Hadamard products with power functions and multipliers of Hardy spaces

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Abstract

We consider Hadamard products of power functions $P(z) = (1 - z)^{-b}$ with functions analytic in the open unit disk in the complex plane, and an integral representation is obtained when $0 < \Re b < 2$. Let $\mu_n = \int_{\Delta} \zeta^n d\mu(\zeta)$ where $\mu$ is a complex-valued measure on the closed unit disk $\bar{\Delta}$. Such sequences are shown to be multipliers of $H^p$ for $1 \leq p \leq \infty$. Moreover, if the support of $\mu$ is contained in a finite set of Stolz angles with vertices on the unit circle, we prove that $\{\mu_n\}$ is a multiplier of $H^p$ for every $p > 0$. When the support of $\mu$ is $[0, 1]$ we get the multiplier sequence $\int_0^1 t^n d\mu(t)$, which provides more concrete applications. We show that if the sequences $\{\mu_n\}$ and $\{\nu_n\}$ are related by an asymptotic expansion

$$\lim_{n \to \infty} \frac{\nu_n}{\mu_n} = \sum_{k=0}^{\infty} \frac{A_k}{n^k}$$

and $\mu_n$ is a multiplier of $H^p$ into $H^q$, then so is $\nu_n$. We ask whether $\{(n + 1)^{\beta}\}$ is a multiplier of $H^p$ when $\beta$ is a nonzero real number. It is clear that the question has an affirmative answer when $p = 2$. The answer is shown to be negative when $p = \infty$.

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1. Introduction

Let $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$. If $f$ and $g$ are functions which are analytic in $\Delta$ and have the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then the Hadamard product of $f$ and $g$ is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$$

for $z \in \Delta$.

We consider the Hadamard product of functions with the power function $P(z) = \frac{1}{(1 - z)^b}$, (2)

where $b \in \mathbb{C}$. Throughout powers and logarithms are the principal branches. Theorem 1 asserts that

$$(f * P)(z) = f(z) - \frac{1}{\pi} \sin(\pi b) \int_{0}^{1} \frac{f(z) - f(tz)}{t^{b/(1 - t)^b}} dt$$

(3)

for $0 < \text{Re} \, b < 2$. The derivation of (3) depends on transforming a certain contour integral on $\partial \Delta$ to an integral on $[0, 1]$. Equation (3) relates to a formula about hypergeometric functions.

Let $\mathcal{F}$ and $\mathcal{G}$ denote two families of functions analytic in $\Delta$. A sequence $\{ \mu_n \}_{n=0}^{\infty}$ is called a multiplier of $\mathcal{F}$ into $\mathcal{G}$ provided that $g \in G$ whenever $f \in F$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} \mu_n a_n z^n$. When $G = F$, $\{ \mu_n \}$ is called a multiplier of $\mathcal{F}$.

Our interest is in the Hardy spaces $H^p$ and sequences given by the moments $\mu_n = \int_{\Delta} \zeta^n d\mu(\zeta)$, where $\mu$ is a complex-valued measure on $\bar{\Delta}$. Such sequences are shown to be multipliers of $H^p$ for $1 \leq p < \infty$. If the support of $\mu$ is contained in a finite set of Stolz angles with vertices on $\partial \Delta$, then $\{ \mu_n \}$ is a multiplier of $H^p$ for all $p > 0$. The key fact used to prove this result is the maximal theorem of Hardy and Littlewood. When the support of $\mu$ is $[0, 1]$ we get the multiplier sequence $\int_{0}^{1} t^n d\mu(t)$, which provides more concrete applications.

There is an extensive literature on multipliers between various families of analytic functions. Multipliers and Hardy spaces are discussed in [2, pp. 99–106] and references to the related literature are given in [2, p. 107].

We show that if the sequences $\{ \mu_n \}$ and $\{ \nu_n \}$ are related by an asymptotic expansion

$$\frac{\nu_n}{\mu_n} \approx \sum_{k=0}^{\infty} \frac{A_k}{n^k} (n \to \infty)$$

(4)

and $\mu_n$ is a multiplier of $H^p$ into $H^q$, then so is $\nu_n$. Also, the question is asked whether $\{(n+1)^\beta \}$ is a multiplier of $H^p$ when $\beta$ is a nonzero real number. It is easy to see that this question has an affirmative answer when $p = 2$. We show that the answer is negative when $p = \infty$. The argument depends upon using (3) with $b = 1 + i\beta$ and $f(z) = (1 - z)^{i\beta}$. It follows that if $\beta$ is a nonzero real number, then the sequence $\{(n+1)^i\beta \}$ is not the moment sequence of a complex-valued measure on $\bar{\Delta}$. 


2. Hadamard products with power functions

**Theorem 1.** Suppose that the function \( f : \Delta \to \mathbb{C} \) is analytic. Let \( P(z) = 1/(1-z)^b \), where \( b \in \mathbb{C} \), and let \( g = f * P \). If \( 0 < \text{Re} b < 2 \) then

\[
g(z) = f(z) - \frac{1}{\pi} \sin(\pi b) \int_0^1 \frac{f(z) - f(tz)}{t^{1-b}(1-t)^b} \, dt \tag{5}
\]

for \( z \in \Delta \).

**Proof.** Let \( \alpha = \text{Re} \, b \) and \( \beta = \text{Im} \, b \). Since \( g(0) = f(0) \), (5) holds when \( z = 0 \). Henceforth assume that \( z \neq 0 \) and write \( z = \rho e^{i\phi} \), where \( 0 < \rho < 1 \) and \( 0 \leq \phi < 2\pi \). For \( 0 \leq t < 1 \) let \( L(t) \) denote the closed line segment from \( tz \) to \( z \). Then \( f(z) - f(tz) = \int_{L(t)} f'(w) \, dw \) and thus \( |f(z) - f(tz)| \leq \sup_{|w| \leq \rho} |f'(w)|(1-t) \). Hence

\[
\left| \frac{f(z) - f(tz)}{t^{1-b}(1-t)^b} \right| \leq \sup_{|w| \leq \rho} |f'(w)| t^{\alpha-1} (1-t)^{1-\alpha}.
\]

The assumption \( 0 < \alpha < 2 \) implies that \( \int_0^1 t^{\alpha-1} (1-t)^{1-\alpha} \, dt \) exists. Therefore the integral in (5) exists.

To prove (5), first assume that \( f \) is analytic in \( \bar{\Delta} \). Then

\[
g(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(e^{i\theta})P(z e^{-i\theta})}{w} \, d\theta,
\]

which can be written

\[
g(z) = \frac{1}{2\pi i} \int_{|w|=1} F(w) \, dw, \tag{6}
\]

where

\[
F(w) = \frac{f(w)}{w(1-z/w)^b}. \tag{7}
\]

The function \( w \mapsto (1-z/w)^b \) is analytic and nonzero in \( \mathbb{C} \setminus L(0) \) and hence \( F \) is analytic in \( \bar{\Delta} \setminus L(0) \). Therefore \( \int_{|w|=1} F(w) \, dw = \int_{C} F(w) \, dw \) for any closed curve \( C \) in \( \bar{\Delta} \setminus L(0) \) which is homologous to \( \partial \Delta \). In particular, we may let \( C = C_1 + C_2 + C_3 + C_4 \), where \( C_1 \) and \( C_3 \) are arcs of circles and \( C_2 \) and \( C_4 \) are line segments described in Fig. 1. We require that \( 0 < \delta < \min\{1-\rho, \rho\}, 0 < \epsilon < \rho - \delta, C_2 \) and \( C_4 \) are parallel to \( L(0) \), and both \( C_2 \) and \( C_4 \) have the distance \( \eta \) from \( L(0) \), where \( 0 < \eta < \min\{\epsilon, \delta\} \).

We note that \( F \) extends continuously at each interior point of \( L(0) \) with respect to approach from each side of \( L(0) \). By letting \( \eta \to 0 \) we see that \( \int_{C_1} F(w) \, dw \to \int_{\Gamma_1} F(w) \, dw \) and \( \int_{C_3} F(w) \, dw \to \int_{\Gamma_3} F(w) \, dw \), where \( \Gamma_1 \) is the circle \( w = \varepsilon e^{i\phi}, \phi \leq \theta \leq \phi + 2\pi \), and \( \Gamma_3 \) is the circle \( w = z + \delta e^{i\phi}, \phi - \pi \leq \theta \leq \phi + \pi \). Also, as \( \eta \to 0 \), \( \int_{C_2} F(w) \, dw \)}
and $\int_{C_4} F(w) \, dw$ tend to integrals along certain line segments on $L(0)$, which we denote by $\Gamma_2$ and $\Gamma_4$, respectively. Thus (6) yields

$$g(z) = \frac{1}{2\pi i} \sum_{k=1}^{4} I_k,$$  \hspace{1cm} (8)

where

$$I_k = \int_{C_k} F(w) \, dw.$$  \hspace{1cm} (9)

If $w \in \Gamma_1$ then

$$|(1-z/w)^b| = |1-z/w|^a \exp\left(-\beta \arg(1-z/w)\right) \geq \frac{(\rho - \epsilon)^a}{\epsilon^a} \exp(-|\beta|\pi).$$

Hence

$$|I_1| \leq 2\pi \exp(-|\beta|\pi) \frac{\epsilon^a}{(\rho - \epsilon)^a} \sup_{w \in \Gamma_1} |f(w)|.$$

Since $\alpha > 0$ this implies that

$$\lim_{\epsilon \to 0^+} I_1 = 0.$$  \hspace{1cm} (10)

By using appropriate limits of the power function on $\Gamma_2$ and $\Gamma_4$ we find $I_2$ and $I_4$ which yields

$$I_2 + I_4 = [e^{\theta \pi i} - e^{-\theta \pi i}] \int_{\epsilon}^{\rho - \delta} \frac{f(r e^{i\varphi})}{r^{1-b}(\rho - r)^b} \, dr.$$  \hspace{1cm} (11)

For $0 < r < \rho$ let $G(r) = f(r e^{i\varphi})/(r^{1-b}(\rho - r)^b)$. Then $G$ is integrable on $[0, \rho - \delta]$ and thus

$$\lim_{\epsilon \to 0^+} \int_{\epsilon}^{\rho - \delta} G(r) \, dr = \int_{0}^{\rho - \delta} G(r) \, dr.$$  \hspace{1cm} (12)

Therefore (8), (10), and (11) imply that

$$g(z) = \frac{1}{\pi} \sin(\pi b) \int_{0}^{\rho - \delta} \frac{f(r e^{i\varphi})}{r^{1-b}(\rho - r)^b} \, dr + \frac{1}{2\pi i} I_3.$$  \hspace{1cm} (12)
By setting $\psi = \varphi - \theta$ and $s = \rho/\delta$ we obtain

$$I_3 = i \int_{-\pi}^{\pi} f \left( z + \frac{z}{s} e^{-i\psi} \right) (1 + se^{i\psi})^{b-1} d\psi. \quad (13)$$

For $-\pi \leq \psi \leq \pi$ and large positive $s$ we have

$$|\left( 1 + se^{i\psi} \right)^{b-1} - (se^{i\psi})^{b-1}| \leq |(se^{i\psi})^{b-1}| (2/s)|b - 1| \leq \frac{2e^{\beta|\pi|}|b - 1|}{s^{2-\alpha}}.$$ 

Hence (13) yields

$$\left| I_3 - i \int_{-\pi}^{\pi} f \left( z + \frac{z}{s} e^{-i\psi} \right) (se^{i\psi})^{b-1} d\psi \right| \leq 4\pi \sup_{w \in \Gamma} |f(w)| \left[ \frac{e^{\beta|\pi|}|b - 1|}{s^{2-\alpha}} \right].$$

By letting $s \to \infty$ and noting that $\alpha < 2$, we get

$$\lim_{\delta \to 0^+} \left[ I_3 + 2i \int_{-\pi}^{\pi} f(z + \delta e^{i(\varphi - \psi)}) \left( \frac{\rho e^{i\varphi}}{\delta} \right)^{b-1} d\psi - f(z) \right] = 0. \quad (14)$$

For small positive $\delta$,

$$\left| \int_{-\pi}^{\pi} f(z + \delta e^{i(\varphi - \psi)}) \left( \frac{\rho e^{i\varphi}}{\delta} \right)^{b-1} d\psi - f(z) \right|$$

is bounded above by $4\pi |f'(z)| e^{\beta|\pi|} \rho^{\alpha-1} \delta^{2-\alpha}$, and this last expression tends to 0 as $\delta \to 0^+$. Also

$$\int_{-\pi}^{\pi} \left( \frac{\rho e^{i\varphi}}{\delta} \right)^{b-1} d\psi = \frac{2}{b-1} \left( \frac{\rho}{\delta} \right)^{b-1} \sin(\pi(b - 1)).$$

and thus (14) implies that

$$\lim_{\delta \to 0^+} \left[ I_3 + 2i \int_{-\pi}^{\pi} f(z) \left( \frac{\rho e^{i\varphi}}{\delta} \right)^{b-1} \sin(\pi(b - 1)) d\psi \right] = 0. \quad (15)$$

Equations (12) and (15) yield

$$g(z) = \lim_{\delta \to 0^+} \left[ \frac{1}{\pi} \sin(\pi b) \int_{0}^{\rho-\delta} \frac{f(r e^{i\varphi})}{r^{1-b}(\rho - r)^b} dr - \frac{1}{\pi} \left( \frac{\rho}{\delta} \right)^{b-1} \sin(\pi(b - 1)) f(z) \right]. \quad (16)$$

In particular, when $f \equiv 1$ (16) becomes

$$1 = \lim_{\delta \to 0^+} \left[ \frac{1}{\pi} \sin(\pi b) \int_{0}^{\rho-\delta} \frac{dr}{r^{1-b}(\rho - r)^b} - \frac{1}{\pi} \left( \frac{\rho}{\delta} \right)^{b-1} \sin(\pi(b - 1)) \right]. \quad (17)$$
Using (17) we can rewrite (16) as
\[
g(z) = f(z) + \frac{1}{\pi} \sin(\pi b) \lim_{\delta \to 0^+} \int_0^{\rho - \delta} \frac{f(re^{i\theta}) - f(z)}{r^{1-b}(\rho - r)^b} \, dr.
\]
With the change of variable \( t = r/\rho \), this becomes
\[
g(z) = f(z) - \frac{1}{\pi} \sin(\pi b) \lim_{\delta \to 0^+} \int_0^{1-\delta/\rho} \frac{f(z) - f(tz)}{t^{1-b}(1-t)^b} \, dt.
\]
(18)

Since the integral
\[
\int_0^1 \frac{f(z) - f(tz)}{t^{1-b}(1-t)^b} \, dt
\]
exists, letting \( \delta \to 0^+ \) in (18) we obtain (5). This completes the proof for \( f \) analytic in \( \bar{\Delta} \).

Now suppose that \( f \) is analytic in \( \Delta \). For \( 0 < r < 1 \) let \( f_r(z) = f(rz) \). Then \( (f_r * P)(z) = g(rz) \). If we apply (5) to \( f_r \) we obtain
\[
g(rz) = f(rz) - \frac{1}{\pi} \sin(\pi b) \int_0^{1-\delta/\rho} \frac{f(z) - f(tz)}{t^{1-b}(1-t)^b} \, dt.
\]
(19)

A simple argument using the Lebesgue dominated convergence theorem allows us to let \( r \to 1^- \) in (19), which yields (5). ✷

Equation (5) relates to a formula about the hypergeometric function \( F(a, b, c; z) \) when \( a = 1 \) and \( c = 2 \). By definition,
\[
F(1, b, 2; z) = 1 + \sum_{n=1}^{\infty} \frac{b(b+1) \cdots (b+n-1)}{(n+1)!} z^n
\]
(20)

for \( z \in \Delta \). If \( 0 < \Re b < 2 \), then [5, p. 159]
\[
F(1, b, 2; z) = \frac{1}{\Gamma(b) \Gamma(2-b)} \int_0^1 \frac{t^{b-1}(1-t)^{1-b}}{1-tz} \, dt,
\]
(21)

where \( \Gamma \) denotes the Gamma function.

If we let \( f(z) = 1/(1-z) \) then \( f * P = P \), and (5) implies that
\[
\int_0^1 \frac{t^{b-1}(1-t)^{1-b}}{1-tz} \, dt = \frac{\pi}{\sin(\pi b)} \left[ \frac{1}{z} \left( 1 - \frac{1}{(1-z)^{b-1}} \right) \right]
\]
(22)

for \( z \in \Delta \) and \( 0 < \Re b < 2 \). By expressing the right side of (22) as a power series and using the relation \( \Gamma(b) \Gamma(2-b) = (1-b)\pi/\sin(\pi b) \), we find that (22) and (21) are equivalent. In other words, the special case of (5) where \( f(z) = 1/(1-z) \) gives a formula equivalent to (21). Also, beginning with (21) it is possible to provide another derivation of (5).
3. Multipliers of Hardy spaces

Theorem 2. Suppose that $\mu$ is a complex-valued measure on $\tilde{\Delta}$, and for each nonnegative integer $n$ let

$$\mu_n = \int_{\tilde{\Delta}} \zeta^n d\mu(\zeta).$$

(23)

Then $\{\mu_n\}$ is a multiplier of $H^p$ for $1 \leq p \leq \infty$.

Proof. Suppose that $f$ is analytic in $\Delta$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Suppose $\{\mu_n\}$ is defined by (23) and let $g(z) = \sum_{n=0}^{\infty} \mu_n a_n z^n$ for $z \in \Delta$. Since $\{\mu_n\}$ is bounded, $g$ is analytic in $\Delta$.

We have

$$g(z) = \sum_{n=0}^{\infty} \left( \int_{\tilde{\Delta}} \zeta^n d\mu(\zeta) \right) a_n z^n = \int_{\tilde{\Delta}} \left( \sum_{n=0}^{\infty} a_n \zeta^n z^n \right) d\mu(\zeta),$$

that is,

$$g(z) = \int_{\tilde{\Delta}} f(\zeta z) d\mu(\zeta)$$

(24)

for $z \in \Delta$. If $f$ is bounded then (24) implies $|g(z)| \leq \|f\|_{H^\infty} \|\mu\| < \infty$. Hence $g$ is bounded. This proves the theorem in the case $p = \infty$.

Now suppose that $1 \leq p < \infty$ and $f \in H^p$. The continuous form of Minkowski’s inequality applied to (24) gives

$$\left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta})|^p d\theta \right]^{1/p} \leq \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\zeta r e^{i\theta})|^p d\theta \right]^{1/p} d|\mu|(\zeta)$$

(25)

for $0 < r < 1$. Periodicity implies that

$$\int_{-\pi}^{\pi} |f(\zeta r e^{i\theta})|^p d\theta = \int_{-\pi}^{\pi} |f(|\zeta| r e^{i\theta})|^p d\theta \leq 2\pi \|f\|_{H^p}^p.$$

Hence (25) yields

$$\left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta})|^p d\theta \right]^{1/p} \leq \|f\|_{H^p} \|\mu\| < \infty.$$

Thus $g \in H^p$. Therefore $\{\mu_n\}$ is a multiplier of $H^p$. $\square$

Theorem 3. Suppose that $\mu$ is a complex-valued measure on $\tilde{\Delta}$ with support $\Lambda$, and that $\Lambda$ contains at most a finite number of points on $\partial\Delta$ and they can only be approached nontangentially by points in $\Lambda$. Then sequence (23) is a multiplier of $H^p$ for $0 < p \leq \infty$. 
Proof. Let \( g \) be defined as in the proof of Theorem 2. If \( \Lambda \cap \partial \Delta = \emptyset \) it follows from (24) that \( g \) is analytic in \( \Delta \). Hence \( g \in H^\infty \).

Now suppose that \( \Lambda \cap \partial \Delta \neq \emptyset \). For \( \sigma = 1, 0 < \gamma < \pi \), and \( 0 < r < 1 \), let \( S(\sigma, \gamma, r) = \{ w = re^{i\theta}: r \leq \rho \leq 1 \text{ and } |\arg(w - \sigma)| \leq \gamma/2 \} \). There exist a positive integer \( k \), points \( \sigma_j (j = 1, 2, \ldots, k) \) on \( \partial \Delta \), and \( 0 < r_0 < 1 \) such that \( \Lambda \subseteq \{ z: |z| < r_0 \} \bigcup_{j=1}^{k} S(\sigma_j, \gamma_j, r_0) \) for suitable \( \gamma_j \). By letting \( \gamma = \max\{\gamma_j: 1 \leq j \leq k\} \) and choosing \( r_0 \) larger if necessary, we may assume that \( \gamma_j = \gamma \) and that the sets \( S_j = S(\sigma_j, \gamma_j, r_0) \) are pairwise disjoint. Suppose that \( 0 < p < \infty \) and \( f \in H^p \). From (24) we can write

\[
g = g_1 + g_2. \tag{26}
\]

where

\[
g_1(z) = \int_{|\zeta| \leq r_0} f(\zeta z) d\mu(\zeta) \tag{27}
\]

and

\[
g_2(z) = \sum_{j=1}^{k} \int_{S_j} f(\zeta z) d\mu(\zeta) \tag{28}
\]

for \( z \in \Delta \). Then \( g_1 \) is analytic in \( \Delta \) and hence \( g_1 \in H^\infty \). Also,

\[
|g_2(z)| \leq \left\{ \sum_{j=1}^{k} \sup_{\zeta \in S_j} |f(\zeta z)| \right\} \|\mu\|. \tag{29}
\]

There is a constant \( A > 0 \) depending only on \( p \) and \( k \) such that \((c_1 + c_2 + \cdots + c_k)^p \leq A(c_1^p + c_2^p + \cdots + c_k^p)\) for \( c_j \geq 0 (j = 1, 2, \ldots, k) \). Hence (29) implies that

\[
\int_{-\pi}^{\pi} |g_2(re^{i\theta})|^p d\theta \leq A\|\mu\|^p \sum_{j=1}^{k} \int_{-\pi}^{\pi} \left[ \sup_{\zeta \in S_j} |f(\zeta re^{i\theta})| \right]^p d\theta \tag{30}
\]

for \( 0 < r < 1 \). Let \( F(\theta) = \sup_{w \in S(e^{i\theta}, \gamma)} |f(w)| \) for \(-\pi \leq \theta \leq \pi \). Then (30) and periodicity yield

\[
\int_{-\pi}^{\pi} |g_2(re^{i\theta})|^p d\theta \leq Ak\|\mu\|^p \int_{-\pi}^{\pi} F^p(\theta) d\theta. \tag{31}
\]

The Hardy–Littlewood maximal theorem [3, p. 114] asserts that \( F \in L^p([-\pi, \pi]) \) and there is a constant \( B > 0 \) depending only on \( p \) such that

\[
\left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} F^p(\theta) d\theta \right]^{1/p} \leq B\|f\|_{H^p}.
\]

Hence (31) implies that
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |g_2(re^{i\theta})|^p d\theta \leq Ak\|\mu\|^p B^p \|f\|^p \|_{H^p} < \infty.
\]

Therefore \( g_2 \in H^p \).

Since \( g_1 \in H^\infty \) and \( g_2 \in H^p \), (26) yields \( g \in H^p \). Therefore \( \{\mu_n\} \) is a multiplier of \( H^p \). \( \Box \)

Corollary 1. If \( \mu \) is a complex-valued measure on \([0, 1] \) and \( \mu_n = \int_0^1 t^n d\mu(t) \), then \( \{\mu_n\}_{n=0}^\infty \) is a multiplier of \( H^p \) for \( 0 < p \leq \infty \).

Suppose that \( 0 < \Re b < 2 \) and let \( \mu_0 = 1 \) and \( \mu_n = b(b + 1) \ldots (b + n - 1)/(n + 1)! \) for \( n = 1, 2, \ldots \). From (20) and (21) we see that \( \mu_n = \int_0^1 t^n d\mu(t) \), where

\[
d\mu(t) = \frac{t^{b-1}(1-t)^{1-b}}{\Gamma(b) \Gamma(2-b)} dt.
\]

Corollary 1 implies that \( \{\mu_n\} \) is a multiplier of \( H^p \). In the case \( b = 1 \) this shows that \( \{1/(n + 1)\} \) is a multiplier of \( H^p \) and hence so is \( \{1/(n + 1)^k\} \) for each positive integer \( k \). Actually, the sequences \( \{1/(n + 1)^k\} \) are multipliers of \( H^p \) into \( H^q \) for suitable \( q > p \), but that information is not needed in the next theorem. Direct applications of Corollary 1 become interesting when \( d\mu(t) = F(t) dt \), \( F \in L^1([0, 1]) \), and \( F \) has a singularity at \( t = 1 \) which is stronger than the singularity given by \( t \mapsto 1/(1-t)^c \), where \( \Re c < 1 \). For example, when \( F(t) = (1-t)^{-1}(\log 2/(1-t))^{-b} \) and \( \Re b > 1 \), a multiplier of \( H^p \) is obtained which is asymptotic to \( A[\log(n+2)]^{1-b} \) as \( n \to \infty \), where \( A \) is a constant.

Theorem 4. Suppose that \( \{a_n\} \) and \( \{b_n\} \) are sequences such that \( a_n \neq 0 \) for any \( n \), and \( b_n/a_n \) has an asymptotic expansion

\[
\frac{b_n}{a_n} \approx \sum_{k=0}^{\infty} \frac{A_k}{n^k} \quad (n \to \infty).
\]

(a) If \( \sum_{n=0}^{\infty} a_n z^n \) defines a function in \( H^p \) then so does \( \sum_{n=0}^{\infty} b_n z^n \).

(b) If \( \{a_n\} \) is a multiplier of \( H^p \) into \( H^q \) then so is \( \{b_n\} \).

Proof. Let \( 0 < p \leq \infty \) and assume that \( f \in H^p \), where \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( z \in \Delta \). There is a positive constant \( A \) such that for each nonnegative integer \( n \),

\[
|a_n| \leq A(n+1)^{1/p-1}
\]

when \( 0 < p < 1 \) and

\[
|a_n| \leq A
\]

when \( 1 \leq p \leq \infty \) [2, p. 98]. If \( 0 < p < 1 \) let \( m \) denote the smallest positive integer such that \( m > 1/p \) and if \( 1 \leq p \leq \infty \) let \( m = 2 \). The expansion (32) implies an expansion

\[
\frac{b_n}{a_n} \approx \sum_{k=0}^{\infty} \frac{B_k}{(n+1)^k} \quad (n \to \infty).
\]
Hence there is a positive constant $B$ such that
\[
\frac{b_n}{a_n} = \sum_{k=0}^{m-1} \frac{B_k}{(n+1)^k} + R_{m,n}
\]  
(36)
and
\[
|R_{m,n}| \leq \frac{B}{(n+1)^m}
\]  
(37)
for $n = 0, 1, 2, \ldots$. Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Since $f$ is analytic in $\Delta$, and (35) implies that $|b_n| \leq C|a_n|$ for some positive constant $C$, $g$ is also analytic in $\Delta$. We have
\[
g = g_1 + g_2,
\]  
(38)
where
\[
g_1(z) = \sum_{k=0}^{m-1} B_k \left( \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^k} z^n \right)
\]  
(39)
and
\[
g_2(z) = \sum_{n=0}^{\infty} a_n R_{m,n} z^n
\]  
(40)
Since $\{1/(n+1)^k\}$ is a multiplier of $H^p$, the function defined by $\sum_{n=0}^{\infty} a_n z^n/(n+1)^k$ belongs to $H^p$ for $k = 0, 1, 2, \ldots, m - 1$. Therefore $g_1 \in H^p$. If $0 < p < 1$ then (40), (33), and (37) yield $|g_2(z)| \leq AB \sum_{n=0}^{\infty} (n+1)^{-1/p-m-1}$. The last sum is finite because $1/p - m$. Hence $g_2$ is bounded. If $1 \leq p \leq \infty$ then (40), (34), and (37) yield $|g_2(z)| \leq AB \sum_{n=0}^{\infty} (n+1)^{-2} < \infty$. Again, $g_2$ is bounded. From $g_1 \in H^p$, $g_2 \in H^\infty$, and (38) we conclude that $g \in H^p$. This proves part (a) of the theorem.

To show part (b), assume that $\{a_n\}$ is a multiplier of $H^p$ into $H^q$. Suppose that $f \in H^p$ and let $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Choose positive numbers $\epsilon_n$ so that $\epsilon_n \neq \epsilon_n$ for any $n$, $\sum_{n=0}^{\infty} \epsilon_n < \infty$, and $\sum_{n=0}^{\infty} \epsilon_n |b_n| < \infty$. Let $\tilde{c}_n = c_n + \epsilon_n$ and let $f(z) = \sum_{n=0}^{\infty} \tilde{c}_n z^n$. Since $f \in H^p$ and $\sum_{n=0}^{\infty} \tilde{c}_n < \infty$ we have $f \in H^p$. Because $\{a_n\}$ is a multiplier of $H^p$ into $H^q$, this implies that $\sum_{n=0}^{\infty} a_n \tilde{c}_n z^n$ defines a function in $H^q$. Note that $\tilde{c}_n \neq 0$ and $b_n \tilde{c}_n/(a_n \tilde{c}_n) = b_n/a_n$. Hence (32) shows that $b_n \tilde{c}_n/(a_n \tilde{c}_n)$ has such an asymptotic expansion. Applying the result in part (a) we find that $\sum_{n=0}^{\infty} b_n \tilde{c}_n z^n$ defines a function in $H^q$. Since $\sum_{n=0}^{\infty} b_n \tilde{c}_n z^n = \sum_{n=0}^{\infty} b_n c_n z^n + \sum_{n=0}^{\infty} b_n \epsilon_n z^n$ and $\sum_{n=0}^{\infty} |b_n| \epsilon_n < \infty$, this implies that $\sum_{n=0}^{\infty} b_n c_n z^n$ defines a function in $H^q$. Therefore $\{b_n\}$ is a multiplier of $H^p$ into $H^q$. \[\square\]

4. Two conjectures

The following result was proved by Hardy and Littlewood in [4].

**Theorem 5.** If $p > 0$ and $0 < \alpha < 1/p$ then the sequence $\{\Gamma(n+1)/\Gamma(n+1+\alpha)\}$ is a multiplier of $H^p$ into $H^q$, where $q = p/(1-\alpha p)$. 
The asymptotic expansion of the Gamma function

\[ \Gamma(z) \approx e^{-z} z^z \sqrt{\frac{2\pi}{z}} \sum_{k=0}^{\infty} A_k \frac{1}{z^k} \quad (z \to \infty), \]

where \(|\arg z| \leq \gamma < \pi\) yields the asymptotic expansion

\[ \frac{1}{(n+1)\alpha} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)} \approx \sum_{k=0}^{\infty} B_k \frac{1}{n^k} \quad (n \to \infty). \]

Hence (b) in Theorem 4 implies that the assertion in Theorem 5 holds when \(\{\Gamma(n+1)/\Gamma(n+1+\alpha)\}\) is replaced by the simpler sequence \(\{1/(n+1)^\alpha\}\).

Since \(\Gamma(z+1) = z\Gamma(z)\), we find that

\[ \left(\frac{n+\gamma}{n}\right) = \frac{\Gamma(\gamma+n+1)}{\Gamma(\gamma+1)\Gamma(n+1)} \]

when \(\gamma\) is not a negative integer, and hence the asymptotic expansion of \(\Gamma\) yields an expansion

\[ \frac{1}{(n+1)^\gamma} \left(\frac{n+\gamma}{n}\right) \approx \sum_{k=0}^{\infty} C_k \frac{1}{n^k} \quad (n \to \infty). \]

Suppose that \(0 < \Re b < 2\) and \(\{\mu_n\}\) is the sequence discussed after Corollary 1. Then

\[ \mu_n = \frac{1}{n+1} \left(\frac{n+b-1}{n}\right) \]

and we obtain the expansion

\[ \frac{1}{(n+1)^{2-b}} \cdot \frac{1}{\mu_n} \approx \sum_{k=0}^{\infty} D_k \frac{1}{n^k} \quad (n \to \infty). \]

Therefore \(\{1/(n+1)^{2-b}\}\) is a multiplier of \(H^p\). In other words, \(\{1/(n+1)^c\}\) is a multiplier of \(H^p\) if \(0 < \Re c < 2\). It is easy to see that this result implies that \(\{1/(n+1)^c\}\) is a multiplier of \(H^p\) for all \(c\), where \(\Re c > 0\). A consequence of this is that the assertions in Theorem 4 hold more generally, where (32) is replaced by the expansion

\[ \frac{b_n}{a_n} \approx \sum_{k=0}^{\infty} \frac{A_k}{n^k} \quad (n \to \infty) \]

and \(\{c_k\}\) is any sequence of complex numbers such that \(\Re c_k > 0\) and \(\Re c_k \to \infty\) as \(k \to \infty\).

We ask whether \(\{1/(n+1)^c\}\) is a multiplier of \(H^p\) when \(\Re c = 0\). This question can be formulated as follows.

**Conjecture 1.** If \(\beta\) is a nonzero real number then \(\{(n+1)^\beta\}\) is a multiplier of \(H^p\) for \(0 < p < \infty\).
Conjecture 1 holds in the case $p = 2$ because the sequence $\{(n + 1)^{i\beta}\}$ is bounded and a function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. In Section 5 we show that the assertion in Conjecture 1 is false when $p = \infty$.

Conjecture 2. If $p > 0$ and $0 < \alpha < 1/p$ then the sequence $\{1/(n + 1)^{i\beta}\}$ is a multiplier of $H^p$ into $H^q$, where $q = p/(1-\alpha p)$ and $\alpha = \Re b$.

Conjecture 2 represents a generalization of Theorem 5 for complex values of $\alpha$. Clearly, Conjecture 1 implies Conjecture 2. Conjecture 2 holds if $0 < p \leq 2$ and $1/p - 1/2 < \alpha < 1/p$ as a consequence of Theorem 5 and the fact that Conjecture 1 holds when $p = 2$. This also is implied by results of Duren in [1], where it is determined exactly when the assumption $\mu_n = O(n^{-\alpha})$ implies that $\mu_n$ is a multiplier of $H^p$ into $H^q$ and $q = p/(1-\alpha p)$. It can be shown that the assertion of Conjecture 2 holds if $q = p/(1-\alpha p)$ is replaced by $q < p/(1-\alpha p)$.

5. A counterexample in the case $p = \infty$

Theorem 6. If $\beta$ is a nonzero real number then $\{(n + 1)^{i\beta}\}$ is not a multiplier of $H^\infty$.

Proof. Suppose that $\beta$ is a nonzero real number and $f \in H^\infty$. Let $P(z) = 1/(1 - z)^{1+i\beta}$ for $z \in \Delta$ and let $g = f* P$. We rewrite (5) as

$$g(z) = \frac{1}{\pi} \sin(\pi \beta) h(z) + \left[ f(z) + \frac{1}{\pi} \sin(\pi \beta) \ell(z) \right], \quad (41)$$

where

$$h(z) = \int_0^1 \frac{f(z) - f(tz)}{(1-t)^{1+i\beta}} dt, \quad k(z) = \int_0^1 \frac{f(z) - f(tz)}{(1-t)^{i\beta}} \left( \frac{t}{1-t} \right)^{i\beta} dt, \quad (42)$$

and $\ell = k - h$. Since $|(1-t)^{-i\beta}| = 1$ and there is a constant $A > 0$ such that $|t^{i\beta} - 1| \leq A(1-t)$ for $0 < t < 1$, we see that $|\ell(z)| \leq 2A \|f\|_{H^\infty}$ for $|z| < 1$. Hence $\ell \in H^\infty$ and (41) shows that $g \in H^\infty$ if and only if $h \in H^\infty$.

Let $f(z) = (1 - z)^{i\beta}$ for $z \in \Delta$. Then $|f(z)| < \exp(|\beta|\pi/2)$ for $|z| < 1$ and hence $f \in H^\infty$. We will show that with this choice of $f$, the function $h$ defined by (42) is unbounded and hence so is $g$. Let $\gamma = |\beta|$ and for each nonnegative integer $n$ let

$$y_n = 1 - \exp\left( -\frac{n2\pi}{\gamma} \right). \quad (43)$$

Then $0 \leq y_n < 1$, so $\{y_n\}$ is increasing, and $y_n \to 1$. Note that $f(y_n) = 1$ for all $n$. Let the sequence $\{u_n\}^\infty_{n=0}$ be defined by

$$u_n = \frac{1}{2} \log u_n = \frac{n2\pi}{\gamma}. \quad (44)$$

Then $u_n > 0$, so $\{u_n\}$ is increasing, and $u_n \to \infty$. 

The change of variable $x = \log 1/(1 - t)$ used in (42) yields
\[
h(z) = \int_0^\infty [f(z) - f((1 - e^{-x})z)]e^{i\beta x} \, dx.
\] (45)

Hence $h(y_n) = \int_0^\infty [1 - f((1 - e^{-x})y_n)]e^{i\beta x} \, dx$. This can be expressed as
\[
h(y_n) = I_n + J_n + K_n,
\] (46)
where
\[
I_n = \int_0^{u_n} [f(1 - e^{-x}) - f((1 - e^{-x})y_n)]e^{i\beta x} \, dx,
\] (47)
\[
J_n = \int_0^{u_n} [1 - f(1 - e^{-x})]e^{i\beta x} \, dx,
\] (48)
and
\[
K_n = \int_{u_n}^\infty [1 - f((1 - e^{-x})y_n)]e^{i\beta x} \, dx.
\] (49)

Suppose that $0 \leq x \leq u_n$. Then (44) yields $(e^x - 1)e^{-n2\pi/\gamma} \leq e^u - e^{-n2\pi/\gamma} = 1/\sqrt{u_n}$. For $w$ sufficiently small, $|1 - (1 + w)^{i\beta}| \leq 2\gamma |w|$. Hence, from (43) and the fact that $u_n \to \infty$ we obtain
\[
|f(1 - e^{-x}) - f((1 - e^{-x})y_n)| = |1 - [1 + e^{-n2\pi/\gamma (e^x - 1)]^{i\beta}}| \leq 2\gamma e^{-n2\pi/\gamma}
\]
for $n$ sufficiently large. Thus $|I_n| \leq \int_0^{u_n} 2\gamma e^{-n2\pi/\gamma} \, dx \leq 2\gamma e^{u_n - n2\pi/\gamma}$. Therefore
\[
|I_n| \leq \frac{2\gamma}{\sqrt{u_n}}
\]
(50)
for all large $n$. Since $|1 - f(1 - e^{-x})]e^{i\beta x} = e^{i\beta x} - 1$,
\[
J_n = \frac{e^{i\beta u_n}}{i\beta} - \frac{1}{i\beta} = u_n.
\]

Therefore
\[
|J_n| \geq u_n - \frac{2\gamma}{\sqrt{u_n}}.
\]
(51)

Note that $|f'(z)| \leq B/(1 - |z|)$ for some positive constant $B$, and hence
\[
|f(z_2) - f(z_1)| \leq \frac{B|z_2 - z_1|}{1 - r},
\]
(52)
where $r = \max(|z_1|, |z_2|)$, and $z_1, z_2 \in \Delta$. From (49), $f(y_n) = 1$, and (52) we obtain
\[
|K_n| \leq \int_{u_n}^\infty \frac{B}{1 - y_n} |y_n - (1 - e^{-x})y_n| \, dx \leq \frac{B}{1 - y_n} \int_{u_n}^\infty e^{-x} \, dx = \frac{Be^{-u_n}}{1 - y_n}.
\]
Hence (43) and (44) yield
\[ |K_n| \leq B \sqrt{u_n}. \]  

Equations (46), (50), (51), and (53) and the fact that \( u_n \to \infty \) imply that \( |h(y_n)| \to \infty \). Therefore \( h \) is unbounded. This completes the proof that \( g \) is unbounded.

Because \( g \) is unbounded, the sequence of Taylor coefficients of \( P \) is not a multiplier of \( H^\infty \). If \( P(z) = \sum_{n=0}^{\infty} p_n z^n \) then \( p_n = \binom{n+i\beta}{n} \), and hence from earlier remarks about the gamma function we find that there is an asymptotic expansion
\[ \frac{p_n}{(n+1)^{i\beta}} \approx \sum_{k=0}^{\infty} \frac{A_k}{n^k} \text{ as } n \to \infty. \]

Since \( \{p_n\} \) is not a multiplier of \( H^\infty \), Theorem 4 implies that \( \{(n+1)^{i\beta}\} \) is not a multiplier of \( H^\infty \).

A consequence of Theorems 2 and 6 is the following assertion: If \( \beta \) is a nonzero real number, then there is no complex-valued measure \( \mu \) on \( \Delta \) such that \( \int_{\Delta} \zeta^n d\mu(\zeta) = (n+1)^{i\beta} \) for \( n = 0, 1, 2, \ldots \). This is also a consequence of the fact that if \( f(z) = \int_{\Delta} (1/(1-\zeta z)) d\mu(\zeta) \) for \( z \in \Delta \) and \( \mu \) is a complex-valued measure on \( \Delta \), then the curve \( w = (1-t)f(t) \), \( 0 \leq t < 1 \), is rectifiable.

References