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A practical approach to fuzzy utilities comparison in fuzzy multicriteria analysis

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Abstract

Comparing fuzzy utilities of the decision alternatives for determining their ranking plays a critical role in fuzzy multicriteria analysis. Existing fuzzy ranking methods may not always be suitable for practical decision problems of large size, due to counter-intuitive ranking outcomes or considerable computational effort. This paper presents a practical approach to address the fuzzy ranking problem. The approach combines the merits of two prominent concepts individually used in the literature: the fuzzy reference set and the degree of dominance. As a result, the decisive information emitted by the fuzzy utilities is sensibly used, and satisfactory ranking outcomes can be achieved. The approach is computationally simple and its underlying concepts are logically sound and comprehensible. A comparative study is conducted on all benchmark cases used in the literature to examine its performance on rationality and discriminatory ability. A new performance measure is introduced to examine its discriminating performance in differentiating between fuzzy utilities. The comparison result shows that the practical approach compares favorably with comparable methods examined.

Keywords: Fuzzy numbers; Fuzzy ranking; Multicriteria analysis; Fuzzy reference sets

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1. Introduction

Decision making often takes place in a fuzzy environment where the information available is imprecise or uncertain. For fuzzy decision problems of prioritizing or evaluating a finite set of alternatives involving multiple criteria, the application of fuzzy set theory to multicriteria analysis models under the framework of utility theory has proven to be an effective approach [5,8,15,23,35–37]. In fuzzy multicriteria analysis models, the overall utility of an alternative with respect to all criteria is often represented by a fuzzy number, referred to as fuzzy utility. In this case, the ranking of the alternatives is based on the comparison of their corresponding fuzzy utilities.

Quite a few fuzzy ranking methods for comparing fuzzy numbers appear in the literature. Bortoland and Degani [3], Chen and Hwang [8], Dubois and Prade [12,13], Li and Lee [22], Nakamura [24], Tseng et al. [27], and Zimmermann [35] give an extensive investigation of fuzzy ranking methods based on various classification schemes. No single existing ranking method dominates performance comparisons in all situations for which the method can be used. With the nature of the fuzzy ranking problem, each method has to be judged by its own merit, since an overall evaluation of the methods based on specific criteria would be subjective to some degree [35]. In practical applications, a rational decision maker (DM) [8,37] only accepts fuzzy ranking methods that can produce outcomes consistent with human's intuition [24,33].

To help the DM make a confident choice between similar alternatives, ranking methods require a high degree of discriminatory ability. The discriminatory ability of a ranking method refers to its capability to differentiate alternatives characterized by similar fuzzy utilities, which are of interest to the DM [3,33,35]. The need for comparing similar fuzzy utilities is likely to grow as the problem size increases [1,26]. In addition, the computations required must not involve any sophistication, so that the comparison of a large number of fuzzy utilities can be carried out within a practical time frame. All of these seem to suggest that the rationality, discriminatory ability, and computational simplicity are of crucial importance for the successful implementation of any fuzzy ranking method on actual decision problems of large size.

Although most fuzzy ranking methods in the literature produce satisfactory results for clear-cut problems, they may generate counter-intuitive outcomes or are not discriminatory enough under certain circumstances [3,8,35]. In addition, most of them require considerable computational effort, which is obviously not desirable for handling large-scale fuzzy multicriteria analysis problems. It is evident that a fuzzy ranking method that can produce rational ranking results using sound logic and simple computations is desirable for practical fuzzy decision problems.

In this paper, we propose a novel practical fuzzy ranking approach by combining the two prominent concepts individually used in the fuzzy ranking research: the fuzzy reference set and the degree of dominance. The practical approach is computationally simple and its underlying concepts are logically sound and comprehensible. In the following, we first discuss these two concepts to pave the way for the development of the practical approach. The detailed algorithm of the practical approach is then given to show its implementability on practical problems. A comparative study is followed to demonstrate its performance on rationality and discriminatory ability. The comparison result shows that it has practical advantages over other comparable methods examined.

2. The fuzzy reference set

Let $\{A_i\}$ ($i \in N = \{1, 2, \dots, n\}$) be the set of n normal fuzzy numbers (fuzzy utilities) on the real line $\boldsymbol{R}$ to be compared for ranking the corresponding set of n alternatives. The assumption of normality does not cause any loss of generality as any type of fuzzy numbers can be easily transformed into a normal one [8,34,35]. Each fuzzy number A_i is represented as $A_i = \{(x, \mu_{Ai}(x)),\ x \in \boldsymbol{R}\}$, where $\boldsymbol{R}$ is the universe of discourse and $\mu_{A_i}(x)$ indicates the degree of membership of x in A_i.

To allow these n fuzzy numbers to be compared in a straightforward manner, fuzzy reference sets such as the fuzzy maximum and the fuzzy minimum are often defined [7,17,18,20,24,29,30]. These definitions are based on part of the following information: (a) the absolute position and the relative position of fuzzy numbers on the real line $\boldsymbol{R}$, (b) the shape (the increasing left part and decreasing right part), (c) the spread, and (d) the area below the graph of each fuzzy number. The fuzzy reference sets, denoted as $Y = \{(y, \mu_Y(y)),\ y \in \boldsymbol{R}\}$ serve as the common comparison base, with which each fuzzy number is compared for determining the overall ranking of the corresponding decision alternatives.

Jain [17,18] first uses a fuzzy maximum (called the maximizing set) as the fuzzy reference set. The fuzzy maximum is defined as $Y_{\max} = \{(y, \mu_{Y_{\max}}(y)),$ $\mu_{Y_{\max}}(y) = (y/x_{\max})^k,\ y \in R\}$, where k is an integer that can be assigned in a given context to indicate a DM's attitude towards risk, and $x_{\max}(x_{\max} \neq 0)$ is the largest value of all supports $S(A_i)$ of fuzzy numbers A_i ($i = 1, 2, \dots, n$), determined by

$$x_{\max} = \sup\left\{ \cup_{i=1}^{n} S(A_i)\right\}; \quad S(A_i) = \{x, \mu_{A_i}(x) > 0,\ x \in R\}, \tag{1}$$

where $\sup\{\cdot\}$ is the supremum of all $S(A_i)$. Fig. 1(a) illustrates this definition. With the fuzzy maximum $Y_{\max}$ defined, fuzzy numbers are ranked based on a ranking index value given by

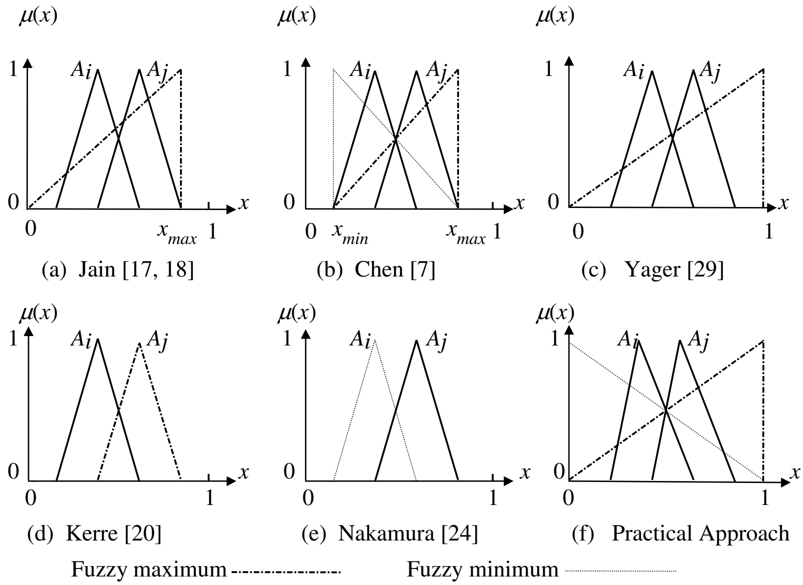


Fig. 1. Various definitions of fuzzy reference sets.

$$Vi = \sup_{x \in R} \left\{ \min(\mu_{Y_{\max}}(x), \mu_{A_i}(x)) \right\}. \tag{2}$$

This ranking method is not logically sound as only the information on the decreasing right part of the fuzzy numbers is used. This results in counter-intuitive ranking in some cases [3,7–9,35]. In addition, the membership value of Jain's fuzzy maximum may be negative which contradicts the definition of the membership function, $0 \leqslant \mu_{Y_{\max}}(x) \leqslant 1$, if negative support is contained in some fuzzy numbers and k is an odd integer [7]. If k is an even integer, and if the absolute value of the negative support x is greater than the absolute value of $x_{\max}$, then $\mu_{Y_{\max}}(x) > 1$. This also conflicts with the definition of the membership function.

To avoid the negative membership problem associated with Jain's definition, Chen [7] defines the fuzzy maximum as $Y_{\max} = \{(y, \mu_{Y_{\max}}(y)), \mu_{Y_{\max}}(y) = \left(\frac{y - x_{\min}}{x_{\max} - x_{\min}}\right)^k, \ y \in R\}$ (see Fig. 1(b)). The parameter k is used in the same way as in Jain's [16,17]. This definition uses both the largest and smallest values of all supports of the fuzzy numbers ($x_{\max}$ and $x_{\min}$ respectively), i.e. the relative position of the fuzzy numbers is considered. To achieve a better degree of discriminatory ability for similar fuzzy numbers, a fuzzy minimum is defined as $Y_{\min} = \{(y, \mu_{Y_{\min}}(y)), \mu_{Y_{\min}}(y) = \left(\frac{x_{\max} - y}{x_{\max} - x_{\min}}\right)^k, \ y \in R\}$, which makes use of the information from the increasing left part of the fuzzy numbers. As a result, both right and left values of each fuzzy number contribute to its relative ranking.

However, Chen's approach may still produce unreasonable ranking results under certain situations [8]. This is due to the use of the same process as Jain [16,17] for obtaining the ranking value. In addition, the ignorance of the absolute position of the fuzzy numbers on the real line results in the same ranking value for different sets of fuzzy numbers that have the same relative position [8]. The definition of the fuzzy maximum and the fuzzy minimum by considering only the relative position of the fuzzy numbers prevents it from being applicable for situations where the comparison between different sets of fuzzy numbers (e.g. different groups of decision alternatives) is required.

Yager [29] defines the fuzzy maximum $Y_{\max} = \{(y, \mu_{Y_{\max}}(y)), \mu_{Y_{\max}}(y) = y, y \in \boldsymbol{R}\}$ as the fuzzy reference set (see Fig. 1(c)). This definition considers the absolute position of the fuzzy numbers on the real line when they are compared with the fuzzy maximum. The relative ranking is based on the closeness of each fuzzy number to the fuzzy maximum measured by the Hamming distance between $Y_{\max}$ and each fuzzy number. The definition of the relationship between $Y_{\max}$ and A_i by the Hamming distance is solely based on area measurement, which ignores the relative position of the fuzzy numbers on the real line. As a result, this method conflicts with intuition in some cases [8].

Kerre [20] uses the concept where the best alternative has the maximum gain, to define the fuzzy maximum between two fuzzy numbers A_i and A_j $(i, j \in N,\ i \neq j)$ as $Y_{\max} = \{(y, \mu_{Y_{\max}}(y)), \mu_{Y_{\max}}(y) = \sup_{y=(x_i \vee x_j)}[\mu_{A_i}(x_i) \wedge \mu_{A_j}(x_j)], y \in R\}$, shown as in Fig. 1(d). The fuzzy number with a smaller Hamming distance to the fuzzy maximum is preferred. Following a similar concept, Nakamura [24] defines a fuzzy minimum between two fuzzy numbers A_i and A_j $(i, j \in N,\ i \neq j)$ as $Y_{\min} = \{(y, \mu_{Y_{\min}}(y)), \mu_{Y_{\min}}(y) = \sup_{y=(x_i \wedge x_j)}[\mu_{A_i}(x_i) \wedge \mu_{A_j}(x_j)], y \in \boldsymbol{R}\}$, shown as in Fig. 1(e). A fuzzy number that is farther from the fuzzy minimum is considered larger. Both methods require considerable computational efforts for fuzzy numbers with a continuous membership function and may produce unsatisfactory results [8].

The above studies suggest that the performance of the ranking method using the concept of the fuzzy reference set is highly influenced by (a) its definition and (b) the way it is compared. A good fuzzy ranking method thus needs to logically link the definition of the fuzzy reference set with the ranking index in order to use all decisive information for distinguishing between fuzzy numbers.

In the practical approach, we take advantage of Yager's definition [29] (also given in [8]) to define the fuzzy maximum and the fuzzy minimum (see Fig. 1(f)), based on the following grounds: (a) it considers the absolute position of the fuzzy numbers, (b) it involves no computations as the definition is independent of the fuzzy numbers to be compared, thus being applicable to all ranking problems, and (c) it permits the comparison between different sets of fuzzy numbers (e.g. different groups of decision alternatives) as they are all compared with the same fuzzy reference sets, resulting in comparable ranking values.

To use the relative position of the fuzzy numbers and other information for achieving a rational ranking outcome, the dominance concept is applied. This allows us to make best use of the concept of the fuzzy reference set.

3. The degree of dominance

To compare two fuzzy numbers A_i and A_j $(i, j \in N,\ i \neq j)$ as to how much larger A_i is over A_j, the dominance concept is introduced. The most widely used definition is based on the maximum grade of membership of fuzzy number A_i in A_j, given by $\max_{x \in R}(\min(\mu_{A_i}(x), \mu_{A_j}(x)))$ [1,2,6,8,21,25,28]. This concept is similar to Jain's definition on the degree of optimality [16,17], although it is applied in a different context. For example, in Baas and Kwakernaak [1], this indicates the degree to which a fuzzy number A_i is ranked first when compared with the best fuzzy number A_j $(j \neq i)$, identified with the use of a conditional fuzzy set. This concept is shared by Watson et al. [28] who describe each of pairwise comparisons of fuzzy numbers as a fuzzy implication. Along the same line, Baldwin and Guild [2] present a better method by defining the better of two fuzzy numbers using a two-dimensional fuzzy preference relation. However, the problem with these methods, mainly resulting from the definition of dominance, is illustrated in [8,21]. Tong and Bonissone [25] directly use this definition to represent the dominance of A_i over A_j, although their method produces a linguistic solution rather than a numerical ranking. Despite their intention for reflecting the separation between two fuzzy numbers, the information about the overall shape of fuzzy numbers is ignored.

Departing from the above concept, Tseng and Klein [26] define the degree of dominance between two fuzzy numbers by comparing their overlap and non-overlap areas, in line with the concept of Hamming distance. This method shows an advantage over some existing methods. However, the relative demerits of this method include that (a) the computation of the areas is not straightforward, and (b) an additional, often tedious, pairwise comparison process is needed for comparing a large set of fuzzy numbers.

Yuan [33] defines the degree of dominance of A_i over A_j as their arithmetic difference, that is, $A_i - A_j$, from which an improved ranking method to Nakamura [24] and Baas and Kwakernaak [1] is derived. Although Yuan's method performs better than some methods compared in terms of rationality and discriminatory ability, it requires considerable computational effort.

In the practical approach, we use Yuan's definition to determine the fuzzy set difference between A_i and A_j, as it allows all their possibly occurring combinations to be compared [33]. However, we apply it in a different context to indicate the relative closeness between A_i and A_j, which is similar in concept to the crisp number comparison. When more than two crisp numbers are compared, the arithmetic difference between any pair of numbers can be regarded as an indication of how much larger (positive or negative) one is over the other. This value highlights the relative closeness of one pair of crisp numbers in comparison with other pairs. This relative closeness can be regarded as a ranking index when comparing a large set of fuzzy numbers with common fuzzy reference sets on the same universe of discourse, where the difference between the fuzzy reference set and each fuzzy number is measured in relative terms.

Fuzzy arithmetic has been well developed to perform standard arithmetic operations on fuzzy numbers [19]. The fuzzy set difference D_{i-j} between A_i and A_j can be calculated by fuzzy subtraction, given as

$$D_{i-j} = A_i - A_j = \{(z, \mu_{D_{i-j}}(z)),\ z \in R\}, \tag{3}$$

where the membership function of D_{i-j} is defined as

$$\mu_{D_{i-j}}(z) = \sup_{z=x_i-x_j} (\min(\mu_{A_i}(x_i), \mu_{A_j}(x_j)),\ x_i, x_j \in R). \tag{4}$$

To determine how much larger A_i is over A_j a defuzzification process is required for extracting a single scalar value from D_{i-j} that can best represent D_{i-j}. Numerous defuzzification methods have appeared in the literature [8,9,35,36]. In the practical approach, we use the average of mid-points of all α-cuts on the fuzzy number [4,11,14,32]. This method is (a) consistent with the fuzzy subtraction used in (3), (b) simple and comprehensible in concept, (c) efficient in computation, and (d) able to incorporate the DM's attitude towards risk in practical decision settings.

By applying the method of the mean value of fuzzy numbers to the fuzzy set difference D_{i-j}, the degree of dominance of A_i over A_j is determined by

$$d(A_i - A_j) = \int_0^1 D_{i-j}(\alpha)\,\mathrm{d}\alpha, \tag{5}$$

$$D_{i-j}(\alpha) = \begin{cases} (1-\lambda)d_{i-j}^{\mathrm{L}\alpha} + \lambda d_{i-j}^{\mathrm{R}\alpha}, & 0 \leqslant \alpha \leqslant 1, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where λ is an optimism index representing the DM's attitude towards risk, and $d_{i-j}^{\mathrm{L}\alpha}$ and $d_{i-j}^{\mathrm{R}\alpha}$ are the lower bound and upper bound of the interval $[d_{i-j}^{\mathrm{L}\alpha}, d_{i-j}^{\mathrm{R}\alpha}]$ respectively, resulting from the α cut on D_{i-j} $(0 \leqslant \alpha \leqslant 1)$ [11]. In practical applications, $\lambda = 1$, $\lambda = 0.5$, or $\lambda = 0$ can be used to indicate that the DM involved is optimistic, moderate, or pessimistic about the decision outcome respectively. The larger the value of $d(A_i - A_j)$, the higher the degree of dominance A_i over A_j. The value of $d(A_i - A_j)$ indicates the degree of relative closeness between A_i and A_j, when comparing a set of fuzzy numbers on the same universe of discourse.

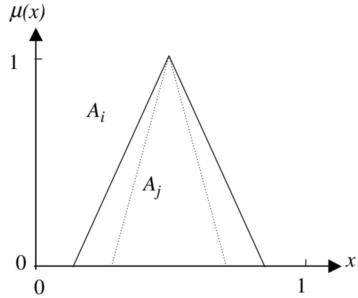


Fig. 2. An indifference case.

A special indifference situation is given in Fig. 2, where $d(A_i - A_j) = 0$. In this case, fuzzy numbers A_i and A_j share the same central tendency with different amounts of dispersion. Generally speaking, human intuition would favor a fuzzy number with a larger mean value and a smaller dispersion [21]. However, the standard deviation and mean value cannot be used as the sole basis for comparing two fuzzy numbers respectively as they do not always agree with each other [8,35].

To effectively handle this situation, a coefficient of variation is used as the preference index for fuzzy number A_i, defined as

$$CV(A_i) = \frac{m_{A_i}}{\sigma_{A_i}}, \tag{7}$$

where σ_{A_i} and m_{A_i} are the standard deviation and the mean value of fuzzy number A_i respectively, given by

$$m_{A_i} = \frac{\int_{S(A_i)} x\mu_{A_i}(x)\mathrm{d}x}{\int_{S(A_i)} \mu_{A_i}(x)\mathrm{d}x}, \tag{8}$$

$$\sigma_{A_i} = \left[\frac{\int_{S(A_i)} x^2\mu_{A_i}(x)\mathrm{d}x}{\int_{S(A_i)} \mu_{A_i}(x)\mathrm{d}x} - \left(\frac{\int_{S(A_i)} x\mu_{A_i}(x)\mathrm{d}x}{\int_{S(A_i)} \mu_{A_i}(x)\mathrm{d}x} \right)^2 \right]^{1/2}, \tag{9}$$

where $S(A_i) = \{x, \mu_{A_i}(x) > 0,\ x \in R\}$, being the support of fuzzy number A_i. The larger the value of $CV(A_i)$, the more preferred the fuzzy number A_i.

The method presented above for calculating the degree of dominance between fuzzy numbers has a number of useful properties. These properties, listed below, are mostly intuitive and derivative based on [4,11,14,16,31].

Proposition 3.1. *If $a, b \in \boldsymbol{R}$, then $d(a - b) = a - b$.*

Proposition 3.2. *Let A_i and A_j $(i,j \in N,\ i \neq j)$ be two fuzzy numbers, if $d(A_i - A_j) > 0$, then $A_i > A_j$.*

Proposition 3.3. *Let A_i and A_j $(i,j \in N,\ i \neq j)$ be two fuzzy numbers, if $d(A_i - A_j) < 0$, then $A_i < A_j$.*

Proposition 3.4. *Let A_i and A_j $(i,j \in N,\ i \neq j)$ be two fuzzy numbers, if $d(A_i - A_j) = d(A_j - A_i)$, then $A_i = A_j$.*

Proposition 3.5. *Let A_i, A_j, and A_k $(i,j,k \in N,\ i \neq j \neq k)$ be three fuzzy numbers, if $d(A_i - A_j) > 0$ and $d(A_j - A_k) > 0$, then $d(A_i - A_k) > 0$.*

4. The practical approach

The new, practical fuzzy ranking approach presented in this section incorporates the two concepts discussed in the previous two sections. The approach involves the use of two fuzzy reference sets: the fuzzy maximum and the fuzzy minimum. The rationale of the approach is that a fuzzy number is preferred if it is dominated by the fuzzy maximum by a smaller degree (i.e. closer to the fuzzy maximum), and at the same time dominates the fuzzy minimum by a larger degree (i.e. farther away from the fuzzy minimum) [7,10]. The approach makes use of all decisive information associated with the fuzzy numbers. The definition of the two fuzzy reference sets uses the absolute position of the fuzzy numbers. The use of fuzzy subtraction for defining the relationship between the fuzzy reference sets and the fuzzy numbers considers the relative position, and the shape (increasing left part and decreasing right part) and the area of each fuzzy number.

In the practical approach, the fuzzy maximum $(Y_{\max} = (y, \mu_{Y_{\max}}(y)),\ y \in \boldsymbol{R})$ and the fuzzy minimum $(Y_{\min} = (y, \mu_{Y_{\min}}(y)),\ y \in \boldsymbol{R})$ are always given by

$$\mu_{Y_{\max}}(y) = \begin{cases} y, & \text{if } 0 \leqslant y \leqslant 1, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

$$\mu_{Y_{\min}}(y) = \begin{cases} 1 - y, & \text{if } 0 \leqslant y \leqslant 1, \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

Based on our definition in (3)–(5), the degree to which the fuzzy maximum dominates each fuzzy number A_i $(i = 1, 2, \ldots, n)$ can be expressed in general form as

$$d_i^+ = d(Y_{\max} - A_i) = \int_0^1 D_{\max - i}(\alpha)\,\mathrm{d}\alpha, \tag{12}$$

where $D_{\max - i} = Y_{\max} - A_i$, and $D_{\max - i}(\alpha)$ is determined by (6) based on the α-cut on $D_{\max - i}$. The value of d_i^+ indicates the degree of closeness of each fuzzy number A_i to the fuzzy maximum $Y_{\max}$. A rational DM would prefer a fuzzy number closer to the fuzzy maximum, indicated by a smaller value of d_i^+.

Similarly, the degree of dominance of each fuzzy number A_i over the fuzzy minimum $Y_{\min}$ is given as

$$d_i^- = d(A_i - Y_{\min}) = \int_0^1 D_{i-\min}(\alpha)\,\mathrm{d}\alpha, \tag{13}$$

where $D_{i-\min} = A_i - Y_{\min}$, and $D_{i-\min}(\alpha)$ is determined by (6) based on the α-cut on $D_{i-\min}$. The value of d_i^- represents the degree of closeness of each fuzzy number A_i to the fuzzy minimum. Fuzzy numbers that are farther away from the fuzzy minimum, indicated by a larger value of d_i^-, are considered preferred.

An overall preference index for each fuzzy number A_i $(i = 1, 2, \ldots, n)$ as the relative ranking value is obtained by

$$P_i = \frac{(d_i^-)^2}{(d_i^+)^2 + (d_i^-)^2}. \tag{14}$$

The larger the preference index P_i, the more preferred the fuzzy number A_i. Clearly, the smaller the d_i^+ or the larger the d_i^-, the larger the P_i. This implies that a fuzzy number A_i is preferred (a larger P_i) if it is closer to the fuzzy maximum (a smaller d_i^+) and it is farther away from the fuzzy minimum (a larger d_i^-).

The algorithm of the practical approach for comparing n fuzzy numbers (fuzzy utilities) $\{A_i\}$ $(i = 1, 2, \ldots, n)$ can be summarized as follows.

Step 0. Determine the fuzzy maximum $Y_{\max}$ and the fuzzy minimum $Y_{\min}$ by (10) and (11).
Step 1. Calculate the fuzzy set difference $D_{\max-i}$ between $Y_{\max}$ and A_i and the fuzzy set difference $D_{i-\min}$ between A_i and $Y_{\min}$ respectively by (6).
Step 2. Determine the λ value based on the DM's attitude towards risk.
Step 3. Determine the degree of dominance (d_i^+) of $Y_{\max}$ over A_i by (12), based on $D_{\max-i}$ derived at Step 1.
Step 4. Determine the degree of dominance (d_i^-) of A_i over $Y_{\min}$ by (13), based on $D_{i-\min}$ obtained at Step 1.
Step 5. Calculate the overall preference index P_i of A_i by (14).
Step 6. Rank the fuzzy numbers A_i $(i = 1, 2, \ldots, n$) in descending order of P_i.
Step 7. If $P_j = P_k$ $(j, k \in N, j \neq k)$, calculate the coefficients of variation $CV(A_j)$ and $CV(A_k)$ for fuzzy numbers A_j and A_k by (7)–(9). $A_j \geqslant A_k$, if $CV(A_j) \geqslant CV(A_k)$.

5. Comparative study

The algorithm presented in the previous section shows the practical advantages of the practical approach over other ranking methods in terms of computational simplicity, when triangular fuzzy numbers (widely used in practical applications) are compared. To examine its performance on rationality and discriminatory ability, the approach is compared with comparable ranking methods which use one of the concepts discussed above. All the problems or cases used in the literature are examined. The examination shows that the approach always produces satisfactory results for all the cases in terms of rationality and discriminatory ability. It is noteworthy that the approach also gives satisfactory results if Chen's fuzzy maximum and fuzzy minimum [7] (see Fig. 1(b)) are used as the fuzzy reference sets.

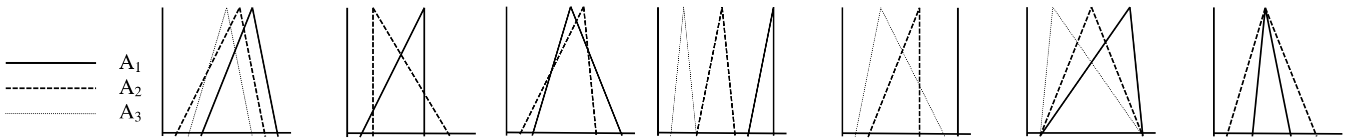
To demonstrate how the practical approach compares favorably with comparable methods, we present only seven benchmark cases here as an example. Table 1 shows the ranking results of the representative methods examined. In Table 1, some results are adopted from [3,8], and some are calculated by this study. For the practical approach, the DM's attitude towards risk is assumed to be moderate (i.e. $\lambda = 0.5$). For easy comparison, ranking values produced by Kerre [20] and Yager [29] are modified by subtracting the original index value from the crisp number 1, so that they are listed in descending order. The first row indicates the rational ranking suggested by the literature [3,8,21,22,27,35]. The last row shows the ranking values given by the practical approach. All methods, except for the practical approach, give unsatisfactory results for one case or more.

To examine how effective the practical approach is in differentiating between fuzzy utilities, we suggest a new performance measure, called the discrimination index. This index is calculated based on the final ranking values of fuzzy utilities produced by a ranking method. It can be regarded as the degree to which the fuzzy utilities are distinguished, which is of interest to the DM. This is based on our perception that if an alternative is to be selected, the DM will have much more confidence if its ranking value is much larger than that of other alternatives. This also applies to the selection of multiple alternatives from a set of alternatives. For example, if one of the two alternatives is to be selected, the discrimination index will have the maximum value of 1 if their normalized ranking values are 1 and 0 respectively. To select 2 out of 3 alternatives, the ranking values of 1, 0.5 and 0 will yield a discrimination index of 1. Thus, for a set of n normalized ranking values P_i $(i = 1, 2, \ldots, n)$ listed in descending preference order, the discrimination index ($DI_k \in [0, 1]$) is defined as

$$DI_k = \frac{\sum_{i=1}^{k} \dfrac{P_i - P_{i+1}}{\max\left(\frac{1}{k}, (P_i - P_{i+1})\right)}}{k}, \quad k \in \{1, 2, \ldots, n-1\}, \tag{15}$$

where k is an integer indicating the number of alternatives the DM wants to select out of n alternatives. DI_k is measured by the average ratio of the

Table 1
Comparison results



	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Ranking	$A_1 > A_2 > A_3$	$A_1 > A_2$	$A_1 > A_2$	$A_1 > A_2 > A_3$	$A_1 > A_2 > A_3$	$A_1 > A_2 > A_3$	$A_1 > A_2$
Bass–Kwakernaak [1]	1.00 > 0.74 > 0.60	1.00 > 0.84	0.80 < 1.00	1.00 > 0.00 = 0.00	1.00 > 0.00 = 0.00	1.00 > 0.74 > 0.60	1. 00 = 1.00
Chen [6]	0.47 > 0.38 > 0.34	0.58 > 0.35	0.55 = 0.55	0.93 > 0.46 > 0.15	1.00 > 0.45 > 0.35	0.70 > 0.50 > 0.33	0.50 = 0.50
Kerre [18]	1.00 > 0.942 > 0.94	0.11 > 0.04	0.88 > 0.65	1.00 > 0.72 < 0.85	1.00 > 0.90 > 0.65	0.24 > 0.14 > 0.00	0.09 = 0.09
Yager's HDI [27]	0.58 > 0.53 > 0.52	0.54 > 0.53	0.78 < 0.81	0.53 > 0.48 > 0.40	0.50 < 0.55 < 0.56	0.79 > 0.73 > 0.67	0.55 < 0.59
Yager's CI [29]	0.45 > 0.37 > 0.33	0.42 = 0.42	0.64 > 0.48	0.95 > 0.55 > 0.20	1.00 > 0.50 > 0.22	0.76 > 0.70 > 0.63	0.50 = 0.50
Practical approach	0.26 > 0.07 > 0.03	0.31 > 0.14	0.92 > 0.90	0.92 > 0.65 > 0.01	0.90 > 0.50 > 0.10	1.00 > 0.98 > 0.84	16.7 > 5.56

HDI = hamming distance index, CI = centroid index.

difference between each pair of adjacent ranking values P_i and P_{i+1} ($i \in \{1, 2, \ldots, k\}$) and an ideal difference. The ideal difference is determined by the value of k. Only the first k ranking values are the DM's concern because there is no need to distinguish between $(n - k)$ alternatives that are not to be selected. The larger the discrimination index is, the better the discriminating performance of the method is, and the more confidence the DM has in making decisions based on the ranking values produced by the method.

Table 2 shows the discrimination index of the ranking values produced by the ranking methods for the seven cases in Table 1. The practical approach has a high discriminating performance, as compared with most of the methods examined.

The result of the comparative study upholds the belief held by Chen and Hwang [8] that a flawless ranking approach may only be obtained by combining some of the good ideas from available ranking methods into one algorithm.

6. Conclusion

The importance of fuzzy number comparison has been realized by some applications using fuzzy set theory [7]. Fuzzy multicriteria analysis often requires comparing fuzzy utilities (fuzzy numbers) to rank decision alternatives. Although existing fuzzy ranking methods have shown their merits, they are not always practically capable of comparing similar or a large set of fuzzy utilities. To ensure that a reliable decision outcome is always obtained, a rational fuzzy ranking method with good discriminatory ability is required, especially for large-scale problems. To make the decision outcome acceptable, the underlying concepts used by the method must be logically sound and comprehensible. To ensure its implementability in practical problems, the computational process must be simple and fast. To this end, we have presented a practical approach for ranking fuzzy utilities in practical multicriteria analysis problems.

To effectively compare a set of fuzzy utilities, we have to make use of decisive information that can distinguish between them. Research has shown that the more the information is used, the better the method performs in terms of its rationality and discriminatory ability. In this respect, the practical approach presented compares favorably with other comparable methods examined on all benchmark cases. This is mainly because it sensibly uses all the decisive information, while other methods use only part of it. The experimental performance of the practical approach shows its practical advantages for comparing fuzzy utilities in fuzzy decision problems.

Table 2
Comparison of the discrimination index (DI_k) between ranking methods

Method	Case										
	(a)		(b)	(c)	(d)		(e)		(f)		(g)
	$k=1$	$k=2$	$k=1$	$k=1$	$k=1$	$k=2$	$k=1$	$k=2$	$k=1$	$k=2$	$k=1$
Bass–Kwakernaak [1]	0.26	0.40	0.16	–	1.00	–	1.00	–	0.26	0.40	–
Chen [6]	0.09	0.13	0.23	–	0.47	0.78	0.55	0.65	0.20	0.37	–
Kerre [18]	0.06	0.06	0.07	0.23	0.28	–	0.10	0.35	0.10	0.24	–
Yager-HD index [27]	0.05	0.06	0.01	–	0.05	0.13	–	–	0.06	0.12	–
Yager-centroid index [29]	0.08	0.12	–	0.16	0.40	0.75	0.50	0.78	0.06	0.13	–
Practical approach	0.19	0.23	0.17	0.02	0.27	0.77	0.40	0.90	0.02	0.16	1.00

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