International Journal of Solids and Structures 48 (2011) 2755–2763

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Spatial behavior in the electromagnetic theory of microstretch elasticity

C. Galeş *

Faculty of Mathematics, Al. I. Cuza University of Iaşi, 700506 Iaşi, Romania

ARTICLE INFO

Article history:
Received 11 March 2011
Received in revised form 25 May 2011
Available online 22 June 2011

Keywords:
Microstretch magnetoelectroelastic solids
Spatial behavior
Domain of influence
Uniqueness

ABSTRACT

This paper is concerned with the electromagnetic theory of microstretch elasticity. First, the initial boundary value problem is formulated in the framework of the linear dynamic theory of microstretch magnetoelectroelastic solids. Then, the spatial behavior of solutions is studied in both bounded and unbounded regions. The obtained result gives an exact idea of the domain of influence, in the sense that for each fixed time in a given interval, the entire activity vanishes at distanced from the support of the given data greater than a time-dependent threshold value. The study of spatial behavior is completed by an exponential decay estimate inside the domain of influence. As a by product a uniqueness result holding for both bounded and unbounded bodies is derived. Finally, the effect of a concentrated microstretch body force is studied.

1. Introduction

The interaction of electromagnetic fields with deformable bodies has been the subject of many theoretical investigations in continuum mechanics (see for example the books of Tiersten, 1969; Eringen and Maugin, 1990; Zhou, 1999; Yang, 2005).

Several recent works (Eringen, 1999, 2003, 2004; Lee et al., 2004) are dedicated to formulate electromagnetic theories for elastic bodies with inner structure. Thus, in the papers by Eringen (2003) and Lee et al. (2004) it was introduced a continuum theory of micromorphic electromagnetic thermoelastic solids, while Eringen (2004) derived the electromagnetic theory of microstretch elasticity. The intended applications of these theories are to porous elastic bodies such as bones and ceramics, synthetic materials containing microscopic components (e.g., nanocomposites), solids with microcracks, etc. (see Eringen, 1999).

Special cases of the field equations are the theory of piezoelectricity and the theory of magnetoelasticity. These theories consider only static or quasi-static electromagnetic fields. Thus, the mechanical equations are dynamic while the electromagnetic equations are static and the electric field and the magnetic field are not dynamically coupled. We recall that the linear theory of microstretch piezoelectricity was studied by Ieşan (2006) and Quintanilla (2008), while in the paper by Ieşan and Quintanilla (2007) some important theorems have been proven for microstretch thermopiezoelectricity. Moreover, the basic equations governing the bending of microstretch piezoelectric plates have been treated by Ieşan (2008a), and a linear theory of microstretch thermopiezoelectricity without energy dissipation has been presented by Ieşan (2008b).

Here we consider the full electromagnetic theory of microstretch elasticity (Eringen, 2004). Our goal is to investigate the spatial behavior of solutions to the magnetoelectroelastic initial boundary value problem. It is worth to note that in the framework of microstretch piezoelectricity, the problem of spatial behavior of solutions has been tackled by Quintanilla (2008). He derived a spatial decay estimate for the solution to the problem of a homogeneous and isotropic semi-infinite cylinder in motion, subject to homogeneous initial and boundary data except for that prescribed on the base. Quintanilla (2008) utilized a measure of solution which leads to a polynomial decay estimate in terms of the distance from the loaded end of the cylinder. The reason for which the result is not of exponential type is due to the quasi-static feature of the considered problem. In piezoelectricity the electric fields are considered quasi-static, although the mechanical equations are dynamic. Or in mathematical terms, the theory of piezoelectricity combines hyperbolic with elliptic equations.

The purpose of this paper is to show that if the full electromagnetic theory of microstretch elasticity is considered then a stronger result can be obtained, in contrast with the special case of quasi-static piezoelectricity. Thus, introducing an adequate measure of solutions and utilizing its properties we get both a domain of influence and an exponential decay estimate inside the domain of influence (see Chiriţă and Ciarletta, 1999 for corresponding results in elasticity and viscoelasticity). The result is proved in the general context of anisotropic and inhomogeneous magnetoelectroelastic microstretch bodies. And clearly, the result holds for dynamic piezoelectricity (or dynamic magnetoelasticity).

Such studies are motivated by the rapid development of smart structures technology and the current models introduced to

* Tel.: +40 232 201226; fax: +40 232 201160.
E-mail address: cgales@uaic.ro

0020-7683/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2011.05.025

describe the behavior of solids with inner structures, such as animal bones, solids with microcracks, foams and other synthetic materials. For example, when the electromagnetic theory of microstretch elasticity was introduced in the paper by Eringen (2004), the following application coming from medicine has been given as motivation: *"a physical exercise therapy program designed for bone healing is based on deformations and motions of bones under the application of a mild amount of stress. Clinically, it is also known that an electromagnetic field applied to bone hastens the healing processes,* Satter et al. (1999)*. These processes involve interactions of electromagnetic fields and mechanical deformations of porous solids, namely bones."* Going further with this reasoning, we may ask, for example, what happens during the therapy program with the healthy bone tissue? How do external loads (of mechanical or/and electromagnetic origins) applied to a region which need to be cured affect the rest of the bone, which let say is healthy and must not be deformed? The domain of influence provides a clear answer for this problem.

In the next Section we set down the basic equations and formulate the initial boundary value problem. Then, we discuss the restrictions imposed on the constitutive coefficients and establish some preliminary estimates in Section 3. The spatial behavior of solutions is described in the Section 4. Following Chiriţă and Ciarletta (1999), we introduce first the so called "support" of the given data in a fixed interval of time $[0,T]$, that is the set of all points for which at least one of the given data in $[0,T]$ (boundary or initial data or body loads) is nonzero. We assume that this set is bounded and included in a bounded regular set. Then, we consider an appropriate time-weighted surface power function. Using its properties we get the domain of influence and an exponential decay estimate in terms of the distance from the support of the given data inside the domain of influence. A uniqueness result is obtained. Finally, Section 5 is concerned with the problem of a concentrated microstretch body force that acts in an unbounded domain. Using the properties of Laplace transform an approximate solution useful for small times was obtained.

2. Basic formulation

We consider a body that at time $t = 0$ occupies the regular region $\mathcal{B}$ of Euclidean three-dimensional space whose boundary is the regular surface $\partial\mathcal{B}$. We refer the motion of the body to a fixed system of rectangular Cartesian axes $0x_k$ ($k = 1,2,3$). We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over integer (1,2,3), summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation.

We consider the linear theory of microstretch magnetoelectroelasticity. The basic equations of the dynamic theory consist of the following equations (see Eringen, 2004; Ieşan, 2006):

– the equations of motion

$$\begin{aligned} &t_{ji,j} + \rho_0 f_i = \rho_0 \ddot{u}_i, \\ &m_{ji,j} + \epsilon_{ijk} t_{jk} + \rho_0 l_i = \rho_0 I_{ij} \ddot{\varphi}_j, \\ &\pi_{k,k} - \sigma + \rho_0 l = \rho_0 \jmath_0 \ddot{\varphi}; \end{aligned} \tag{1}$$

– the Maxwell's equations

$$\epsilon_{ijk} E_{k,j} = -\frac{1}{c}\dot{B}_i, \quad \epsilon_{ijk} H_{k,j} = \frac{1}{c}\dot{D}_i; \tag{2}$$

– the constitutive equations

$$\begin{aligned} &t_{ij} = A_{ijrs} e_{rs} + B_{ijrs}\kappa_{rs} + D_{ij}\varphi + F_{ijk}\zeta_k + \chi^{(1)}_{ijk} E_k + \mu^{(1)}_{ijk} B_k, \\ &m_{ij} = B_{rsij} e_{rs} + C_{ijrs}\kappa_{rs} + E_{ij}\varphi + G_{ijk}\zeta_k + \chi^{(2)}_{ijk} E_k + \mu^{(2)}_{ijk} B_k, \\ &\sigma = D_{ij} e_{ij} + E_{ij}\kappa_{ij} + \xi\varphi + h_k\zeta_k + \chi^{(3)}_k E_k + \mu^{(3)}_k B_k, \\ &\pi_k = F_{ijk} e_{ij} + G_{ijk}\kappa_{ij} + h_k\varphi + A_{kj}\zeta_j + \chi^{(4)}_{kj} E_j + \mu^{(4)}_{kj} B_j, \\ &D_k = -\chi^{(1)}_{ijk} e_{ij} - \chi^{(2)}_{ijk}\kappa_{ij} - \chi^{(3)}_k\varphi - \chi^{(4)}_{jk}\zeta_j + \chi_{kj} E_j + \alpha_{kj} B_j, \\ &H_k = -\mu^{(1)}_{ijk} e_{ij} - \mu^{(2)}_{ijk}\kappa_{ij} - \mu^{(3)}_k\varphi - \mu^{(4)}_{jk}\zeta_j + \alpha_{jk} E_j + \mu_{kj} B_j; \end{aligned} \tag{3}$$

– and the geometrical equations

$$e_{ij} = u_{j,i} + \epsilon_{jik}\varphi_k, \quad \kappa_{ij} = \varphi_{j,i}, \quad \zeta_j = \varphi_{,j}; \tag{4}$$

where t_{ij} is the stress tensor; f_i is the body force; ρ is the reference mass density; u_i is the mechanical displacement vector; m_{ij} is the couple stress tensor; ϵ_{ijk} is the alternating symbol; l_i is the body couple; I_{ij} is the microinertia tensor; φ_i is the microrotation vector; π_i is the microstretch stress vector; σ is the microstress function; l is the microstretch body force; $\jmath_0$ is the microstretch inertia; φ is the microstretch function; E_i is the electric field vector; B_i is the magnetic induction; c is the speed of light; H_i is the magnetic field intensity; D_i is the dielectric displacement vector; e_{ij}, κ_{ij} and ζ are kinematic strain measures; and $A_{ijrs}, B_{ijrs}, \ldots, \mu_{kj}$ are constitutive coefficients.

We assume the charge density to be absent and we do not consider the Gauss' laws $B_{i,i} = 0$ and $D_{i,i} = 0$ since we regard these equations as consequences of (2) and initial conditions.

Now, let us note that in the formulation of the constitutive equations it is often convenient to choose e_{ij}, κ_{ij}, φ, ζ_i, E_i and H_i as independent variables. Then, Eq. (3) are replaced by

$$\begin{aligned} &t_{ij} = a_{ijrs} e_{rs} + b_{ijrs}\kappa_{rs} + d_{ij}\varphi + f_{ijk}\zeta_k + \lambda^{(1)}_{ijk} E_k + \nu^{(1)}_{ijk} H_k, \\ &m_{ij} = b_{rsij} e_{rs} + c_{ijrs}\kappa_{rs} + \varepsilon_{ij}\varphi + g_{ijk}\zeta_k + \lambda^{(2)}_{ijk} E_k + \nu^{(2)}_{ijk} H_k, \\ &\sigma = d_{ij} e_{ij} + \varepsilon_{ij}\kappa_{ij} + \nu\varphi + \chi_k\zeta_k + \lambda^{(3)}_k E_k + \nu^{(3)}_k H_k, \\ &\pi_k = f_{ijk} e_{ij} + g_{ijk}\kappa_{ij} + \chi_k\varphi + a_{kj}\zeta_j + \lambda^{(4)}_{kj} E_j + \nu^{(4)}_{kj} H_j, \\ &D_k = -\lambda^{(1)}_{ijk} e_{ij} - \lambda^{(2)}_{ijk}\kappa_{ij} - \lambda^{(3)}_k\varphi - \lambda^{(4)}_{jk}\zeta_j + \lambda_{kj} E_j + \beta_{kj} H_j, \\ &B_k = -\nu^{(1)}_{ijk} e_{ij} - \nu^{(2)}_{ijk}\kappa_{ij} - \nu^{(3)}_k\varphi - \nu^{(4)}_{jk}\zeta_j + \beta_{jk} E_j + \nu_{kj} H_j, \end{aligned} \tag{5}$$

where $a_{ijrs}, b_{ijrs}, \ldots, \nu_{kj}$ are constitutive coefficients.

For the specific case of a microstretch piezoelectric medium (or a microstretch piezomagnetic medium), the constitutive relations can be expressed in (5) by deleting the coupling coefficient tensors $\nu^{(1)}_{ijk}$, $\nu^{(2)}_{ijk}$, $\nu^{(3)}_k$, $\nu^{(4)}_{kj}$ and β_{ij} (or $\lambda^{(1)}_{ijk}$, $\lambda^{(2)}_{ijk}$, $\lambda^{(3)}_k$, $\lambda^{(4)}_{kj}$ and β_{ij} correspondingly). Our result is proved for electromagnetic bodies (the constitutive equation (5)) and clearly, in view of the hypotheses presented in the next Section, it also holds for piezoelectric and piezomagnetic mediums.

We suppose the constitutive coefficients and the inertia tensor satisfy the symmetry relations

$$a_{ijrs} = a_{rsij}, \quad c_{ijrs} = c_{rsij}, \quad a_{ij} = a_{ji}, \quad \lambda_{ij} = \lambda_{ji}, \quad \nu_{ij} = \nu_{ji}, \quad I_{ij} = I_{ji}. \tag{6}$$

The components of the surface traction, the components of the surface moment and the surface microforce defined at every regular point of a boundary surface are given by

$$t_i = t_{ji} n_j, \quad m_i = m_{ji} n_j, \quad \tau = \pi_j n_j, \tag{7}$$

where n_j are the components of the outward unit normal vector to the boundary surface of a region.

To the system of field equations we adjoin the following boundary conditions

$$\begin{aligned}
&u_i = \tilde{u}_i \quad \text{on } \overline{S}_1 \times [0,\infty), \quad t_{ji}n_j = \tilde{t}_i \quad \text{on } S_2 \times [0,\infty),\\
&\varphi_i = \tilde{\varphi}_i \quad \text{on } \overline{S}_3 \times [0,\infty), \quad m_{ji}n_j = \tilde{m}_i \quad \text{on } S_4 \times [0,\infty),\\
&\varphi = \tilde{\varphi} \quad \text{on } \overline{S}_5 \times [0,\infty), \quad \pi_j n_j = \tilde{\pi} \quad \text{on } S_6 \times [0,\infty),\\
&\epsilon_{ijk}E_j n_k = \widetilde{E}_i \quad \text{on } \overline{S}_7 \times [0,\infty), \quad \epsilon_{ijk}H_j n_k = \widetilde{H}_i \quad \text{on } S_8 \times [0,\infty),
\end{aligned} \tag{8}$$

where $\tilde{u}_i,\ \tilde{t}_i,\ \tilde{\varphi}_i,\ \tilde{m}_i,\ \tilde{\varphi},\ \tilde{\pi},\ \widetilde{E}_i,\ \widetilde{H}_i$ are prescribed functions and $S_{\mathfrak{A}}$ $(\mathfrak{A}=1,2,\dots 8)$ are subsets of $\partial\mathcal{B}$ such that $\partial\mathcal{B} = \overline{S}_1 \cup S_2 = \overline{S}_3 \cup S_4 = \overline{S}_5 \cup S_6 = \overline{S}_7 \cup S_8$, $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = \emptyset$. Moreover, we adjoin the following initial conditions

$$\begin{aligned}
&u_i(\mathbf{x},0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x},0) = v_i^0(\mathbf{x}), \quad \varphi_i(\mathbf{x},0) = \varphi_i^0(\mathbf{x}),\\
&\quad \dot{\varphi}_i(\mathbf{x},0) = \theta_i^0(\mathbf{x}),\\
&\varphi(\mathbf{x},0) = \varphi^0(\mathbf{x}), \quad \dot{\varphi}(\mathbf{x},0) = \theta^0(\mathbf{x}), \quad D_i(\mathbf{x},0) = D_i^0(\mathbf{x}),\\
&\quad B_i(\mathbf{x},0) = B_i^0(\mathbf{x}), \quad \mathbf{x} \in \overline{\mathcal{B}},
\end{aligned} \tag{9}$$

where $u_i^0,\ v_i^0,\ \varphi_i^0,\ \theta_i^0,\ \varphi^0,\ \theta^0,\ D_i^0$ and B_i^0 are prescribed fields, satisfying $D_{i,i}^0 = 0$ and $B_{i,i}^0 = 0$.

By an external data system we mean the ordered array

$$\mathfrak{J} = \left\{f_i, l_i, l;\ \tilde{u}_i, \tilde{t}_i, \tilde{\varphi}_i, \tilde{m}_i, \tilde{\varphi}, \tilde{\pi}, \widetilde{E}_i, \widetilde{H}_i;\ u_i^0, v_i^0, \varphi_i^0, \theta_i^0, \varphi^0, \theta^0, D_i^0, B_i^0\right\}.$$

We denote by $(\mathcal{P})$ the initial-boundary value problem defined by the field equations (1), (2), (4), (5), the boundary conditions (8) and the initial conditions (9).

Let M and N be nonnegative integers. We say that h is of class $C^{M,N}$ on $\overline{\mathcal{B}} \times [0,\infty)$ if h is continuous on $\overline{\mathcal{B}} \times [0,\infty)$, and the functions

$$\frac{\partial^m}{\partial x_i \partial x_j \cdots \partial x_r}\left(\frac{\partial^n h}{\partial t^n}\right) \quad m \in \{0,1,\dots,M\},\ n \in \{0,1,\dots,N\},$$
$$m + n \leqslant \max\{M,N\}$$

exist and are continuous on $\overline{\mathcal{B}} \times [0,\infty)$. We denote $C^{M,M}$ by C^M.

By an admissible process we mean the ordered array

$$\mathfrak{P} = \{u_i, \varphi_i, \varphi, E_i, H_i, e_{ij}, \kappa_{ij}, \zeta_i, t_{ij}, m_{ij}, \sigma, \pi_i, D_i, B_i\}$$

with the properties (a) u_i, φ_i, φ are of class $C^{1,2}$ on $\overline{\mathcal{B}} \times [0,\infty)$; (b) B_i, $D_i \in C^1$ on $\overline{\mathcal{B}} \times [0,\infty)$; (c) E_i, H_i, t_{ij}, m_{ij}, π_i are of class $C^{1,0}$ on $\overline{\mathcal{B}} \times [0,\infty)$; *d)* e_{ij}, κ_{ij}, ζ_i, $\sigma \in C^0$ on $\overline{\mathcal{B}} \times [0,\infty)$.

A solution of the initial boundary value problem $(\mathcal{P})$ corresponding to external data system $\mathfrak{J}$ is an admissible process $\mathfrak{P}$ that satisfies the field equations (1), (2), (4), (5) on $\overline{\mathcal{B}} \times [0,\infty)$, the boundary conditions (8) and the initial conditions (9).

Given an admissible process $\mathfrak{P}$, then we associate with the strain measures $\Phi = \{e_{ij}, \kappa_{ij}, \varphi, \zeta_i\}$ and the electromagnetic fields $\Xi = \{E_i, H_i\}$ the following quadratic forms:

$$\begin{aligned}
2\mathcal{W}(\Phi) = {}& a_{ijrs}e_{ij}e_{rs} + c_{ijrs}\kappa_{ij}\kappa_{rs} + \nu\varphi^2 + a_{ij}\zeta_i\zeta_j + 2b_{ijrs}e_{ij}\kappa_{rs} + 2d_{ij}e_{ij}\varphi\\
&+ 2f_{ijk}e_{ij}\zeta_k + 2\varepsilon_{ij}\kappa_{ij}\varphi + 2g_{ijk}\kappa_{ij}\zeta_k + 2\chi_i\zeta_i\varphi
\end{aligned} \tag{10}$$

and

$$2\mathcal{E}(\Xi) = \lambda_{ij}E_iE_j + 2\beta_{ij}E_iH_j + \nu_{ij}H_iH_j. \tag{11}$$

3. Hypotheses and preliminary results

In this paper we shall use the hypotheses:

(α) ρ_0 and $\jmath_0$ are strictly positive fields on $\mathcal{B}$, that is

$$\begin{aligned}
&\rho_0(\mathbf{x}) \geqslant \bar{\rho}_0 > 0, \quad \bar{\rho}_0 = const.\\
&\jmath_0(\mathbf{x}) \geqslant \bar{\jmath}_0 > 0, \quad \bar{\jmath}_0 = const.;
\end{aligned} \tag{12}$$

(β) the microinertia tensor I_{ij} is positive definite, that is

$$I^m X_iX_i \leqslant I_{ij}X_iX_j \leqslant I^M X_iX_i, \quad \forall\ X_i \in \mathbb{R}, \tag{13}$$

where $I^m(\mathbf{x}) > 0$ and $I^M(\mathbf{x}) > 0$ are the smallest and the largest eigenvalues of $I_{ij}(\mathbf{x})$;

(γ) there exist $\eta^m(\mathbf{x}) > 0$, $\eta^M(\mathbf{x}) > 0$ such that

$$\begin{aligned}
&\eta^m\left(X_{ij}X_{ij} + I^m Y_{ij}Y_{ij} + w^2 + \jmath_0 Z_iZ_i\right) \leqslant 2\mathcal{W}(S)\\
&\quad \leqslant \eta^M\left(X_{ij}X_{ij} + I^m Y_{ij}Y_{ij} + w^2 + \jmath_0 Z_iZ_i\right),
\end{aligned} \tag{14}$$

for all $S = \{X_{ij}, Y_{ij}, w, Z_i\}$, where the arbitrary quantities X_{ij}, Y_{ij}, w, Z_i have the same dimensions as the strain measures e_{ij}, κ_{ij}, φ, ζ_i, respectively;

(δ) there exist $\lambda^m(\mathbf{x}) > 0$, $\nu^m(\mathbf{x}) > 0$, $\lambda^M(\mathbf{x}) > 0$, $\nu^M(\mathbf{x}) > 0$ such that

$$\lambda^m X_iX_i + \nu^m Y_iY_i \leqslant 2\mathcal{E}(M) \leqslant \lambda^m X_iX_i + \nu^m Y_iY_i, \tag{15}$$

where $M = \{X_i, Y_i\}$ and X_i, Y_i are arbitrary quantities having the same dimensions as the electromagnetic fields E_i and H_i.

Given the processes $\mathfrak{P}$ and $\mathfrak{P}^*$, then for the strain measures $\Phi = \{e_{ij}, \kappa_{ij}, \varphi, \zeta_i\}$ and $\Phi^* = \{e_{ij}^*, \kappa_{ij}^*, \varphi^*, \zeta_i^*\}$, respectively, we introduce the following bilinear form:

$$\begin{aligned}
2\mathcal{F}(\Phi,\Phi^*) = {}& a_{ijrs}e_{ij}e_{rs}^* + c_{ijrs}\kappa_{ij}\kappa_{rs}^* + \nu\varphi\varphi^* + a_{ij}\zeta_i\zeta_j^*\\
&+ b_{ijrs}(e_{ij}\kappa_{rs}^* + e_{ij}^*\kappa_{rs}) + d_{ij}(e_{ij}\varphi^* + e_{ij}^*\varphi) + f_{ijk}(e_{ij}\zeta_k^* + e_{ij}^*\zeta_k)\\
&+ \varepsilon_{ij}(\kappa_{ij}\varphi^* + \kappa_{ij}^*\varphi) + g_{ijk}(\kappa_{ij}\zeta_k^* + \kappa_{ij}^*\zeta_k) + \chi_i(\zeta_i\varphi^* + \zeta_i^*\varphi).
\end{aligned} \tag{16}$$

Obviously, in view of (6) it results that $\mathcal{F}(\cdot,\cdot)$ is symmetric. Moreover, on using the hypothesis (γ), it follows the Cauchy–Schwartz's inequality:

$$\mathcal{F}(\Phi,\Phi^*) \leqslant [\mathcal{W}(\Phi)]^{1/2}[\mathcal{W}(\Phi^*)]^{1/2}, \quad \forall\ \Phi, \Phi^*. \tag{17}$$

Now, let us denote by $\hat{\lambda}_M^{(\mathfrak{U})},\ \hat{\nu}_M^{(\mathfrak{U})},\ \mathfrak{U} = 1,2,3,4$, the largest eigenvalues of the symmetric and positive semidefinite tensors

$$\begin{aligned}
&\hat{\lambda}_{kl}^{(1)} = \lambda_{ijk}^{(1)}\lambda_{ijl}^{(1)}, \quad \hat{\lambda}_{kl}^{(2)} = \frac{1}{I^m}\lambda_{ijk}^{(2)}\lambda_{ijl}^{(2)}, \quad \hat{\lambda}_{kl}^{(3)} = \lambda_k^{(3)}\lambda_l^{(3)}, \quad \hat{\lambda}_{kl}^{(4)} = \frac{1}{\jmath_0}\lambda_{jk}^{(4)}\lambda_{jl}^{(4)},\\
&\hat{\nu}_{kl}^{(1)} = \nu_{ijk}^{(1)}\nu_{ijl}^{(1)}, \quad \hat{\nu}_{kl}^{(2)} = \frac{1}{I^m}\nu_{ijk}^{(2)}\nu_{ijl}^{(2)}, \quad \hat{\nu}_{kl}^{(3)} = \nu_k^{(3)}\nu_l^{(3)}, \quad \hat{\nu}_{kl}^{(4)} = \frac{1}{\jmath_0}\nu_{jk}^{(4)}\nu_{jl}^{(4)},
\end{aligned} \tag{18}$$

respectively. Moreover, for later convenience, let us introduce the notations

$$\begin{aligned}
&\hat{\lambda}_M^0 = \left(\sqrt{\hat{\lambda}_M^{(1)}} + \sqrt{\hat{\lambda}_M^{(2)}} + \sqrt{\hat{\lambda}_M^{(3)}} + \sqrt{\hat{\lambda}_M^{(4)}}\right)^2,\\
&\hat{\nu}_M^0 = \left(\sqrt{\hat{\nu}_M^{(1)}} + \sqrt{\hat{\nu}_M^{(2)}} + \sqrt{\hat{\nu}_M^{(3)}} + \sqrt{\hat{\nu}_M^{(4)}}\right)^2
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\bar{\eta}^m = \min_{\mathbf{x}\in\mathcal{B}}\{\eta^m(\mathbf{x})\} > 0, \quad \bar{\eta}^M = \max_{\mathbf{x}\in\mathcal{B}}\{\eta^M(\mathbf{x})\} < \infty,\\
&\bar{\lambda}^m = \min_{\mathbf{x}\in\mathcal{B}}\{\lambda^m(\mathbf{x})\} > 0, \quad \bar{\lambda}^M = \max_{\mathbf{x}\in\mathcal{B}}\{\lambda^M(\mathbf{x})\} < \infty,\\
&\bar{\nu}^m = \min_{\mathbf{x}\in\mathcal{B}}\{\nu^m(\mathbf{x})\} > 0, \quad \bar{\nu}^M = \max_{\mathbf{x}\in\mathcal{B}}\{\nu^M(\mathbf{x})\} < \infty,\\
&\bar{\hat{\lambda}}_M^0 = \max_{\mathbf{x}\in\mathcal{B}}\{\hat{\lambda}_M^0(\mathbf{x})\} < \infty, \quad \bar{\hat{\nu}}_M^0 = \max_{\mathbf{x}\in\mathcal{B}}\{\hat{\nu}_M^0(\mathbf{x})\} < \infty.
\end{aligned} \tag{20}$$

Lemma 1. *Let $\mathfrak{P}$ be an admissible process satisfying the constitutive equation (5) . Suppose the hypotheses (α), (β) and (γ) are fulfilled. Then, the following inequality holds true*

$$\begin{aligned}
t_{ij}t_{ij} + \frac{1}{I^m}m_{ij}m_{ij} + \sigma^2 + \frac{1}{\jmath_0}\pi_i\pi_i \leqslant {}& 2(1 + \vartheta_1 + \vartheta_2)\eta^M\mathcal{W}(\Phi)\\
&+ \left(2 + \frac{1}{\vartheta_1}\right)\hat{\lambda}_M^0 E_iE_i\\
&+ \left(2 + \frac{1}{\vartheta_2}\right)\hat{\nu}_M^0 H_iH_i,
\end{aligned} \tag{21}$$

where $\mathcal{W}(\Phi)$ is given by (10), $\hat{\lambda}_M^0,\ \hat{\nu}_M^0$ are defined in (19) and ϑ_1, ϑ_2 are positive constants.

Proof. From (5) and (16) we obtain

$$
\begin{aligned}
t_{ij}t_{ij} + \frac{1}{I^m} m_{ij}m_{ij} + \sigma^2 + \frac{1}{J_0}\pi_i\pi_i = {} & 2\mathcal{F}(\Phi, \Lambda) + \lambda^{(1)}_{ijk} E_k t_{ij} \\
& + \frac{1}{I^m}\lambda^{(2)}_{ijk} E_k m_{ij} + \lambda^{(3)}_k E_k \sigma \\
& + \frac{1}{J_0}\lambda^{(4)}_{jk} E_k \pi_j + v^{(1)}_{ijk} H_k t_{ij} \\
& + \frac{1}{I^m} v^{(2)}_{ijk} H_k m_{ij} + v^{(3)}_k H_k \sigma \\
& + \frac{1}{J_0} v^{(4)}_{jk} H_k \pi_j,
\end{aligned} \tag{22}
$$

where $\Phi = \{e_{ij}, \kappa_{ij}, \varphi, \zeta_i\}$ and $\Lambda = \{t_{ij}, \frac{m_{ij}}{I^m}, \sigma, \frac{\pi_i}{J_0}\}$.

On using (17) and (18), from (22) we deduce

$$
\begin{aligned}
t_{ij}t_{ij} + \frac{1}{I^m} m_{ij}m_{ij} + \sigma^2 + \frac{1}{J_0}\pi_i\pi_i \leqslant {} & [2\mathcal{W}(\Phi)]^{\frac{1}{2}}[2\mathcal{W}(\Lambda)]^{\frac{1}{2}} \\
& + \left(\hat{\lambda}^{(1)}_{kl} E_k E_l\right)^{\frac{1}{2}} (t_{ij}t_{ij})^{\frac{1}{2}} \\
& + \left(\frac{1}{I^m}\hat{\lambda}^{(2)}_{kl} E_k E_l\right)^{\frac{1}{2}} \left(\frac{1}{I^m} m_{ij}m_{ij}\right)^{\frac{1}{2}} \\
& + \left(\hat{\lambda}^{(3)}_{kl} E_k E_l\right)^{\frac{1}{2}} (\sigma^2)^{\frac{1}{2}} \\
& + \left(\frac{1}{J_0}\hat{\lambda}^{(4)}_{kl} E_k E_l\right)^{\frac{1}{2}} \left(\frac{1}{J_0}\pi_i\pi_i\right)^{\frac{1}{2}} \\
& + \left(\hat{v}^{(1)}_{kl} H_k H_l\right)^{\frac{1}{2}} (t_{ij}t_{ij})^{\frac{1}{2}} \\
& + \left(\frac{1}{I^m}\hat{v}^{(2)}_{kl} H_k H_l\right)^{\frac{1}{2}} \left(\frac{1}{I^m} m_{ij}m_{ij}\right)^{\frac{1}{2}} \\
& + \left(\hat{v}^{(3)}_{kl} H_k H_l\right)^{\frac{1}{2}} (\sigma^2)^{\frac{1}{2}} \\
& + \left(\frac{1}{J_0}\hat{v}^{(4)}_{kl} H_k H_l\right)^{\frac{1}{2}} \left(\frac{1}{J_0}\pi_i\pi_i\right)^{\frac{1}{2}}.
\end{aligned} \tag{23}
$$

Moreover, taking into account the relations (14) and (19) we obtain

$$
\begin{aligned}
& t_{ij}t_{ij} + \frac{1}{I^m} m_{ij}m_{ij} + \sigma^2 + \frac{1}{J_0}\pi_i\pi_i \\
& \quad \leqslant \left\{ \left[2\eta^M \mathcal{W}(\Phi)\right]^{1/2} + \left(\hat{\lambda}^0_M E_i E_i\right)^{1/2} + \left(\hat{v}^0_M H_i H_i\right)^{1/2} \right\}^2.
\end{aligned} \tag{24}
$$

By means of the arithmetic–geometric mean inequality, from (24) we get the inequality (21) and the proof is complete. □

Lemma 2. *Let $\mathfrak{P}$ be an admissible process satisfying the constitutive equation (5). Suppose the hypotheses (α), (β) and (γ) are fulfilled. Then, the following inequalities hold true*

$$
\begin{aligned}
t_i t_i + \frac{1}{I^m} m_i m_i + \frac{1}{J_0}\tau^2 \leqslant {} & 2(1 + \vartheta_1 + \vartheta_2)\eta^M \mathcal{W}(\Phi) \\
& + \left(2 + \frac{1}{\vartheta_1}\right)\hat{\lambda}^0_M E_i E_i + \left(2 + \frac{1}{\vartheta_2}\right)\hat{v}^0_M H_i H_i
\end{aligned} \tag{25}
$$

and

$$
\epsilon_{ijk} H_k E_i n_j \leqslant \frac{1}{2}\left(\vartheta_3 H_i H_i + \frac{2}{\vartheta_3} E_i E_i\right), \tag{26}
$$

where t_i, m_i, τ are defined by (7) and ϑ_1, ϑ_2, ϑ_3 are arbitrary positive constants.

Proof. The inequality (25) follows from (7), (21) and the inequality

$$
M_{ij} M_{kj} a_i a_k \leqslant M_{ij} M_{ij}, \tag{27}
$$

which holds for any tensor M_{ij} and any vector a_i with $a_i a_i = 1$.

Regarding the second inequality, let us note that the arithmetic–geometric mean inequality leads to

$$
\epsilon_{ijk} H_k E_i n_j \leqslant \frac{1}{2}\left(\vartheta_3 H_i H_i + \frac{1}{\vartheta_3}\epsilon_{ijk} E_i n_j \epsilon_{mnk} E_m n_n\right), \quad \vartheta_3 > 0. \tag{28}
$$

Thus, applying (27) with $M_{ij} = \epsilon_{mij} E_m$, $a_i = n_i$ and by using the identity $\epsilon_{ijk}\epsilon_{mjk} = 2\delta_{im}$, where δ_{im} is the Kronecker delta, we deduce (26) and the proof is complete. □

4. Spatial behavior

In this section we prove the main theorem of the paper. First, we introduce the so called "support" of the external given data and we establish some properties of an appropriate time–weighted surface power function associated with the solution at issue. Then, we obtain the domain of influence in the sense described by Chiriţă and Ciarletta (1999) and we establish an estimate suggesting exponential decay of activity inside the domain of influence.

Let us fix a time $T \in [0, \infty)$. Given a solution $\mathfrak{P} = \{u_i, \varphi_i, \varphi, E_i, H_i, e_{ij}, \kappa_{ij}, \zeta_i, t_{ij}, m_{ij}, \sigma, \pi_i, D_i, B_i\}$ corresponding to the external data system $\mathfrak{J} = \{f_i, l_i, l; \tilde{u}_i, \tilde{t}_i, \tilde{\varphi}_i, \tilde{m}_i, \tilde{\varphi}, \tilde{\pi}, \tilde{E}_i, \tilde{H}_i; u^0_i, v^0_i, \varphi^0_i, \theta^0_i, \varphi^0, \theta^0, D^0_i, B^0_i\}$, we introduce the set $\widehat{\Omega}_T$ of all points $\mathbf{x} \in \overline{\mathcal{B}}$ such that:

(i) if $\mathbf{x} \in \mathcal{B}$ then

$$
\begin{aligned}
& u_i(\mathbf{x}, 0) \neq 0 \quad \text{or} \quad \dot{u}_i(\mathbf{x}, 0) \neq 0 \quad \text{or} \quad \varphi_i(\mathbf{x}, 0) \neq 0 \quad \text{or} \\
& \dot{\varphi}_i(\mathbf{x}, 0) \neq 0 \quad \text{or} \quad \varphi(\mathbf{x}, 0) \neq 0 \quad \text{or} \quad \dot{\varphi}(\mathbf{x}, 0) \neq 0 \quad \text{or} \\
& D_i(\mathbf{x}, 0) \neq 0 \quad \text{or} \quad B_i(\mathbf{x}, 0) \neq 0
\end{aligned} \tag{29}
$$

or

$$
f_i(\mathbf{x}, \tau) \neq 0 \quad \text{or} \quad l_i(\mathbf{x}, \tau) \neq 0 \quad \text{or} \quad l(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T], \tag{30}
$$

(ii) if $\mathbf{x} \in \partial\mathcal{B}$ then

$$
\begin{aligned}
& t_i(\mathbf{x}, \tau)\dot{u}_i(\mathbf{x}, \tau) \neq 0 \quad \text{or} \quad m_i(\mathbf{x}, \tau)\dot{\varphi}_i(\mathbf{x}, \tau) \neq 0 \quad \text{or} \\
& \tau(\mathbf{x}, \tau)\dot{\varphi}(\mathbf{x}, \tau) \neq 0 \quad \text{or} \quad \epsilon_{ijk} E_i(\mathbf{x}, \tau) H_k(\mathbf{x}, \tau) n_j \neq 0 \quad \text{for some} \\
& \tau \in [0, T].
\end{aligned} \tag{31}
$$

Roughly speaking, $\widehat{\Omega}_T$ represents the support of the external given data on the time interval $[0,T]$. If the region $\mathcal{B}$ is unbounded, then we shall assume that $\widehat{\Omega}_T$ is a bounded region.

We consider next a nonempty set $\widehat{\Omega}^\star_T$ of $\overline{\mathcal{B}}$ such that $\widehat{\Omega}_T \subset \widehat{\Omega}^\star_T \subset \overline{\mathcal{B}}$ and

(1) if $\widehat{\Omega}_T \cap \mathcal{B} \neq \emptyset$, we choose $\widehat{\Omega}^\star_T$ to be the smallest regular region in $\overline{\mathcal{B}}$ that includes $\widehat{\Omega}_T$; in particular, we set $\widehat{\Omega}^\star_T = \widehat{\Omega}_T$ if $\widehat{\Omega}_T$ happens to be a regular region;

(2) if $\emptyset \neq \widehat{\Omega}_T \subset \partial\mathcal{B}$, we choose $\widehat{\Omega}^\star_T$ to be the smallest regular subsurface of $\partial\mathcal{B}$ that includes $\widehat{\Omega}_T$; in particular, we set $\widehat{\Omega}^\star_T = \widehat{\Omega}_T$ if $\widehat{\Omega}_T$ is a regular subsurface of $\partial\mathcal{B}$;

(3) if $\widehat{\Omega}_T$ is empty, then we choose $\widehat{\Omega}^\star_T$ to be an arbitrary regular subsurface of $\partial\mathcal{B}$.

On this basis we introduce the set Ω_r, $r \geqslant 0$ by

$$
\Omega_r = \left\{\mathbf{x} \in \overline{\mathcal{B}};\ \widehat{\Omega}^\star_T \cap \overline{\Sigma(\mathbf{x}, r)} \neq \emptyset\right\}, \tag{32}
$$

where $\Sigma(\mathbf{x}, r)$ is the open ball with radius r and center $\mathbf{x}$. We shall use the notation $\mathcal{B}(r)$ for the part of $\mathcal{B}$ contained in $\mathcal{B} \setminus \Omega_r$ and we set $\mathcal{B}(r_1, r_2) = \mathcal{B}(r_2) \setminus \mathcal{B}(r_1)$, $r_1 > r_2$. Moreover, we shall denote by S_r the subsurface of $\partial\mathcal{B}(r)$ contained into inside of $\mathcal{B}$ and whose outward unit normal vector is forwarded to the exterior of Ω_r.

We note that for a bounded body r ranges over $[0,L]$, where

$$L=\max\left\{\min\left\{[(x_i-y_i)(x_i-y_i)]^{1/2}:\mathbf{y}\in\widehat{\Omega}_T^{\star}\right\}:\mathbf{x}\in\overline{\mathcal{B}}\right\}. \tag{33}$$

We associate with the solution $\mathfrak{P}$ the following time–weighted surface power function

$$Q(r,t)=-\int_0^t\int_{S_r}e^{-\gamma z}[t_i(z)\dot{u}_i(z)+m_i(z)\dot{\varphi}_i(z)+\tau(z)\dot{\varphi}(z)$$
$$+\epsilon_{ijk}cH_k(z)E_i(z)n_j]\,da\,dz,\quad r\geqslant 0,\ t\in[0,T], \tag{34}$$

where γ is a prescribed positive parameter.

Lemma 3. *Let $\mathfrak{P}=\{u_i,\varphi_i,\varphi,E_i,H_i,e_{ij},\kappa_{ij},\zeta_i,t_{ij},m_{ij},\sigma,\pi_i,D_i,B_i\}$ be a solution of the problem $(\mathcal{P})$ and let $\widehat{\Omega}_T$ be the bounded support of the external data on the interval [0,T]. Then, $Q(r,t)$ is a continuous differentiable function on $r\geqslant 0$, $t\in[0,T]$ and*

$$\frac{\partial}{\partial t}Q(r,t)=-\int_{S_r}e^{-\gamma t}[t_i(t)\dot{u}_i(t)+m_i(t)\dot{\varphi}_i(t)+\tau(t)\dot{\varphi}(t)$$
$$+\epsilon_{ijk}cH_k(t)E_i(t)n_j]\,da, \tag{35}$$

$$\frac{\partial}{\partial r}Q(r,t)=-\frac{1}{2}\int_{S_r}e^{-\gamma t}\left[\rho_0\left(\dot{u}_i(t)\dot{u}_i(t)+I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t)+\jmath_0\dot{\varphi}^2(t)\right)\right.$$
$$+2\mathcal{W}(\Phi(t))+2\mathcal{E}(\Xi(t))]da-\frac{\gamma}{2}\int_0^t\int_{S_r}e^{-\gamma z}[\rho_0(\dot{u}_i(z)\dot{u}_i(z)$$
$$\left.+I_{ij}\dot{\varphi}_i(z)\dot{\varphi}_j(z)+\jmath_0\dot{\varphi}^2(z)\right)+2\mathcal{W}(\Phi(z))+2\mathcal{E}(\Xi(z))]\,da\,dz, \tag{36}$$

where $\mathcal{W}(\Phi)$ and $\mathcal{E}(\Xi)$ are given by (10) and (11). Moreover, if the hypotheses (α), (β), (γ) and (δ) are fulfilled, then at any fixed $t\in[0,T]$, $Q(r,t)$ is a nonincreasing function with respect to r.

Proof. In view of (7), the definition of $\widehat{\Omega}_T$ and the divergence theorem we get

$$Q(r_1,t)-Q(r_2,t)=-\int_0^t\int_{\partial\mathcal{B}(r_1,r_2)}e^{-\gamma z}[t_i(z)\dot{u}_i(z)+m_i(z)\dot{\varphi}_i(z)$$
$$+\tau(z)\dot{\varphi}(z)+\epsilon_{ijk}cH_k(z)E_i(z)n_j]\,da\,dz=-\int_0^t\int_{\mathcal{B}(r_1,r_2)}e^{-\gamma z}$$
$$\times\left[t_{ji}(z)\dot{u}_{i,j}(z)+t_{ji,j}(z)\dot{u}_i(z)+m_{ji}(z)\dot{\varphi}_{i,j}(z)+m_{ji,j}(z)\dot{\varphi}_i(z)\right.$$
$$+\pi_k(z)\dot{\varphi}_{,k}(z)+\pi_{k,k}(z)\dot{\varphi}(z)+\epsilon_{ijk}cH_{k,j}(z)E_i(z)$$
$$\left.+\epsilon_{ijk}cE_{i,j}(z)H_k(z)\right]dv\,dz\quad 0\leqslant r_2\leqslant r_1. \tag{37}$$

Further, on using the basic equations (1), (2) and (4) we obtain

$$Q(r_1,t)-Q(r_2,t)=-\int_0^t\int_{\mathcal{B}(r_1,r_2)}e^{-\gamma z}\left[\rho_0\ddot{u}_i(z)\dot{u}_i(z)+\rho_0I_{ij}\ddot{\varphi}_j(z)\dot{\varphi}_i(z)\right.$$
$$+\rho_0\jmath_0\ddot{\varphi}(z)\dot{\varphi}(z)+t_{ij}(z)\dot{e}_{ij}(z)+m_{ij}(z)\dot{\kappa}_{ij}(z)+\sigma(z)\dot{\varphi}(z)$$
$$\left.+\pi_k(z)\dot{\zeta}_k(z)+\dot{D}_i(z)E_i(z)+\dot{B}_i(z)H_i(z)\right]dv\,dz. \tag{38}$$

Using the relations (5), (6), (10) and (11) we have

$$t_{ij}(z)\dot{e}_{ij}(z)+m_{ij}(z)\dot{\kappa}_{ij}(z)+\sigma(z)\dot{\varphi}(z)+\pi_k(z)\dot{\zeta}_k(z)+\dot{D}_i(z)E_i(z)$$
$$+\dot{B}_i(z)H_i(z)=\frac{\partial}{\partial z}[\mathcal{W}(\Phi(z))+\mathcal{E}(\Xi(z))]. \tag{39}$$

If we substitute (39) in (38) then by means of an integration by parts and the definition of $\widehat{\Omega}_T$ we get the identity

$$Q(r_1,t)-Q(r_2,t)=-\frac{1}{2}\int_{\mathcal{B}(r_1,r_2)}e^{-\gamma t}\left[\rho_0\left(\dot{u}_i(t)\dot{u}_i(t)+I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t)\right.\right.$$
$$\left.+\jmath_0\dot{\varphi}^2(t)\right)+2\mathcal{W}(\Phi(t))+2\mathcal{E}(\Xi(t))]\,dv$$
$$-\frac{\gamma}{2}\int_0^t\int_{\mathcal{B}(r_1,r_2)}e^{-\gamma z}\left[\rho_0\left(\dot{u}_i(z)\dot{u}_i(z)+I_{ij}\dot{\varphi}_i(z)\dot{\varphi}_j(z)+\jmath_0\dot{\varphi}^2(z)\right)\right.$$
$$\left.+2\mathcal{W}(\Phi(z))+2\mathcal{E}(\Xi(z))\right]dv\,dz,\quad 0\leqslant r_2\leqslant r_1. \tag{40}$$

The identities (35) and (36) follow from (34) and (40). Finally, from (12)–(15) and (36) we find that $Q(r,t)$ is a nonincreasing function with respect to r. The proof is complete. □

Lemma 4. *Assume that the hypotheses (α), (β), (γ) and (δ) are fulfilled. Let $\mathfrak{P}$ be solution of the problem $(\mathcal{P})$ and let $\widehat{\Omega}_T$ be the bounded support of the external data system $\mathfrak{J}$ on the interval [0,T]. Then for any $r\geqslant 0$ and $t\in[0,T]$, $Q(r,t)$ satisfies the following differential inequalities*

$$\frac{1}{\mathcal{C}}\left|\frac{\partial}{\partial t}Q(r,t)\right|+\frac{\partial}{\partial r}Q(r,t)\leqslant 0, \tag{41}$$

$$\frac{\gamma}{\mathcal{C}}|Q(r,t)|+\frac{\partial}{\partial r}Q(r,t)\leqslant 0, \tag{42}$$

where

$$\mathcal{C}=\sqrt{\frac{(1+2\vartheta)\bar{\eta}^M}{\bar{\rho}_0}} \tag{43}$$

and ϑ is the positive root of the algebraic equation

$$\vartheta^3-\left[\frac{1}{\bar{\eta}^M}\left(\frac{\bar{\hat{\lambda}}{}^0_M}{\bar{\lambda}^m}+\frac{\bar{\bar{\nu}}{}^0_M}{\bar{\nu}^m}\right)+\frac{c^2\bar{\rho}_0}{\bar{\lambda}^m\bar{\nu}^m\bar{\eta}^M}-\frac{1}{2}\right]\vartheta^2-\left[\frac{1}{2\bar{\eta}^M}\left(\frac{\bar{\hat{\lambda}}{}^0_M}{\bar{\lambda}^m}+\frac{\bar{\bar{\nu}}{}^0_M}{\bar{\nu}^m}\right)\right.$$
$$\left.-\frac{\bar{\hat{\lambda}}{}^0_M\bar{\bar{\nu}}{}^0_M}{\bar{\lambda}^m\bar{\nu}^m(\bar{\eta}^M)^2}\right]\vartheta+\frac{\bar{\hat{\lambda}}{}^0_M\bar{\bar{\nu}}{}^0_M}{2\bar{\lambda}^m\bar{\nu}^m(\bar{\eta}^M)^2}=0. \tag{44}$$

Proof. Let us prove the inequality (41). Using the Schwarz's inequality, the arithmetic–geometric mean inequality, (20), (25) and (26), from (35) we obtain

$$\left|\frac{\partial}{\partial t}Q(r,t)\right|\leqslant\int_{S_r}e^{-\gamma t}\left\{\frac{\vartheta_4}{2\bar{\rho}_0}\left[t_i(t)t_i(t)+\frac{1}{I^m}m_i(t)m_i(t)+\frac{1}{\jmath_0}\tau^2(t)\right]\right.$$
$$+\frac{\rho_0}{2\vartheta_4}\left[\dot{u}_i(t)\dot{u}_i(t)+I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t)+\jmath_0\dot{\varphi}^2(t)\right]$$
$$\left.+|\epsilon_{ijk}cH_k(t)E_i(t)n_j|\right\}da$$
$$\leqslant\int_{S_r}e^{-\gamma t}\left\{\frac{1}{\vartheta_4}\left[\frac{\rho_0}{2}\dot{u}_i(t)\dot{u}_i(t)+\frac{\rho_0}{2}I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t)+\frac{\rho_0}{2}\jmath_0\dot{\varphi}^2(t)\right]\right.$$
$$+\frac{\vartheta_4(1+\vartheta_1+\vartheta_2)\bar{\eta}^M}{\bar{\rho}_0}\mathcal{W}(\Phi(t))$$
$$+\frac{1}{\bar{\lambda}^m}\left[\frac{\vartheta_4}{\bar{\rho}_0}\left(2+\frac{1}{\vartheta_1}\right)\bar{\hat{\lambda}}{}^0_M+\frac{2c}{\vartheta_3}\right]\frac{\lambda^mE_i(t)E_i(t)}{2}$$
$$\left.+\frac{1}{\bar{\nu}^m}\left[\frac{\vartheta_4}{\bar{\rho}_0}\left(2+\frac{1}{\vartheta_2}\right)\bar{\bar{\nu}}{}^0_M+c\vartheta_3\right]\frac{\nu^mH_i(t)H_i(t)}{2}\right\}da, \tag{45}$$

for all $r\geqslant 0$, $t\in[0,T]$, $\vartheta_1>0$, $\vartheta_2>0$, $\vartheta_3>0$, $\vartheta_4>0$. We equate the coefficients of the various energetic terms in the integral of (45), namely we set

$$\frac{1}{\vartheta_4}=\frac{\vartheta_4(1+\vartheta_1+\vartheta_2)\bar{\eta}^M}{\bar{\rho}_0}=\frac{1}{\bar{\lambda}^m}\left[\frac{\vartheta_4}{\bar{\rho}_0}\left(2+\frac{1}{\vartheta_1}\right)\bar{\hat{\lambda}}{}^0_M+\frac{2c}{\vartheta_3}\right]$$
$$=\frac{1}{\bar{\nu}^m}\left[\frac{\vartheta_4}{\bar{\rho}_0}\left(2+\frac{1}{\vartheta_2}\right)\bar{\bar{\nu}}{}^0_M+c\vartheta_3\right]. \tag{46}$$

Thus, choosing

$$\vartheta_1=\vartheta,\quad\vartheta_2=\vartheta,\quad\vartheta_3=\frac{\mathcal{C}}{c\vartheta\bar{\eta}^M}\left(\bar{\nu}^m\bar{\eta}^M\vartheta-\bar{\bar{\nu}}{}^0_M\right),\quad\vartheta_4=\frac{1}{\mathcal{C}} \tag{47}$$

where $\mathcal{C}$ is given by (43) and ϑ is the algebraic root of Eq. (44), from (15), (36) and (45) we obtain (41). The inequality (42) is obtained in a very similar manner. The proof is complete. □

Lemma 5. *Let $\mathfrak{P}$ be solution of the problem $(\mathcal{P})$ and let $\widehat{\Omega}_T$ be the bounded support of the external data system $\mathfrak{I}$ on the interval [0,T]. If (α), (β), (γ), (δ) are satisfied, then for any $r \geqslant 0$ and $t \in [0,T]$, the corresponding time–weighted surface power function may be written in the form*

$$Q(r,t)=\frac{1}{2}\int_{\mathcal{B}(r)} e^{-\gamma t}\left[\rho_0\left(\dot{u}_i(t)\dot{u}_i(t)+I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t)+\jmath_0\dot{\varphi}^2(t)\right)\right.$$
$$\left.+2\mathcal{W}(\Phi(t))+2\mathcal{E}(\Xi(t))\right]dv+\frac{\gamma}{2}\int_0^t\int_{\mathcal{B}(r)} e^{-\gamma z}\left[\rho_0\left(\dot{u}_i(z)\dot{u}_i(z)\right.\right.$$
$$\left.\left.+I_{ij}\dot{\varphi}_i(z)\dot{\varphi}_j(z)+\jmath_0\dot{\varphi}^2(z)\right)+2\mathcal{W}(\Phi(z))+2\mathcal{E}(\Xi(z))\right]dv\,dz. \quad (48)$$

Therefore $Q(r,t)$ is an acceptable measure of the solution $\mathfrak{P}$, in the sense that it is positive for all $\mathfrak{P} \neq 0$ in $\mathcal{B}(r)\times[0,T]$ and it vanishes only when $u_i = 0$, $\varphi_i = 0$, $\varphi = 0$, $E_i = 0$, $H_i = 0$, $e_{ij} = 0$, $\kappa_{ij} = 0$, $\zeta_i = 0$, $t_{ij} = 0$, $m_{ij} = 0$, $\sigma = 0$, $\pi_i = 0$, $D_i = 0$, $B_i = 0$ in $\mathcal{B}(r)\times[0,T]$.

Proof. Let us prove first that $Q(r,t)$ is a nonnegative function on $r \geqslant 0$, $t\in[0,T]$. If $\mathcal{B}$ is bounded then r ranges on $[0,L]$, where L is defined by (33). Then, the relation (34) together with the definition of $\widehat{\Omega}_T$ yield

$$Q(L,t)=0, \quad \forall\ t\in[0,T]. \quad (49)$$

Since $Q(r,t)$ is a nonincreasing function with respect to r, from (49) we get that $Q(r,t)\geqslant 0$, for all $r\in[0,L]$ and $t\in[0,T]$.

Let us now consider the case of an unbounded body, namely r ranges on $[0,\infty)$. We note that (41) is equivalent to the following first order differential inequalities

$$\frac{1}{\mathcal{C}}\frac{\partial}{\partial t}Q(r,t)+\frac{\partial}{\partial r}Q(r,t)\leqslant 0 \quad (50)$$

and

$$-\frac{1}{\mathcal{C}}\frac{\partial}{\partial t}Q(r,t)+\frac{\partial}{\partial r}Q(r,t)\leqslant 0. \quad (51)$$

Let us fix $t^\star\in[0,T]$ and let us consider $r^\star\geqslant \mathcal{C}t^\star$. Putting $t=t^\star+[(r-r^\star)/\mathcal{C}]$ in (50), we get

$$\frac{d}{dr}\left[Q\left(r,t^\star+\frac{r-r^\star}{\mathcal{C}}\right)\right]\leqslant 0. \quad (52)$$

Since $0\leqslant r=r^\star-\mathcal{C}t^\star\leqslant r^\star$, from (52) we obtain

$$Q(r^\star,t^\star)\leqslant Q(r^\star-\mathcal{C}t^\star,0). \quad (53)$$

Similarly, setting $t=t^\star-[(r-r^\star)/\mathcal{C}]$ in (51), we deduce

$$\frac{d}{dr}\left[Q\left(r,t^\star-\frac{r-r^\star}{\mathcal{C}}\right)\right]\leqslant 0. \quad (54)$$

so that

$$Q(r^\star+\mathcal{C}t^\star,0)\leqslant Q(r^\star,t^\star). \quad (55)$$

Making $r^\star$ to tend to infinity in the inequalities (53) and (55) and taking into account that $Q(r^\star-\mathcal{C}t^\star,0)=Q(r^\star+\mathcal{C}t^\star,0)=0$, we get

$$Q(\infty,t^\star)=\lim_{r\to\infty}Q(r,t^\star)=0, \quad \forall\ t^\star\in[0,T]. \quad (56)$$

Thus, it follows that $Q(r,t)$ is a nonnegative function on $r\in[0,\infty)$, $t\in[0,T]$.

The relation (48) is obtained from (40), (49) and (56). Clearly, if $Q(r,t)=0$, then (48) yields $\dot{u}_i=0, \dot{\varphi}_i=0$, $\varphi=0$, $e_{ij}=0$, $\kappa_{ij}=0$, $\zeta_i=0$, $E_i=0$ and $H_i=0$ in $\mathcal{B}(r)\times[0,T]$. By (5) and the definition of $\widehat{\Omega}_T$ we obtain $\mathfrak{P}=0$ in $\mathcal{B}(r)\times[0,T]$ and the proof is complete. □

Theorem 1 (Spatial behavior). *Assume that (α), (β), (γ) and (δ) are satisfied. Let $\mathfrak{P}=\{u_i,\varphi_i,\varphi,E_i,H_i,e_{ij},\kappa_{ij},\zeta_i,t_{ij},m_{ij},\sigma,\pi_i,D_i,B_i\}$ be a solution of the problem $(\mathcal{P})$. Let $\widehat{\Omega}_T$ be the bounded support of the external data on the interval [0,T] and let $Q(r,t)$ be the time weighted surface power measure associated with $\mathfrak{P}$. Then for each fixed $t\in[0,T]$, the spatial behavior of $\mathfrak{P}$ outside the bounded support $\widehat{\Omega}_T$ is controlled by the followings:*

(i) For $r\geqslant \mathcal{C}t$ we have

$$\begin{aligned}
&u_i(\mathbf{x},t)=0,\ \varphi_i(\mathbf{x},t)=0,\ \varphi(\mathbf{x},t)=0,\ E_i(\mathbf{x},t)=0,\\
&H_i(\mathbf{x},t)=0,\ e_{ij}(\mathbf{x},t)=0,\ \kappa_{ij}(\mathbf{x},t)=0,\ \zeta_i(\mathbf{x},t)=0,\\
&t_{ij}(\mathbf{x},t)=0,\ m_{ij}(\mathbf{x},t)=0,\ \sigma(\mathbf{x},t)=0,\ \pi_i(\mathbf{x},t)=0,\\
&D_i(\mathbf{x},t)=0,\ B_i(\mathbf{x},t)=0;
\end{aligned} \quad (57)$$

(ii) For $0\leqslant r\leqslant \mathcal{C}t$ we have

$$Q(r,t)\leqslant Q(0,t)\exp\left(-\frac{\gamma}{\mathcal{C}}r\right). \quad (58)$$

Proof. Let us consider the case (i). If we consider $t\in[0,T]$ and set $r=\mathcal{C}t$ in (50), then we deduce

$$\frac{d}{dt}[Q(\mathcal{C}t,\ t)]\leqslant 0, \quad (59)$$

so that, we obtain

$$Q(\mathcal{C}t,\ t)\leqslant Q(0,0)=0, \quad t\in[0,T]. \quad (60)$$

Since $Q(r,t)$ is a nonincreasing function with respect to r, from (60) we deduce

$$Q(r,t)=0, \quad t\in[0,T], \quad r\geqslant \mathcal{C}t. \quad (61)$$

From (61) and Lemma 5 we conclude that (57) holds true.

We consider now the second part. We note that the inequality (42) may be written in the form

$$\frac{\partial}{\partial r}\left[\exp\left(\frac{\gamma}{\mathcal{C}}r\right)Q(r,t)\right]\leqslant 0, \quad t\in[0,T], \quad 0\leqslant r\leqslant \mathcal{C}t. \quad (62)$$

The relation (62) leads to (58) and the proof is complete. □

A direct consequence of this theorem is the following result, holding both for bounded and unbounded bodies:

Theorem 2 (*Uniqueness*). *In the hypotheses (α), (β), (γ) and (δ) the problem $(\mathcal{P})$ has at most one solution.*

Remark 1. Theorem 1 is obtained for a sufficiently general domain that may be particularized for various bounded or unbounded regions. For example, $\mathcal{B}$ could be a right cylinder, finite or semi-infinite, subject to homogeneous initial and boundary data except for that prescribed on a base. In this case, Theorem 1 describes the spatial behavior of solution with respect to distance from the loaded end of the cylinder. Another example is provided by a thick-walled spherical shell, whose inner surface is subject to a given loading while the other external data are homogeneous. Theorem 1 provides a characterization of the spatial behavior in terms of the distance from the inner boundary of the shell. A third example is the three dimensional Euclidean space subject to homogeneous initial data and vanishing body loads except a bounded region on which acts nonzero prescribed body forces. In fact, the next Section studies a problem of this latter type.

5. The effect of a concentrated microstretch body force

In this section, we study the effect of a concentrated microstretch body force, acting in an unbounded microstretch piezoelectric medium. We consider an isotropic and homogeneous microstretch piezoelectric body. Thus, the constitutive equation (5) reduce to (see Eringen, 2004; Ieşan, 2006)

$$\begin{aligned}
t_{ij}&=\lambda e_{rr}\delta_{ij}+(\mu+k)e_{ij}+\mu e_{ji}+\lambda_0\varphi\delta_{ij},\\
m_{ij}&=\alpha\kappa_{rr}\delta_{ij}+\beta\kappa_{ji}+\gamma\kappa_{ij}+b_0\epsilon_{ijk}\zeta_k+\lambda_1\epsilon_{ijk}E_k,\\
\sigma&=\lambda_0 e_{rr}+\xi_0\varphi,\\
\pi_i&=a_0\zeta_i+b_0\epsilon_{rsi}\kappa_{rs}+\lambda_2 E_i,\\
D_k&=-\lambda_1\epsilon_{jik}\kappa_{ij}-\lambda_2\zeta_k+\lambda^\star E_k,\\
B_k&=\nu^\star H_k,
\end{aligned} \quad (63)$$

where λ, μ, k, λ_0, α, β, γ, b_0, λ_1, ξ_0, a_0, λ_2, $\lambda^\star$, $\nu^\star$ are constitutive coefficients. The positiveness conditions (14) and (15) imply that

$$\begin{aligned}&(3\lambda+2\mu+k)\xi_0>3\lambda_0^2,\quad 2\mu+k>0,\quad k>0,\\&3\alpha+\beta+\gamma>0,\quad \gamma+\beta>0,\quad \gamma-\beta>0,\\&\xi_0>0,\quad a_0>0,\quad \lambda^\star>0,\quad \nu^\star>0.\end{aligned}\tag{64}$$

It follows from (1), (2), (4) and (63) that the field equations of the theory of homogeneous and isotropic bodies can be expressed as

$$\begin{aligned}&(\mu+k)\Delta u_i+(\lambda+\mu)u_{j,ji}+k\epsilon_{ijk}\varphi_{k,j}+\lambda_0\varphi_{,i}+\rho_0 f_i=\rho_0\ddot{u}_i,\\&\gamma\Delta\varphi_i+(\alpha+\beta)\varphi_{j,ji}+k\epsilon_{ijk}u_{k,j}-2k\varphi_i+\lambda_1\epsilon_{ijk}E_{k,j}+\rho_0 l_i=\rho_0 I\ddot{\varphi}_i,\\&(a_0\Delta-\xi_0)\varphi+\lambda_2 E_{i,i}-\lambda_0 u_{j,j}+\rho_0 l=\rho_0\jmath_0\ddot{\varphi},\\&\epsilon_{ijk}E_{k,j}=-\frac{1}{c}\dot{B}_i,\\&\epsilon_{ijk}B_{k,j}=\frac{\nu^\star}{c}\left(-\lambda_1\epsilon_{ijk}\dot{\varphi}_{j,k}-\lambda_2\dot{\varphi}_{,i}+\lambda^\star\dot{E}_i\right),\end{aligned}\tag{65}$$

where Δ is the Laplacian.

We consider a body occupying the entire three-dimensional Euclidian space and assume that the body loads have the form

$$f_i=0,\quad l_i=0,\quad \rho_0 l=g(r,t),\tag{66}$$

where $r=|\mathbf{x}-\mathbf{y}|$, $\mathbf{y}$ is a fixed point and g is a prescribed function. We consider the initial conditions

$$\begin{aligned}&u_i(\mathbf{x},0)=0,\quad \dot{u}_i(\mathbf{x},0)=0,\quad \varphi_i(\mathbf{x},0)=0,\quad \dot{\varphi}_i(\mathbf{x},0)=0,\\&\varphi(\mathbf{x},0)=0,\quad \dot{\varphi}(\mathbf{x},0)=0,\quad E_i(\mathbf{x},0)=0,\quad B_i(\mathbf{x},0)=0,\quad \mathbf{x}\in\mathbb{R}^3,\end{aligned}\tag{67}$$

and the following conditions at infinity

$$\begin{aligned}&u_i\to0,\quad u_{i,j}\to0,\quad \varphi_i\to0,\quad \varphi_{i,j}\to0,\\&\varphi\to0,\quad \varphi_{,i}\to0,\quad E_i\to0,\quad B_i\to0\quad\text{for}\quad r\to\infty.\end{aligned}\tag{68}$$

We seek the solution in the form

$$u_i=U_{,i},\quad \varphi_i=0,\quad \varphi=\Upsilon(r,t),\quad E_i=-\psi_{,i},\quad B_i=0,\tag{69}$$

where U, Υ and ψ are unknown functions that depend only on the variables r and t.

Clearly, the field equation (65) are satisfied if the functions U, Υ and ψ satisfy the equations

$$\begin{aligned}&(\lambda+2\mu)\Delta U+\lambda_0\Upsilon=\rho_0\ddot{U},\\&(a_0\Delta-\xi_0)\Upsilon-\lambda_2\Delta\psi-\lambda_0\Delta U+g=\rho_0\jmath_0\ddot{\Upsilon},\\&\lambda_2\Upsilon+\lambda^\star\psi=0.\end{aligned}\tag{70}$$

Introducing the notations

$$\begin{aligned}&c_1^2=\frac{\lambda+2\mu}{\rho_0},\quad c_2^2=\frac{a_0+\frac{\lambda_2^2}{\lambda^\star}}{\rho_0\jmath_0},\quad \beta^*=\frac{\xi_0}{a_0+\frac{\lambda_2^2}{\lambda^\star}},\\&\alpha_1^*=\frac{\lambda_0}{\lambda+2\mu},\quad \alpha_2^*=\frac{\lambda_0}{a_0+\frac{\lambda_2^2}{\lambda^\star}},\quad \gamma^*=\frac{\lambda_2}{\lambda^\star},\end{aligned}\tag{71}$$

then Eq. (70) may be written in the form

$$\begin{aligned}&\left(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)U+\alpha_1^*\Upsilon=0,\\&\left(\Delta-\beta^*-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\Upsilon-\alpha_2^*\Delta U=-\frac{1}{\rho_0\jmath_0c_2^2}g\end{aligned}\tag{72}$$

and

$$\psi=-\gamma^*\Upsilon.\tag{73}$$

Let us define the operator Ω by

$$\Omega=\left(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\left(\Delta-\beta^*-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)+\alpha_1^*\alpha_2^*\Delta.\tag{74}$$

Then it is easy to verify that if we take

$$U=-\alpha_1^*h,\quad \Upsilon=\left(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)h,\tag{75}$$

where the function h satisfies the equation

$$\Omega h=-\frac{1}{\rho_0\jmath_0c_2^2}g,\tag{76}$$

then U and Υ satisfy (72). The initial conditions for the function h are

$$\frac{\partial^q h}{\partial t^q}(\mathbf{x},0)=0,\quad q=0,1,2,3,\quad \mathbf{x}\in\mathbb{R}^3.\tag{77}$$

These conditions imply the initial conditions (67).

If we denote by $\bar{f}$ the Laplace transform with respect to t of the function f, that is

$$\bar{f}(\mathbf{x},p)=\mathcal{L}[f(\mathbf{x},t)]=\int_0^\infty f(\mathbf{x},t)\exp(-pt)dt,\tag{78}$$

then from (75) and (76) we find that

$$\overline{U}=-\alpha_1^*\bar{h},\quad \overline{\Upsilon}=\left(\Delta-\frac{p^2}{c_1^2}\right)\bar{h},\tag{79}$$

where $\bar{h}$ satisfies the equation

$$\left[\left(\Delta-\frac{p^2}{c_1^2}\right)\left(\Delta-\beta^*-\frac{p^2}{c_2^2}\right)+\alpha_1^*\alpha_2^*\Delta\right]\bar{h}=-\frac{1}{\rho_0\jmath_0c_2^2}\bar{g}.\tag{80}$$

This equation may be written in the form

$$(\Delta-k_1^2)(\Delta-k_2^2)\bar{h}=-\frac{1}{\rho_0\jmath_0c_2^2}\bar{g},\tag{81}$$

where

$$k_{1,2}^2=\frac{1}{2}\left\{\Gamma_1p^2+\beta^*-\alpha_1^*\alpha_2^*\mp\sqrt{\Gamma_2^2p^4+2\left[(\beta^*-\alpha_1^*\alpha_2^*)\Gamma_1-\frac{2\beta^*}{c_1^2}\right]p^2+(\beta^*-\alpha_1^*\alpha_2^*)^2}\right\}\tag{82}$$

and the constants Γ_1, Γ_2 are defined by

$$\Gamma_1=\frac{1}{c_2^2}+\frac{1}{c_1^2},\quad \Gamma_2=\frac{1}{c_2^2}-\frac{1}{c_1^2}.\tag{83}$$

It is easy to verify that if the coupling coefficient λ_0 is vanishing (or equivalently $\alpha_1^*=0$, $\alpha_2^*=0$), then we have $k_1=\frac{p}{c_1}$ and $k_2=\sqrt{\beta^*+\frac{p^2}{c_2^2}}$.

Let us suppose that the functions $\bar{h}_1$ and $\bar{h}_2$ satisfy the equations

$$(\Delta-k_1^2)\bar{h}_1=-\frac{1}{\rho_0\jmath_0c_2^2}\bar{g},\quad (\Delta-k_2^2)\bar{h}_2=-\frac{1}{\rho_0\jmath_0c_2^2}\bar{g}.\tag{84}$$

Then, it is easy to find that the solution of Eq. (81) can be written in the form

$$\bar{h}=\frac{1}{k_1^2-k_2^2}(\bar{h}_1-\bar{h}_2).\tag{85}$$

Let us consider that the microstretch body force g has the form

$$g=g^*\delta(\mathbf{x}-\mathbf{y})H(t),\tag{86}$$

where g^* is a given constant, $\delta(\cdot)$ is the Dirac delta and H is the Heaviside unit step function, i.e. $H(t)=0$ for $t\leqslant0$ and $H(t)=1$ for

$t > 0$. By using the conditions at infinity, from (84) and (86) we obtain

$$\bar{h}_1 = \frac{C_0}{rp}\exp(-k_1 r), \quad \bar{h}_2 = \frac{C_0}{rp}\exp(-k_2 r), \tag{87}$$

where

$$C_0 = \frac{g^*}{4\pi\rho_0 \jmath_0 c_2^2}. \tag{88}$$

Thus, the function $\bar{h}$ has the form

$$\bar{h} = \frac{C_0}{rp(k_1^2 - k_2^2)}[\exp(-k_1 r) - \exp(-k_2 r)], \tag{89}$$

and from (69), (73) and (79) we deduce the Laplace transform of the unknowns functions u_i, φ, E_i, namely

$$\begin{aligned}
\bar{u}_i &= \frac{\alpha_1^* C_0 x_i}{r^2 p(k_1^2 - k_2^2)}\left[\left(k_1 + \frac{1}{r}\right)\exp(-k_1 r) - \left(k_2 + \frac{1}{r}\right)\exp(-k_2 r)\right],\\
\bar{\varphi} &= \frac{C_0}{c_1^2 p(k_1^2 - k_2^2)}\left[\frac{c_1^2 k_1^2 - p^2}{r}\exp(-k_1 r) - \frac{c_1^2 k_2^2 - p^2}{r}\exp(-k_2 r)\right],\\
\overline{E}_i &= -\frac{\gamma^* C_0 x_i}{r^2 c_1^2 p(k_1^2 - k_2^2)}\left[(c_1^2 k_1^2 - p^2)\left(k_1 + \frac{1}{r}\right)\exp(-k_1 r) - (c_1^2 k_2^2 - p^2)\right.\\
&\quad \left.\left(k_2 + \frac{1}{r}\right)\exp(-k_2 r)\right].
\end{aligned} \tag{90}$$

The calculation of the inverse transforms is very complicated and the exact solution of the problem defined by (65)–(68) and (86) is difficult to find. However, following similar arguments as those used in thermoelasticity (see Hetnarski, 1961, 1964; Ieşan and Scalia, 1996) we present an approximate solution which gives a reliable description for small times. Thus, according to the initial value theorem (see for example: Schiff, 1999, p. 88), if $\bar{f}$ is the Laplace transform of the function f, then we have

$$\lim_{t\to 0} f(t) = \lim_{p\to\infty} p\bar{f}(p)$$

and so for large values of p correspond small values of t. This fact allows the inversion of Laplace transforms for small times. Supposing that $c_1 \neq c_2$, then by developing in power series with respect to $\frac{1}{p}$, we can write

$$\begin{aligned}
k_1 &= \frac{1}{c_1}p + \frac{A_1}{2}\frac{1}{p} + O\left(\frac{1}{p^3}\right), \quad k_2 = \frac{1}{c_2}p + \frac{A_2}{2}\frac{1}{p} + O\left(\frac{1}{p^3}\right),\\
\frac{k_1}{p(k_1^2 - k_2^2)} &= -\frac{1}{c_1 \Gamma_2}\frac{1}{p^2} + O\left(\frac{1}{p^4}\right),\\
\frac{k_2}{p(k_1^2 - k_2^2)} &= -\frac{1}{c_2 \Gamma_2}\frac{1}{p^2} + O\left(\frac{1}{p^4}\right),\\
\frac{c_1^2 k_1^2 - p^2}{p(k_1^2 - k_2^2)} &= O\left(\frac{1}{p^3}\right),\\
\frac{c_1^2 k_2^2 - p^2}{p(k_1^2 - k_2^2)} &= -c_1^2\frac{1}{p} + O\left(\frac{1}{p^3}\right),\\
\frac{(c_1^2 k_1^2 - p^2)k_1}{p(k_1^2 - k_2^2)} &= -\frac{A_1}{\Gamma_2}\frac{1}{p^2} + O\left(\frac{1}{p^4}\right),\\
\frac{(c_1^2 k_2^2 - p^2)k_2}{p(k_1^2 - k_2^2)} &= -\frac{c_1^2}{c_2} - \Lambda\frac{1}{p^2} + O\left(\frac{1}{p^4}\right),
\end{aligned} \tag{91}$$

where

$$\begin{aligned}
A_1 &= \frac{\alpha_1^*\alpha_2^*}{c_1 \Gamma_2}, \quad A_2 = \beta^* c_2 - \frac{\alpha_1^*\alpha_2^*}{c_2 \Gamma_2},\\
\Lambda &= \frac{c_1^2 c_2}{2}\left[\beta^* + \alpha_1^*\alpha_2^*\left(\frac{\Gamma_1^2}{\Gamma_2^2} - \frac{2}{c_2^4 \Gamma_2^2}\right)\right].
\end{aligned} \tag{92}$$

Moreover, from (91) we have

$$\begin{aligned}
\exp(-k_1 r) &\approx \exp\left(-\frac{pr}{c_1}\right)\exp\left(-\frac{A_1 r}{2p}\right) \approx \exp\left(-\frac{pr}{c_1}\right)\left[1 - \frac{A_1 r}{2p} + \frac{A_1^2 r^2}{8p^2}\right],\\
\exp(-k_2 r) &\approx \exp\left(-\frac{pr}{c_2}\right)\exp\left(-\frac{A_2 r}{2p}\right) \approx \exp\left(-\frac{pr}{c_2}\right)\left[1 - \frac{A_2 r}{2p} + \frac{A_2^2 r^2}{8p^2}\right].
\end{aligned} \tag{93}$$

If we use the relations

$$\begin{aligned}
&\mathcal{L}^{-1}[\exp(-px)] = \delta(t - x),\\
&\mathcal{L}^{-1}[p^{-1}\exp(-px)] = H(t - x), \quad \mathcal{L}^{-1}[p^{-2}\exp(-px)] = (t - x)H(t - x),
\end{aligned} \tag{94}$$

then from (90), (91) and (93) we find

$$\begin{aligned}
u_i &= \frac{\alpha_1^* C_0 x_i}{\Gamma_2 r^2}\left[\frac{1}{c_2}\left(t - \frac{r}{c_2}\right)H\left(t - \frac{r}{c_2}\right) - \frac{1}{c_1}\left(t - \frac{r}{c_1}\right)H\left(t - \frac{r}{c_1}\right)\right],\\
\varphi &= \frac{C_0}{r}\left[H\left(t - \frac{r}{c_2}\right) - \frac{A_2 r}{2}\left(t - \frac{r}{c_2}\right)H\left(t - \frac{r}{c_2}\right)\right],\\
E_i &= -\frac{\gamma^* C_0 x_i}{r^2}\left[\frac{1}{c_2}\delta\left(t - \frac{r}{c_2}\right) + \left(\frac{1}{r} - \frac{A_2 r}{2c_2}\right)H\left(t - \frac{r}{c_2}\right) + \left(\frac{\Lambda}{c_1^2} - \frac{A_2}{2} + \frac{A_2^2 r^2}{8c_2}\right)\right.\\
&\quad \left.\times\left(t - \frac{r}{c_2}\right)H\left(t - \frac{r}{c_2}\right) - \frac{A_1}{c_1^2 \Gamma_2}\left(t - \frac{r}{c_1}\right)H\left(t - \frac{r}{c_1}\right)\right].
\end{aligned} \tag{95}$$

6. Concluding remarks

(i) Derived by Eringen (2004), the electromagnetic theory of microstretch elasticity provides the mathematical apparatus needed to describe the interaction between electromagnetic fields and mechanical deformations of porous bodies such as bones, ceramics, solids with microcracks and synthetic materials with microreinforcements. In this paper we formulated the initial boundary value problem for the linear electromagnetic theory of microstretch elasticity and we studied the problem of spatial behavior of solutions. We got the domain of influence and an exponential decay estimate inside the domain of influence. A direct consequence is the uniqueness of solutions. Our result completes the study given previously by Quintanilla (2008) for quasi-static piezoelectricity.

(ii) In our investigation we considered the charge density to be absent and moreover, in order to be consistent with the law of conservation of charge, we assumed that there is no electric current J_i. However, the result continuous to hold if we change the Ampère's equation $(2)_2$ with

$$\epsilon_{ijk} H_{k,j} = \frac{1}{c}\dot{D}_i + \frac{1}{c}J_i \tag{96}$$

and consider that the electric current satisfies the dissipation inequality (see Eringen, 2004, the relation (5.5))

$$J_i E_i \geqslant 0. \tag{97}$$

The presence of the electric current J_i in (96) results in the relation (36) which becomes

$$\begin{aligned}
\frac{\partial}{\partial r}Q(r,t) = &-\frac{1}{2}\int_{S_r} e^{-\gamma t}\left[\rho_0(\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + \jmath_0\dot{\varphi}^2(t))\right.\\
&\left. + 2\mathcal{W}(\Phi(t)) + 2\mathcal{E}(\Xi(t))\right]da\\
&-\frac{\gamma}{2}\int_0^t\int_{S_r} e^{-\gamma z}\left[\rho_0(\dot{u}_i(z)\dot{u}_i(z) + I_{ij}\dot{\varphi}_i(z)\dot{\varphi}_j(z)\right.\\
&+ \jmath_0\dot{\varphi}^2(z)) + 2\mathcal{W}(\Phi(z)) + 2\mathcal{E}(\Xi(z))\\
&\left. + \frac{2}{\gamma}J_i(z)E_i(z)\right]da\,dz.
\end{aligned} \tag{98}$$

Clearly, in view of the (97) the first order differential inequalities (41) and (42) remain valid and thus, the domain of influence and the estimate inside the domain of influence are obtained.

(iii) In order to verify and validate our result, in Section 5 we studied the problem of a concentrated microstretch body force acting in a microstretch piezoelectric body that occupy the entire three-dimensional Euclidean space. To solve the problem, we adopted a semi-inverse method and we utilized the properties of Laplace transform. Since the calculation of inverse Laplace transforms is very complicated, we presented an approximate solution useful for small times. Although we cannot apply directly the Theorem 1, since in Section 5 the problem is formulated in the weak sense and the Theorem 1 deals with classical solutions, it is clear from (95) that even in this case we have a domain of influence. The solution being expressed in terms of the Heaviside and Dirac delta functions of $t-\frac{r}{c_1}$ and $t-\frac{r}{c_2}$, it follows that the whole activity vanishes for $r \geqslant t\max\{c_1, c_2\}$.

Acknowledgements

The author is grateful to the reviewers for useful observations leading to improvements of the paper. This work was supported by the CNCSIS Grant TE-184, No. 86/30.07.2010 and the POSDRU/89/1.5/49944 Project.

References

Chiriţă, S., Ciarletta, M., 1999. Time–weighted surface power function method for the study of spatial behaviour in dynamics of continua. European Journal of Mechanics A/Solids 18, 915–933.
Eringen, A.C., 1999. Microcontinuum Field Theories. Foundations and Solids. Springer, New York.
Eringen, A.C., 2003. Continuum theory of micromorphic electromagnetic thermoelastic solids. International Journal of Engineering Science 41, 653–665.
Eringen, A.C., 2004. Electromagnetic theory of microstretch elasticity and bone modeling. International Journal of Engineering Science 42, 231–242.
Eringen, A.C., Maugin, G.A., 1990. Electrodynamics of Continua, vol. I. Springer-Verlag, New York.
Hetnarski, R.B., 1961. Coupled one-dimensional thermal shock problem for small times. Archiwum Mechaniki Stosowanej 13, 295–306.
Hetnarski, R.B., 1964. The fundamental solution of the coupled thermoelastic problem for small times. Archiwum Mechaniki Stosowanej 16, 23–31.
Ieşan, D., 2006. On the microstretch piezoelectricity. International Journal of Engineering Science 44, 819–829.
Ieşan, D., 2008a. On the bending of piezoelectric plates with microstructure. Acta Mechanica 198, 191–208.
Ieşan, D., 2008b. Thermopiezoelectricity without energy dissipation. Proceedings of the Royal Society A 464, 631–657.
Ieşan, D., Quintanilla, R., 2007. Some theorems in the theory of microstretch thermopiezoelectricity. International Journal of Engineering Science 45, 1–16.
Ieşan, D., Scalia, A., 1996. Thermoelastic Deformations. Kluwer Academic Publishers, Dordrecht.
Lee, J.D., Chen, Y., Eskandarian, A., 2004. A micromorphic electromagnetic theory. International Journal of Solids and Structures 41, 2099–2110.
Quintanilla, R., 2008. A spatial decay in the linear theory of microstretch piezoelectricity. Mathematical and Computer Modelling 47, 1117–1124.
Satter, S.A., Islam, M.S., Rabbani, K.S., Talukder, M.S., 1999. Pulsed electromagnetic fields for the treatment of bone fracture. Bangladesh Medical Research Council Bulletin 25, 6–10.
Schiff, J.L., 1999. The Laplace Transform: Theory and Applications. Springer-Verlag, New York.
Tiersten, H.F., 1969. Linear Piezoelectric Plate Vibrations. Plenum, New York.
Yang, J., 2005. An Introduction to the Theory of Piezoelectricity. Springer Science + Business Media, Inc., Boston.
Zhou, S.A., 1999. Electrodynamics of Solids and Microwave Superconductivity. John Wiley and Sons, Inc., New York.