_+$ is a morphism of $\mathbb{N}^*$-sets. This formulation by means of a pair of maps (u_1, u_2) is the one usually employed to represent interval orders; cf. [8, Theorem 2.7]. Let us remark that in the general (i.e., not necessarily homothetic) theory of interval orders, there is a priori no possible uniqueness result for the pair (u_1, u_2). As we will see in Section 7, for homothetic interval orders the result is quite different.

7. Let A be a $\mathbb{N}^*$-set endowed with a binary relation $\succ$. We denote $\succ_1$ and $\succ_2$ the binary relations on A defined by:

- $x \succ_1 y \Leftrightarrow (mx \succ z \succsim my, \exists (z, m) \in A \times \mathbb{N}^*)$,
- $x \succ_2 y \Leftrightarrow (mx \succsim z \succ my, \exists (z, m) \in A \times \mathbb{N}^*)$.

7.1. Lemma. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$. Then for $i = 1, 2$, $\succ_i$ is a non-empty homothetic weak order.*

Proof. Let a pair $(u, \sigma) \in \mathscr{E}_0(A, \succ)$ satisfying 6.1-(1). We may (and do) suppose that $\sigma = \sigma^a_\succ$ for an element $a \in A^*$. For $(x, y) \in A^2$, we have $x \succ y \Rightarrow x \succ_i y$ $(i = 1, 2)$. Therefore, the relations $\succ_1$ and $\succ_2$ are non-empty. Let us prove that $\succ_1$ is a homothetic weak order. Let $(x, y) \in A^2$ such that $x \succ_1 y$, and let $(z, m) \in A \times \mathbb{N}^*$ such that $mx \succ z \succsim my$. Thus, we have $x \in A^*$. First of all suppose that $(y, z) \in (A^*)^2$. Hence we have $\sigma(x, z)u(mx) > u(z) \geqslant \sigma(x, y)u(my)$. We obtain

$$r_{z,x} \frac{u(z)}{u(x)} u(mx) > u(z) \geqslant r_{z,y} \frac{u(z)}{u(y)} u(my),$$

hence $r_{z,x} > r_{z,y}$. But from 3.4, we have $r_{z,x} = r_{z,a} s_{a,a} r_{a,x}$ and $r_{z,y} = r_{z,a} s_{a,a} r_{a,y}$. From which we deduce that $r_{a,x} > r_{a,y}$. Now if $(y, z) \in (A \backslash A^*) \times A$, then this last inequality remains true: we have $r_{a,x} > 0$ and $r_{a,y} = 0$. At last, if $(y, z) \in A^* \times (A \backslash A^*)$, then replacing z by a in the calculation above, we still obtain $r_{a,x} > r_{a,y}$.

Conversely, let $(x, y) \in A^2$ such that $r_{a,x} > r_{a,y}$. Then $x \in A^*$, and $\exists (m, n) \in (\mathbb{N}^*)^2$ such that $r_{a,x} > n/m \geqslant r_{a,y}$. Because $(1/n) r_{a,t} = r_{na,t}$ $(t \in A)$, we have $mr_{na,x} > 1 \geqslant mr_{na,y}$. First of all suppose that $y \in A^*$. Then we obtain $\sigma(x, a)u(mx) > u(na) \geqslant \sigma(y, a)u(my)$; i.e., $mx \succ na \succsim my$. Thus, we have $x \succ_1 y$. Now if $y \in A \backslash A^*$, then $\forall m \in \mathbb{N}^*$ such that $m > s_{x,a}$, we have $mx \succ a \succ my$; therefore $x \succ_1 y$.

So we proved that the morphism of $\mathbb{N}^*$-sets $u_1 : A \to \mathbb{R}_+,\ x \mapsto r_{a,x}$ represents the relation $\succ_1$. Then it is easy to check (and left to the reader) that $\succ_1$ is a homothetic weak order.

Let $(x, y) \in A^2$ such that $x \succ_2 y$, and let $(z, m) \in A \times \mathbb{N}^*$ such that $mx \succsim z \succ my$. Then $z \in A^*$, $u(mx) \geqslant \sigma(z, x)u(z)$ and $\sigma(z, y)u(z) > u(my)$, from which we obtain $\sigma(z, x)^{-1}u(mx) \geqslant u(z) > \sigma(z, y)^{-1}u(my)$. In particular, we have $x \in A^*$. First of all suppose that $y \in A^*$. Like for $\succ_1$, we obtain $s_{a,x} > s_{a,y}$; and this inequality remains true for $y \in A \backslash A^*$. Conversely, like for $\succ_1$ we prove that if $(x, y) \in A^2$ is such that $s_{a,x} > s_{a,y}$, then $x \succ_2 y$. Hence the morphism of $\mathbb{N}^*$-sets $u_2 : A \to \mathbb{R}_+, x \mapsto s_{a,x}$ represents $\succ_2$. And like for $\succ_1$, it is easy to check that $\succ_2$ is a homothetic weak order. □

7.2. Proposition. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. The two following conditions are equivalent*:

(1) *there exist two morphisms of $\mathbb{N}^*$-sets $u_1, u_2 : A \to \mathbb{R}_+$ such that $u_1 \leqslant u_2$ and $\forall (x, y) \in A^2$, we have $x \succ y \Leftrightarrow u_1(x) > u_2(y)$*;
(2) *$\succ$ is a* homothetic interval order.

Moreover, if $\succ$ is a homothetic interval order, then the pair (u_1, u_2) of (1) *is unique up to multiplication by a positive scalar* (*i.e., up to replacing it by $(\lambda u_1, \lambda u_2)$ for a $\lambda > 0$*); *and for $i = 1, 2$, u_i represents $\succ_i$.*

Proof. Let $u_1, u_2 : A \to \mathbb{R}_+$ be two morphisms of $\mathbb{N}^*$-sets verifying (1). Because $u_1 \leqslant u_2$, $\succ$ satisfies (A); and $\forall (x, y) \in A^2$, we have $x \succsim y \Leftrightarrow u_2(x) \geqslant u_1(y)$. It is easy to check (and left to the reader) that $\succ$ is a homothetic interval order.

Conversely, suppose that $\succ$ is a homothetic interval order. Choose an element $a \in A^*$, and let $u_1^*, u_2^* : A^* \to \mathbb{R}_+^*$ be the morphisms of $\mathbb{N}^*$-sets defined by $u_1^*(x) = s_{a,a} r_{a,x}$ and $u_2^*(x) = s_{a,x}$. For $i = 1, 2$, let $u_i : A \to \mathbb{R}_+$ be the morphism of $\mathbb{N}^*$-sets obtained extending u_i^* by zero on $A \backslash A^*$. For $(x, y) \in (A^*)^2$, we have

$$\begin{aligned} x \succ y &\Leftrightarrow r_{y,x} > 1 \\ &\Leftrightarrow r_{y,a} s_{a,a} r_{a,x} > 1 \\ &\Leftrightarrow u_1(x) > u_2(y). \end{aligned}$$

By construction, we have $u_i^{-1}(u_i(A)^*) = A^*$ $(i = 1, 2)$, therefore, the equivalence above remains true for $y \in A \backslash A^*$. Because $\succ$ satisfies (A), we have $u_1 \leqslant u_2$. From the proof of 7.1, we already know that for $i = 1, 2$, u_i represents $\succ_i$.

Concerning the uniqueness property, let $u_1', u_2' : A \to \mathbb{R}_+$ be two others morphisms of $\mathbb{N}^*$-sets verifying (1). For $(m, n, p) \in (\mathbb{N}^*)^3$, we have $mu_1(x) > nu_2(x) > pu_1(x)$ if and only if $mu_1'(x) > nu_2'(x) > pu_1'(x)$. Thus for $i = 1, 2$, we have $u_i'(x) = 0 \Leftrightarrow u_i(x) = 0$ $(x \in A)$. For $i = 1, 2$, let $\lambda_i : A^* \to \mathbb{R}_+^*$ be the map defined by $\lambda_i(x) = u_i(x)^{-1} u_i'(x)$; because u_i and u_i' are morphisms of $\mathbb{N}^*$-sets, λ_i factorizes through the quotient-set $\bar{A}^*$. Let $f : \bar{A}^* \times \bar{A}^* \to \mathbb{R}_+^*$ be the map defined by $f(x, y) = \lambda_2(y)^{-1} \lambda_1(x)$. Let $(x, y) \in (A^*)^2$, and put $\mu = u_1(x)^{-1} u_2(y)$ and $\alpha = f(x, y)$. For $(m, n) \in (\mathbb{N}^*)^2$, we have $mx \succ ny \Leftrightarrow m/n > \mu$; but we also have $mx \succ ny \Leftrightarrow u_1'(mx) > u_2'(ny) \Leftrightarrow \alpha(m/n) > \mu$. If

$\alpha > 1$, let choose $(m, n) \in (\mathbb{N}^*)^2$ such that $\alpha(m/n) > \mu \geqslant (m/n)$; then we have $mx \succ nx \succsim mx$, contradiction. If $\alpha < 1$, let choose $(m, n) \in (\mathbb{N}^*)^2$ such that $m/n > \mu \geqslant \alpha(m/n)$; then we have $mx \succ nx \succsim mx$, contradiction. Hence $\alpha = 1$. So we proved that $f = 1$. This implies there exists a constant $\lambda > 0$ such that $\lambda_1 = \lambda_2 = \lambda$. This completes the proof of the proposition. □

7.3. Corollary. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$. Let $a \in A^*$ and $u_1, u_2 : A \to \mathbb{R}_+$ be the morphisms of $\mathbb{N}^*$-sets defined by $u_1(x) = s_{a,a}r_{a,x}$ and $u_2(x) = s_{a,x}$. Then the pair (u_1, u_2) verifies* 7.2-(1).

7.4. Corollary. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$.*

(1) *Let $(u, \sigma) \in \mathscr{E}(A, \succ)$ be a pair verifying* 6.1-(1). *Let $u_1, u_2 : A \to \mathbb{R}_+$ be the morphisms of $\mathbb{N}^*$-sets defined by $u_i(A\backslash A^*) = 0$ $(i = 1, 2)$, $u_1(x) = \gamma^*_\succ(x)u(x)$ and $u_2(x) = \gamma^*_\succ(x)^{-1}u(x)$ $(x \in A^*)$. Then the pair (u_1, u_2) verifies* 7.2-(1).
(2) *Let $u_1, u_2 : A \to \mathbb{R}_+$ be two morphisms of $\mathbb{N}^*$-sets verifying* 7.2-(1). *Let $u : A \to \mathbb{R}_+$ be the morphism of $\mathbb{N}^*$-sets defined by $u = (u_1u_2)^{1/2}$, and let $v^* : \bar{A}^* \to \mathbb{R}^*_+$ be the map defined by $v^* = (u_1\overline{u}_2)^{1/2}$ with $\overline{u}_2(x) = u_2(x)^{-1}$. Then u represents $\succ_0$ and $\gamma^*_\succ = v^*$.*

Proof. Choose an element $a \in A^*$ and let $u'_1, u'_2 : A \to \mathbb{R}_+$ be the morphisms of $\mathbb{N}^*$-sets defined by $u'_1(x) = s_{a,a}r_{a,x}$ and $u'_2(x) = s_{a,x}$. For $(x, y) \in (A^*)^2$, we have

$$\begin{aligned} r_{x,y} < r_{y,x} &\Leftrightarrow r_{x,a}s_{a,a}r_{a,y} < r_{y,a}s_{a,a}r_{a,x} \\ &\Leftrightarrow s_{a,a}r_{a,x}s_{a,x} > s_{a,a}r_{a,y}s_{a,y} \\ &\Leftrightarrow (u'_1u'_2)(x) > (u'_1u'_2)(y). \end{aligned}$$

Because for $i = 1, 2$, we have $u'^{-1}_i(u'_i(A)^*) = A^*$, the equivalence above remains true for $(x, y) \in A^2$. Hence $u'_1u'_2$ represents $\succ_0$. Therefore, $u' = (u'_1u'_2)^{1/2}$ represents $\succ_0$, and u' is a morphism of $\mathbb{N}^*$-sets. Moreover, it is easy to check (and left to the reader) that the map $\gamma^*_\succ : \bar{A}^* \to \mathbb{R}^*_+$ is given by $\gamma^*_\succ(x) = u'_1(x)^{1/2}u'_2(x)^{-1/2}$. By construction, for $x \in A^*$, we have $u'_1(x) = \gamma^*_\succ(x)u'(x)$ and $u'_2(x) = \gamma^*_\succ(x)^{-1}u'(x)$. Finally, the uniqueness properties in 6.1 and 7.2 imply the corollary. □

The following proposition characterizes the homothetic semiorders.

7.5. Proposition. *Let A be a $\mathbb{N}^*$-set endowed with a non-empty binary relation $\succ$. The three following conditions are equivalent*:

(1) *there exist a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ and a constant $\alpha \in]0, 1]$ such that $\forall (x, y) \in A^2$, we have $x \succ y \Leftrightarrow \alpha u(x) > u(y)$;*
(2) *$\succ$ is a homothetic interval order such that $\succ_1 = \succ_2$ (in that case, we have $\succ_1 = \succ_0 = \succ_2$);*
(3) *$\succ$ is a homothetic semiorder.*

Moreover, if $\succ$ is a homothetic semiorder, then the pair (u, α) of (1) *is* unique *up to multiplication of u by a positive scalar.*

Proof. Suppose that there exist a morphism of $\mathbb{N}^*$-sets $u : A \to \mathbb{R}_+$ and a constant $\alpha \in]0, 1]$ verifying (1). Let $(x, y) \in A^2$. We have $x \succ_1 y$ if and only if $\exists (z, m) \in A \times \mathbb{N}^*$ such that $\alpha u(mx) > u(z) \geqslant \alpha u(my)$; i.e., (cf. the proof of 7.1), if and only if $u(x) > u(y)$. And we have $x \succ_2 y$ if and only if $\exists (z, m) \in A \times \mathbb{N}^*$ such that $\alpha u(mx) \geqslant \alpha u(z) \geqslant \alpha u(my)$; i.e., if and only if $u(x) > u(y)$. Thus, we have $\succ_1 = \succ_0 = \succ_2$. Now let $(x, y, z, t) \in A^4$ such that $x \succ y \succ z$. Because $\alpha u(x) > u(y) > \alpha^{-1}u(z)$, we have $\alpha^2 u(x) > u(z)$. If $t \succsim x$, we have $u(t) \geqslant \alpha u(x)$ and $\alpha u(t) \geqslant \alpha^2 u(x) > u(z)$, hence $t \succ z$. And if $z \succsim t$, we have $\alpha^{-1}u(z) \geqslant u(t)$ and $\alpha u(x) > \alpha^{-1}u(z) > u(t)$, hence $x \succ t$. Therefore, $\succ$ is a semiorder.

Conversely, suppose that $\succ_1 = \succ_2$. Let $a \in A^*$. From the uniqueness property in 4.1, there exists a (unique) $\beta > 0$ such that $\forall x \in A$, we have $r_{a,x} = \beta s_{a,x}$; taking $x = a$, we obtain $r_{a,a} = \beta s_{a,a}$. From 7.3 and 7.4, we have $\succ_0 = \succ_1$, and

$\forall(x, y) \in (A^*)^2$, we have $\sigma^*_{\succ}(x, y) = \sigma^*_{\succ}(a, a) = r_{a,a}$. Put $\alpha = r_{a,a} \in]0, 1]$. If $u : A \to \mathbb{R}^*_+$ is a morphism of $\mathbb{N}^*$-sets which represents $\succ_0$, then $\forall(x, y) \in A^2$, we have $x \succ y \Leftrightarrow \alpha u(x) > u(y)$.

The implication (1) $\Rightarrow$ (3) and the equivalence (1) $\Leftrightarrow$ (2) are proved. Let us prove the implication (3) $\Rightarrow$ (1). Suppose that $\succ$ is a homothetic semiorder. Let a pair $(u, \sigma) \in \mathscr{E}_0(A, \succ)$ verifying 6.1-(1). We have to prove that $\sigma^* = \sigma^*_{\succ}$ is a constant map. Let $(x, y, z, t) \in (A^*)^4$ such that $x \succ y \succ z$. We have $\sigma(x, y)u(x) > u(y)$ and $\sigma(y, z)u(y) > u(z)$. Mutiplying the first inequality by $\sigma(y, t)$ and the second one by $\sigma(z, t)$, we obtain $\sigma(y, y)\sigma(x, t)u(x) > \sigma(y, t)u(y)$ and $\sigma(z, z)\sigma(y, t)u(y) > \sigma(z, t)u(z)$. From which we deduce that

$$\frac{\sigma(y, y)\sigma(x, t)\sigma(z, z)}{\sigma(z, t)}u(x) > u(z);$$

i.e., that $\sigma(y, y)\sigma(x, z)u(x) > u(z)$. Suppose that σ^* is not a constant map. Then we may (and do) assume that $\sigma(t, t) \neq \sigma(y, y)$. Up to permuting t and y, and replacing x, t, z par some multiples of themselves (in order to have $x \succ t \succ z$), we may (and do) assume that $\sigma(t, t) < \sigma(y, y)$. Put $\mu = (\sigma(y, y)/\sigma(t, t)) > 1$. Because $\mathscr{P}_{x,y} = \mathbb{Q}_{>s_{x,y}}$, $\mathscr{P}_{y,z} = \mathbb{Q}_{>s_{y,z}}$ and $s_{x,y}s_{y,z} = s_{x,y}r^{-1}_{y,y} = s_{x,y}s_{y,y}$, we have $\mathscr{P}_{x,y}\mathscr{P}_{y,z} = \mathbb{Q}_{>s_{x,z}s_{y,y}}$. Thus, we deduce that for every $\varepsilon > 0$, there exists $(m, n, p) \in (\mathbb{N}^*)^3$ such that $mx \succ py \succ nz$ and $s_{x,z}s_{y,y} < m/n < s_{x,z}s_{y,y} + \varepsilon$. So let $(m, n, p) \in (\mathbb{N}^*)^3$ such that $s_{x,z}s_{y,y} < m/n < \mu s_{x,z}s_{y,y}$. Because $\sigma(x, z) = s^{-1}_{x,z}u(x)^{-1}u(z)$, multiplying by $u(x)u(z)^{-1}$, we obtain

$$\frac{1}{\sigma(y, y)\sigma(x, z)} < \frac{u(mx)}{u(nz)} < \frac{\mu}{\sigma(y, y)\sigma(x, z)}.$$

Therefore, up to replacing (x, y, z) by (mx, py, nz), we may (and do) suppose that we have $\sigma(y, y)\sigma(x, z)u(x) > u(z) > \sigma(t, t)\sigma(x, z)u(x)$. Then $\exists(a, b) \in (\mathbb{N}^*)^2$ such that

$$u(z) \geqslant \frac{a}{b}\sigma(t, z)u(t) \geqslant \sigma(t, t)\sigma(x, z)u(x).$$

Again, up to replacing (x, y, z, t) by (bx, by, bz, at), we may (and do) suppose that $a = b = 1$. Thus, we have $z \succsim x$; and $u(t) \geqslant \sigma(t, z)^{-1}\sigma(t, t)\sigma(x, z)u(x) = \sigma(x, t)u(x)$, that is $t \succsim x$. Therefore $\succ$ is not a semiorder, contradiction. So we proved that σ^* is a constant map, which implies (1). □

Let A be a $\mathbb{N}^*$-set endowed with a non-empty homothetic interval order $\succ$. From 7.5, $\succ$ is a semiorder if and only if its invariant $\sigma^*_{\succ}$ is a constant map. And $\succ$ is a weak order if and only if $\sigma^*_{\succ} = 1$. We can see the homothetic interval order $\succ$ as a deformation of its associated homothetic weak order $\succ_0$; the invariant $\sigma^*_{\succ}$ being the expression of this deformation. So the homothetic semiorders are the homothetic interval orders for which the deformation is as simple as possible, that is expressed by a constant invariant.

7.6. Example. Let $A = \mathbb{N}^*x \coprod \mathbb{N}^*y$ be the union of two copies of $\mathbb{N}^*$. Let α, β be two real numbers such that $0 < \alpha, \beta \leqslant 1$, and let $\sigma : \bar{A} \times \bar{A} \to \mathbb{R}^*_+$ be the map defined by $\sigma(x, x) = \alpha$, $\sigma(y, y) = \beta$ and $\sigma(x, y) = \sigma(y, x) = (\alpha\beta)^{1/2}$. Let $u : A \to \mathbb{R}_+$ be the morphism of $\mathbb{N}^*$-sets defined by $u(x) = u(y) = 1$. From 6.1, the binary relation $\succ$ on A defined by $z \succ t \Leftrightarrow \sigma(z, t)u(z) > u(t)$ is a *homothetic interval order*. Remark that we have $A^*_{\succ} = A$. Moreover, $\succ$ is a *semiorder* if and only if $\alpha = \beta$; in which case we have $\sigma^*_{\succ} = \alpha$.

Otherwise, we have $r_{x,x} = \sigma(x, x)$ and $r_{y,y} = \sigma(y, y)$. So if $\alpha \neq \beta$, then the map $A \times A \to \mathbb{R}^*_+$, $(z, t) \mapsto r_{z,t}$ *does not factorizes* through the product-map $u \times u$; which answers the question asked in 6.5.

8. In this section, we generalize Proposition 5.1 to the homothetic interval orders.

8.1. Lemma. *Let $(A, \circ)$ be a commutative semigroup endowed with a non-empty homothetic interval order $\succ$. If $\succ_0$ is $\circ$-independent, then $\succ$ est un semiorder.*

Proof. Suppose that $\succ_0$ is $\circ$-independent. In particular, we have $A^* \circ A \subset A^*$. Let $a \in A^*$. For $(x, y, z) \in A^3$, we have $x \circ z \succ_1 y \circ z \Leftrightarrow r_{a,x\circ z} > r_{a,y\circ z}$. Replacing a by $a \circ z \in A^*$, we obtain

$$x \circ z \succ_1 y \circ z \Leftrightarrow r_{a\circ z,x\circ z} > r_{a\circ z,y\circ z} \Leftrightarrow r_{a,x} > r_{a,y} \Leftrightarrow x \succ_1 y.$$

Thus $\succ_1$ is $\circ$-independent. In the same way, we prove that $\succ_2$ est $\circ$-independent. Let $u_0,\ u_1,\ u_2 : A \to \mathbb{R}_+$ be the morphisms of $\mathbb{N}^*$-sets defined by $u_1(x) = s_{a,a}r_{a,x}$, $u_2(x) = s_{a,x}$ and $u_0 = (u_1u_2)^{1/2}$. From 7.3, for $i = 0, 1, 2$, u_i represents $\succ_i$; and from 5.1, u_i is a morphism of semigroups. For $(x, y)^2 \in A$, we have (easy calculation)

$$\begin{aligned} u_0(x \circ y)^2 &= u_0(x)^2 + u_0(y)^2 + u_1(x)u_2(y) + u_1(y)u_2(x) \\ &= [u_0(x) + u_0(y)]^2 + ([u_1(x)u_2(y)]^{1/2} - [u_1(y)u_2(x)]^{1/2})^2, \end{aligned}$$

from which we deduce that $([u_1(x)u_2(y)]^{1/2} - [u_1(y)u_2(x)]^{1/2})^2 = 0$; i.e., that $u_1(x)u_2(y) = u_1(y)u_2(x)$. That is possible only if $u_2 = \lambda u_1$ for a constant $\lambda > 0$. Hence $\succ$ is a semiorder 7.5. □

8.2. Proposition. *Let $(A, \circ)$ be a commutative semigroup endowed with a non-empty homothetic interval order $\succ$. The two following conditions are equivalent*:

(1) $\succ$ *is $\circ$-pseudoindependent*;
(2) *for $i = 1, 2$, $\succ_i$ is $\circ$-independent.*

Proof. Suppose that $\succ$ is $\circ$-pseudoindependent. Let $a \in A^*$. From the proof of 5.1, for $x, y \in A$, we have $r_{a,x\circ y} = r_{a,x} + r_{a,y}$; and in the same way, we obtain $s_{a,x\circ y} = s_{a,x} + s_{a,y}$. So the implication (1) $\Rightarrow$ (2) is proved.

Conversely, suppose that for $i = 1, 2$, $\succ_i$ is $\circ$-independent. Let $u_1,\ u_2 : A \to \mathbb{R}_+$ be two morphisms of $\mathbb{N}^*$-sets verifying (7.2)-(1). For $i = 1, 2$, because u_i represents $\succ_i$ 7.2, it is a morphism of semigroups (5.1). From this we deduce that for $(x, y, z, t) \in A^4$, we have

$$\begin{cases} (x \succ y,\ z \succ t) \Rightarrow x \circ z \succ y \circ t, \\ (x \succsim y,\ z \succsim t) \Rightarrow x \circ z \succsim y \circ t. \end{cases}$$

Let $(x, y) \in A^* \times (A\backslash A^*)$. If $x \circ y \in A\backslash A^*$, then we have $x \succ x \circ y$, that is $u_1(x) > u_2(x \circ y) = u_2(x) + u_2(y) = u_2(x)$, which is impossible because $u_1 \leqslant u_2$. Hence $\succ$ is $\circ$-pseudoindependent. □

8.3. Corollary. *Let $(A, \circ)$ be a commutative semigroup endowed with a non-empty homothetic interval order $\succ$. The two following conditions are equivalent*:

(1) $\succ_0$ *is $\circ$-independent*;
(2) $\succ$ *is a $\circ$-pseudoindependent semiorder.*

Proof. If $\succ_0$ is $\circ$-independent, then $\succ$ is a semiorder (8.1), therefore, $\succ_1 = \succ_0 = \succ_2$ (7.5) and $\succ$ is $\circ$-pseudoindependent. So we have (1) $\Rightarrow$ (2). Conversely, if $\succ$ is a $\circ$-pseudoindependent semiorder, then we have $\succ_1 = \succ_0 = \succ_2$ (7.5) and $\succ_0$ is $\circ$-independent (8.2). □

8.4. Example. Let $A = \mathbb{N}^*x \times \mathbb{N}^*y$ be the product of two copies of $\mathbb{N}^*$, endowed with the structure of commutative semigroup $\circ$ defined by $(mx, ny) \circ (m'x, n'y) = ((m+m')x, (n+n')y)$. Let $\lambda,\ \mu$ be two real numbers such that $0 < \lambda \leqslant \mu$, and let $u_1, u_2 : A \to \mathbb{R}_+$ be the morphisms of semigroups defined by $u_1(mx, ny) = \lambda m + n$ and $u_2(mx, ny) = \mu m + n$. Then from 7.2 and 8.2, the binary relation $\succ$ on A defined by $z \succ t \Leftrightarrow u_1(z) > u_2(t)$, is a $\circ$-*pseudoindependent homothetic interval order*. But the homothetic weak order $\succ_0$ is $\circ$-*independent* (i.e., $\succ_1 = \succ_2$) if and only if we have $\lambda = \mu$; in which case $\succ$ is a *homothetic weak order*.

We conclude with an abstract definition of a biased balance:

8.5. Definition. A commutative semigroup $(A, \circ)$ endowed with a $\circ$-pseudoindependent homothetic semiorder $\succ$, is called a *biased balance*.

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