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Fractional telegraph equations

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Abstract

We investigate several aspects of the fractional telegraph equations, in an effort to better understand the anomalous diffusion processes observed in blood flow experiments. In the earlier work Eckstein et al. [Electron. J. Differential Equations Conf. 03 (1999) 39–50], the telegraph equation $D^2u + 2aDu + Au = 0$ was used, where $D = d/dt$, and it was shown that, as t tends to infinity, u is approximated by v, where $2aDv + Av = 0$; here $A = -d^2/dx^2$ on $L^2(\mathbb{R})$, or A can be a more general nonnegative selfadjoint operator. In this paper the concern is with the fractional telegraph equation $E^2u + 2aEu + Au = 0$, where $E = D^\gamma$ and $0 < \gamma < 1$; after solving this equation it is shown that u is approximated by v, where $2aEv + Av = 0$.

1. Introduction

Of concern are suspension flows. These combine directed and random motion and are traditionally modelled by parabolic partial differential equations.

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Sometimes they can be better modelled (in terms of fitting the data generated by certain blood flow experiments) by hyperbolic equations such as the telegraph equation, which have parabolic (or analytic) asymptotics. In particular, the experimental results described in [2,3,8] seem to be better modelled by the telegraph equation than by the heat equation. Some of the related mathematics was discussed in [2].

In Section 6 of [2], a fractional telegraph equation

$$\left(D^\gamma\right)^2 u + 2aD^\gamma u + Au = 0 \tag{1}$$

was proposed as an alternative model. Here A is a positive self-adjoint operator on a Hilbert space $\mathcal{H}$, the example we have in mind being

$$A = -\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \quad \text{on } \mathcal{H} = L^2\left(\mathbb{R}^n\right),$$

especially with $n = 1$; also a is a positive constant and D^γ is the fractional derivative with respect to time of order $\gamma \in (0,1)$. When $\gamma = 1$, this becomes the telegraph equation. For $\gamma < 1$ we call (1) the fractional telegraph equation. The motivation for this model comes from experimental considerations. The results look similar on many scales. Such self-similar behavior leads to fractional derivatives. We offer a simple heuristic discussion to motivate the use of fractional derivatives. Let $X_1, \ldots, X_n, \ldots$ be independent, identically distributed symmetric random variables with the property that for each $n = 1, 2, \ldots$ and for suitable positive constants a_n, $a_n \sum_{j=1}^n X_j$ has the same distribution as X_1, and this holds for all n. Thus the distribution of $\sum_{i=1}^n X_i$, suitably scaled, is that of X_1. This is a kind of self-similarity, independent of the sample size. Necessarily (see [7]) the distribution of X_1 has Fourier transform $e^{-c|\xi|^b}$, for $\xi \in \mathbb{R}$, for certain constants $c > 0$ and $b \in (0, 2]$. These are precisely the symmetric stable laws, including the normal distribution ($b = 2$) and the Cauchy distribution ($b = 1$). The infinitesimal generator of the corresponding Feller–Markov semigroup is given by a positive multiple of the fractional Laplacian $-(-d^2/dx^2)^{b/2}$.

Fractional differential equations have been studied extensively in the literature. Independent of the considerations in this paper, it is worth mentioning at least the work [11], in which the authors study a second order ODE of the form $D^2u + 2aD^\gamma u + F(u) = 0$, with a fractional damping term of order $0 < \gamma < 2$. F is a locally Lipshitz function. An interesting and closely related work is [5].

Here is an outline of our paper. Section 2 treats fractional derivatives, fractional ordinary differential equations, and Laplace transform methods. Section 3 deals with well-posedness for fractional telegraph equation. Section 4 treats asymptotics.

2. Reinventing the wheel

We briefly review Laplace transform, fractional calculus and fractional differential equations. While this is all "well-known," it seems worthwhile to collect the results we need in a concise format, since there are various different and inequivalent treatments of fractional differential equations in the literature.

2.1. Laplace transform

The Laplace transform of u is

$$\mathcal{L}(u)(\lambda) = U(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\,dt,$$

defined for complex λ with sufficiently large real part. When we write $\mathcal{L}(u)$ or U, we always assume it exists.

Recall that

$$\mathcal{L}(Du)(\lambda) = \lambda U(\lambda) - u(0), \tag{2}$$

where $Du = u' = du/dt$.

Next we solve the Cauchy problem for the constant coefficient ODE

$$D^2u + 2aDu + bu = h(t), \qquad u(0) = f_1, \qquad Du(0) = f_2, \tag{3}$$

by Laplace transforms. Using (2) we obtain

$$\big(\lambda^2 + 2a\lambda + b\big)U(\lambda) - \lambda f_1 - f_2 - 2af_1 = H(\lambda),$$

where $H = \mathcal{L}(h)$, thus

$$U(\lambda) = \frac{H(\lambda) + (\lambda + 2a)f_1 + f_2}{\lambda^2 + 2a\lambda + b}. \tag{4}$$

Next, we factor $\lambda^2 + 2a\lambda + b = (\lambda - \lambda_+)(\lambda - \lambda_-)$, where

$$\lambda_\pm = -a \pm \sqrt{a^2 - b}. \tag{5}$$

We assume $\lambda_+ \neq \lambda_-$ (i.e., $b \neq a^2$). Then, since

$$\mathcal{L}\big(e^{\alpha t}\big)(\lambda) = \frac{1}{\lambda - \alpha},$$

the solution u of (3) is obtained by inverting (4)

$$u(t) = e^{\lambda_+ t} g_1 + e^{\lambda_- t} g_2 + k(t),$$

where

$$g_1 = \frac{(\sqrt{a^2-b}+a)f_1+f_2}{2\sqrt{a^2-b}}, \qquad g_2 = \frac{(\sqrt{a^2-b}-a)f_1-f_2}{2\sqrt{a^2-b}},$$
$$\mathcal{L}(k)(\lambda) = \frac{H(\lambda)}{\lambda^2+2a\lambda+b},$$

so that $k \equiv 0$ when $h \equiv 0$.

Note also that for suitable choices of function spaces X, Y, $\mathcal{L}$ can be viewed as a continuous map from X to Y. The same applies to $\mathcal{L}^{-1}$, thus $\mathcal{L}$ can be thought of as a linear homeomorphism in certain specific contexts. This bicontinuity property allows one to conclude that initial value problems solved by Laplace transform depend continuously on the initial data (f_1, f_2) and the inhomogeneous term $h(t)$ in a certain precise sense as in [9,10].

2.2. Fractional derivatives and integrals

For $0 < \alpha < 1$ define

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}}\, dy.$$

Here we assume $f \in C([0,\infty))$ for simplicity. As $\alpha \to 1$,

$$D^{-\alpha} f(x) \to D^{-1} f(x) = \int_0^x f(y)\, dy,$$

and

$$DD^{-1} f(x) = f(x), \qquad D^{-1}Df(x) = f(x) - f(0)$$

(where in the latter equality we assume $f \in C^1[0,\infty)$). It is easy to check that $D^{-\alpha}D^{-\beta} = D^{-(\alpha+\beta)}$ for positive α, β with $\alpha + \beta < 1$. Define, for $0 < \gamma < 1$,

$$D^\gamma f = D^{\gamma-1} Df$$

(which is not the same as $DD^{\gamma-1} f$). To be more precise, we define $D^\gamma f$ in this way for $f \in C^1[0,\infty)$. Then D^γ will be a closable operator in the setting of certain function spaces and $D^\gamma f$ will be defined for more general f by closure. For $0 < \alpha, \gamma < 1$,

$$\mathcal{L}\big(D^{-\alpha} f\big)(\lambda) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_0^t \frac{e^{-\lambda t} f(s)}{(t-s)^{1-\alpha}}\, ds\, dt$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_s^\infty \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\, dt\, f(s)\, ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\int_0^\infty \frac{e^{-\lambda r}}{r^{1-\alpha}}\, dr \right) e^{-\lambda s} f(s)\, ds$$
$$= \lambda^{-\alpha} \mathcal{L}(f)(\lambda);$$

also

$$\mathcal{L}\big(D^\gamma f\big)(\lambda) = \mathcal{L}\big(D^{\gamma-1} f'\big)(\lambda) = \lambda^{\gamma-1} \mathcal{L}(f')(\lambda)$$
$$= \lambda^\gamma \mathcal{L}(f)(\lambda) - \lambda^{\gamma-1} f(0), \tag{6}$$

by (2).

2.3. Fractional differential equations

Let u satisfy

$$D^{\gamma_1}\big(D^{\gamma_2} u\big) + 2a D^{\gamma_3} u + bu = h(t), \tag{7}$$

where $0 < \gamma_1, \gamma_2, \gamma_3 < 1$. This equation becomes (3) when each γ_j is 1. Note that, by (2),

$$\mathcal{L}\big(D^{\gamma_1}\big(D^{\gamma_2} u\big)\big)(\lambda) = \lambda^{\gamma_1} \mathcal{L}\big(D^{\gamma_2} u\big)(\lambda) - \lambda^{\gamma_1 - 1}\big(D^{\gamma_2} u\big)(0)$$
$$= \lambda^{\gamma_1+\gamma_2} U(\lambda) - \lambda^{\gamma_1+\gamma_2-1} u(0) - \lambda^{\gamma_1-1}\big(D^{\gamma_2} u\big)(0).$$

Using this calculation, the Laplace transform of (7) becomes

$$\lambda^{\gamma_1+\gamma_2} U(\lambda) + 2a\lambda^{\gamma_3} U(\lambda) + bU(\lambda) - \lambda^{\gamma_1+\gamma_2-1} u(0) - \lambda^{\gamma_1-1}\big(D^{\gamma_2} u\big)(0)$$
$$- 2a\lambda^{\gamma_3-1} u(0) = H(\lambda).$$

Consequently

$$U(\lambda) = \frac{H(\lambda) + \lambda^{\gamma_1+\gamma_2-1} u(0) + \lambda^{\gamma_1-1}(D^{\gamma_2} u)(0) + 2a\lambda^{\gamma_3-1} u(0)}{\lambda^{\gamma_1+\gamma_2} + 2a\lambda^{\gamma_3} + b}. \tag{8}$$

For the special case when $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$, this reduces to

$$U(\lambda) = \frac{H(\lambda) + \lambda^{2\gamma-1} u(0) + \lambda^{\gamma-1}(D^\gamma u)(0) + 2a\lambda^{\gamma-1} u(0)}{\lambda^{2\gamma} + 2a\lambda^\gamma + b}. \tag{9}$$

This formula is exactly (4) for $\gamma = 1$. More generally, (8) (or (9)) gives the unique solution of (7) with the initial conditions $u(0)$ and $D^{\gamma_2} u(0)$ specified. Notice that U (and hence u) depends on the ordered pair (γ_1, γ_2) as well as γ_3. Thus Eq. (7) differs from both

$$D^{\gamma_1+\gamma_2} u + 2a D^{\gamma_3} u + bu = h$$

and

$$D^{\gamma_2}\big(D^{\gamma_1} u\big) + 2a D^{\gamma_3} u + bu = h,$$

which differ from one another (for $\gamma_1 \neq \gamma_2$). In particular, (7) requires *two* initial conditions for uniqueness, even though the order of the equation is $\max\{\gamma_1 + \gamma_2, \gamma_3\}$, which can be any number in the interval $(0, 2)$, including 1 (as well as 0.83).

Return now to the case of $\gamma_j = \gamma$ for all j. Let $f_1 = u(0)$ and $f_2 = D^\gamma u(0)$ be the initial data. The denominator in (9) factors as

$$\big(\lambda^\gamma - \mu_+\big)\big(\lambda^\gamma - \mu_-\big),$$

where $\mu_\pm = -a \pm \sqrt{a^2 - b}$, as was previously the case (see (5)). When $b = a^2$ we have a double root μ. We omit the analysis in this case and assume $b \neq a^2$. Write $U = U_0 + \widetilde{U}$, where

$$U_0(\lambda) = \frac{(\lambda^{2\gamma-1} + 2a\lambda^{\gamma-1}) f_1 + \lambda^{\gamma-1} f_2}{\lambda^{2\gamma} + 2a\lambda^\gamma + b}, \qquad \widetilde{U}(\lambda) = \frac{H(\lambda)}{\lambda^{2\gamma} + 2a\lambda^\gamma + b}.$$

Write

$$U_0(\lambda) = \frac{\lambda^{\gamma-1}(\alpha\lambda^\gamma + \beta)}{(\lambda^\gamma - \mu_+)(\lambda^\gamma - \mu_-)},$$

where $\alpha = f_1$ and $\beta = 2af_1 + f_2$. A partial fraction analysis yields

$$U_0(\lambda) = \frac{\lambda^{\gamma-1} Q_1}{\lambda^\gamma - \mu_+} + \frac{\lambda^{\gamma-1} Q_2}{\lambda^\gamma - \mu_-}, \tag{10}$$

where

$$Q_1 = \frac{\alpha\mu_+ + \beta}{\mu_+ - \mu_-} \quad \text{and} \quad Q_2 = -\frac{\alpha\mu_- + \beta}{\mu_+ - \mu_-}.$$

More explicitly,

$$U_0(\lambda) = \frac{\lambda^{\gamma-1}((\sqrt{a^2-b} + a) f_1 + f_2)}{2\sqrt{a^2-b}(\lambda^\gamma + a - \sqrt{a^2-b})} + \frac{\lambda^{\gamma-1}((\sqrt{a^2-b} - a) f_1 - f_2)}{2\sqrt{a^2-b}(\lambda^\gamma + a + \sqrt{a^2-b})}. \tag{11}$$

Thus finding $u_0 = \mathcal{L}^{-1}(U_0)$ reduces to finding v where

$$\mathcal{L}(v)(\lambda) = V(\lambda) = \frac{\lambda^{\gamma-1}}{\lambda^\gamma - \mu_0},$$

for $\operatorname{Re}\mu_0 < 0$. (We take $a > 0$, $b > 0$, but $a^2 - b$ could be either positive or negative.)

Define the Mittag–Leffler functions $E_{\alpha,\beta}$ for $\alpha, \beta > 0$ by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \text{for } z \in \mathbb{C}.$$

Then, (see [9, p. 21]) we have for $\mu_0 \in \mathbb{C}$,

$$\begin{cases} f(t) = t^{\beta-1} E_{\alpha,\beta}\big(\mu_0 t^\alpha\big), \\ F(\lambda) = \mathcal{L}(f)(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \mu_0}. \end{cases}$$

Using the above formula with $\alpha = \gamma$ and $\beta = 1$ we obtain

$$u_1(t) = Q_1 E_{\gamma,1}\big(\big(-a + \sqrt{a^2 - b}\big)t^\gamma\big) + Q_2 E_{\gamma,1}\big(\big(-a - \sqrt{a^2 - b}\big)t^\gamma\big), \tag{12}$$

where

$$Q_1 = \frac{(\sqrt{a^2 - b} + a) f_1 + f_2}{2\sqrt{a^2 - b}}, \qquad Q_2 = \frac{(\sqrt{a^2 - b} - a) f_1 - f_2}{2\sqrt{a^2 - b}}. \tag{13}$$

Note that formula (12) yields a real valued solution u_1, provided all the data are real. In this case we are adding two complex conjugate quantities.

We then obtain the solution of our Cauchy problem

$$\big(D^\gamma\big)^2 u + 2a D^\gamma u + bu = h(t), \qquad u(0) = f_1, \qquad D^\gamma u(0) = f_2 \tag{14}$$

to be $u(t) = u_1(t) + u_2(t)$, where $u_2(t) = \mathcal{L}^{-1}(U_2)$ needs to be computed separately. From a more general perspective, let C be the operator of complex conjugation. View each of the equations in (14) as $Lu = k$, where L is a linear operator. When a, b, h, f_1, f_2 are all real (or real-valued), then L commutes with C, so that $Cu = \bar{u}$ is a solution whenever u is. By uniqueness, $u = Cu$ is real whenever all the k's are real. This completes our explanation of how to solve (14) uniquely and with suitable continuous dependence on (f_1, f_2, h).

In the next subsection we specialize some of the previous calculations in the special case $\gamma = 1/2$, with the idea of emphasizing the differences from the classical case $\gamma = 1$. We will take advantage of the explicit formulas available for the half-derivative, which will also give a first glimpse into the asymptotics of solutions. The general γ will be treated in Section 4.

2.4. *Special case $\gamma = 1/2$*

Attention will be restricted here to the initial value problem

$$\big(D^{1/2}\big)^2 u + 2a D^{1/2} u + bu = 0, \qquad u(0) = f_1, \qquad D^{1/2} u(0) = f_2. \tag{15}$$

Note that this is a "first order" equation with leading term $(D^{1/2})^2 \neq D$ (see the remark below). The solution is

$$yu(t) = Q_1 E_{1/2,1}\big(\mu_+ t^{1/2}\big) + Q_2 E_{1/2,1}\big(\mu_- t^{1/2}\big),$$

where $\mu_\pm = -a \pm \sqrt{a^2 - b}$ and Q_1, Q_2 are constants given in (13). As in [9], the explicit formula for $E_{1/2,1}$ reads

$$E_{1/2,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k/2 + 1)} = e^{z^2}\big(1 + \mathrm{erf}(z)\big),$$

where $\operatorname{erf}(z)$ is the error function, defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2}\, d\tau.$$

Note that the formula above defines $\operatorname{erf}(z)$ for all $z \in \mathbb{C}$.

For $\mu \in \mathbb{C}$ and $t \geqslant 0$ define

$$\psi_\mu(t) = e^{\mu^2 t}\bigl(1 + \operatorname{erf}\bigl(\mu\sqrt{t}\,\bigr)\bigr) = E_{1/2,1}\bigl(\mu\sqrt{t}\,\bigr).$$

A straightforward calculation shows that

$$D^{1/2} e^{\mu^2 t} = \mu e^{\mu^2 t} \operatorname{erf}\bigl(\mu\sqrt{t}\,\bigr), \qquad D^{1/2} e^{\mu^2 t} \operatorname{erf}\bigl(\mu\sqrt{t}\,\bigr) = e^{\mu^2 t}. \tag{16}$$

Thus $\psi = \psi_\mu$ satisfies

$$D^{1/2}\psi(t) = \mu\psi(t), \quad \psi(0) = 1,$$

i.e., ψ_μ is eigenfunction for $D^{1/2}$ corresponding to the eigenvalue μ. This is equivalent also to the fact that the Laplace transform $\Psi := \mathcal{L}(\psi)$ is given by

$$\Psi(\lambda) = \frac{\lambda^{-1/2}}{\lambda^{1/2} - \mu},$$

which agrees with Section 2.3. Thus we can rewrite the solution of the initial value problem (15) as

$$u(t) = Q_1\psi_+ + Q_2\psi_-,$$

where $D^{1/2}\psi_\pm = \mu_\pm\psi_\pm$, $\psi_\pm(0) = 1$.

It is convenient to record, for later use, the following asymptotic expansion (as z tends to $+\infty$):

$$e^{z^2}\bigl(1 + \operatorname{erf}(-z)\bigr) = \frac{1}{\sqrt{\pi}z} - \frac{1}{2\sqrt{\pi}z^3} + O\left(\frac{1}{z^5}\right),$$

which implies, for $\mu < 0$, $t \to +\infty$,

$$\psi_\mu(t) = -\frac{1}{\sqrt{\pi}\mu t^{1/2}} + \frac{1}{2\sqrt{\pi}\mu^3 t^{3/2}} + O\left(\frac{1}{t^{5/2}}\right).$$

This expansion holds true even for complex numbers μ with $\operatorname{Re}\mu < 0$ (see Section 4).

We conclude this part with a remark that the formulas (16) imply that

$$\bigl(D^{1/2}\bigr)^2 e^{\mu^2 t} = \mu^2 e^{\mu^2 t}, \qquad \bigl(D^{1/2}\bigr)^2 e^{\mu^2 t} \operatorname{erf}\bigl(\mu\sqrt{t}\,\bigr) = \mu^2 e^{\mu^2 t} \operatorname{erf}\bigl(\mu\sqrt{t}\,\bigr).$$

Clearly $(D^{1/2})^2 \neq D$.

For comparison purposes, if one considers, instead of (15), the initial value problem for the fractional differential equation

$$Du + 2aD^{1/2}u + bu = 0,$$

then one notices that only *one* initial datum, $u(0) = f_1$, determines the solution completely, namely $u(t) = f_1(C_1\psi_+ + C_2\psi_-)$, where $C_1 = (1+\mu_+)/(\mu_+ - \mu_-)$ and $C_2 = -(1+\mu_-)/(\mu_+ - \mu_-)$.

3. The fractional telegraph equation

Let A be a positive (i.e., nonnegative and injective) self-adjoint operator on a Hilbert space $\mathcal{H}$. (More generally, we can treat equations involving several commuting normal operators, but we will stick to this simple but useful case.) By the spectral theorem (cf., e.g., [4]), there is an L^2-space $L^2(\Omega, \Sigma, \mu)$ and a unitary operator $\mathcal{U}:\mathcal{H} \to L^2(\Omega)$ such that $\mathcal{U}A\mathcal{U}^{-1}$ is the operator of multiplication by $m:\Omega \to (0,\infty)$. More precisely, there is a Σ-measurable function $m:\Omega \to \mathbb{R}$, unique (modulo changes on sets of μ-measure zero) and positive (μ a.e.) such that

$$\mathcal{U}A\mathcal{U}^{-1}f = mf, \tag{17}$$

for

$$f \in \mathcal{D}\big(\mathcal{U}A\mathcal{U}^{-1}\big) = \big\{f \in L^2(\Omega, \Sigma, \mu)\colon\ mf \in L^2(\Omega, \Sigma, \mu)\big\}.$$

We identify m with the operator M_m of multiplication by m on $L^2(\Omega, \Sigma, \mu)$. Then for any Borel function $G:(0,\infty) \to \mathbb{C}$ we can define $G(A)$ by $G(A) = \mathcal{U}^{-1}M_{G(m)}\mathcal{U}$. From the point of view of differential equations, this effectively allows us to treat A as a fixed positive real number.

We want to solve the Cauchy problem

$$\big(D^\gamma\big)^2u(t) + 2aD^\gamma u(t) + Au(t) = h(t), \tag{18}$$

$$u(0) = f_1, \qquad D^\gamma u(0) = f_2, \tag{19}$$

for a function $u:[0,\infty) \to \mathcal{H}$. More generally, consider the initial value problem in a Banach space X

$$E^2u(t) + 2aEu(t) + Au(t) = 0,$$

$$u(0) = f_1, \qquad Eu(0) = f_2. \tag{IVP, f_1, f_2}$$

Here $E:D(E) \subset Y \to Y$ is a linear operator in $Y = C([0,\infty), X)$, A is a closed densely defined operator on X. In our context $X = \mathcal{H}$, $a > 0$, $E = D^\gamma$, with $0 < \gamma \leqslant 1$. In particular, for $\gamma = 1$, $E = D = d/dt$ is the usual time-derivative.

A *strong solution* of (IVP, f_1, f_2) is a function $u \in Y$ such that Eu, E^2u are in Y, $u(t) \in D(A)$ for each $t \in (0,\infty)$ and (IVP, f_1, f_2) holds. $u \in Y$ is a *mild solution* of (IVP, f_1, f_2) if there exist strong solutions u_n of a sequence of problems (IVP, $f_{1,n}, f_{2,n}$) such that, as $n \to \infty$, $f_{1,n} \to f_1$, $f_{2,n} \to f_2$ and $u_n(t) \to u(t)$, uniformly for t in compact intervals in R_+.

Similar definitions apply to $Eu(t) + Au(t) = 0, u(0) = f$. In this case $\gamma = 1$ and strong solutions correspond to $f \in D(A)$ and mild solutions correspond to $f \in X\ (= \overline{D(A)})$.

For (IVP, f_1, f_2) with $\gamma = 1$ and $A = A^* > 0$, strong solutions correspond to $(f_1, f_2) \in (D(A), D(A^{1/2}))$, while mild solutions correspond to (f_1, f_2) in the energy space $(D(A^{1/2}), X)$.

For general $0 < \gamma < 1$ we do not specify where f_1 and f_2 are, but merely require that the expressions we construct make sense. In fact we are constructing mild solutions which turn out to be strong solutions if $(f_1, f_2) \in (D(A), D(A^{1/2}))$ as above. It seems plausible that in order to guarantee strong solutions, it is enough to have the initial data in some interpolation space.

Let $\mathcal{U}$ be as in (17). Then $u(t) = \mathcal{U}^{-1}(\tilde{u}(t,\cdot))$ where $\tilde{u}(t,\cdot) \in L^2(\Omega, \Sigma, \mu)$ and $\tilde{u}$ satisfies

$$\big(D^\gamma\big)^2 \tilde{u}(t,\omega) + 2aD^\gamma \tilde{u}(t,\omega) + m(\omega)\tilde{u}(t,\omega) = \tilde{h}(t,\omega), \tag{20}$$

$$\tilde{u}(0,\omega) = \tilde{f}_1(\omega), \qquad D^\gamma \tilde{u}(0,\omega) = \tilde{f}_2(\omega), \tag{21}$$

for all $\omega \in \Omega$. This problem is, for fixed ω, exactly the problem considered in the previous section. Taking $h \equiv 0$ and suppressing the ω variable the unique solution is

$$\begin{aligned}\tilde{u}(t) = {} & \frac{(\sqrt{a^2 - m} + a)\tilde{f}_1 + \tilde{f}_2}{2\sqrt{a^2 - m}} E_{\gamma,1}\big(\big(-a + \sqrt{a^2 - m}\big)t^\gamma\big) \\ & + \frac{(\sqrt{a^2 - m} - a)\tilde{f}_1 - \tilde{f}_2}{2\sqrt{a^2 - m}} E_{\gamma,1}\big(\big(-a - \sqrt{a^2 - m}\big)t^\gamma\big)\end{aligned} \tag{22}$$

and the corresponding unique solution of (18), (19) is

$$u(t) = \mathcal{U}^{-1}\tilde{u}(t).$$

Using the regularity theory of Laplace transforms one can give a precise sense in which this solution for (18) (with $h \equiv 0$) depends continuously on (f_1, f_2), but we omit doing so. (A useful reference in this regard is [1].)

4. Asymptotics

Let

$$u(t) = \sum_{j=1}^{2} Q_j E_{\gamma,1}\big(\big(-a + (-1)^{j+1}\sqrt{a^2 - b}\big)t^\gamma\big) \tag{23}$$

be the unique solution for

$$\big(D^\gamma\big)^2 u + 2aD^\gamma u + bu = 0, \qquad u(0) = f_1, \qquad D^\gamma u(0) = f_2. \tag{24}$$

Here Q_j are given by (13). According to [9, p. 34] we note the asymptotic behavior

$$E_{\gamma,1}(z) = -\sum_{k=1}^{N}\left(\frac{1}{\Gamma(1-k\gamma)}\right)\frac{1}{z^k} + O\left(\frac{1}{|z|^{N+1}}\right) \tag{25}$$

for all $N \in \mathbb{N}$ as $|z| \to \infty$ with $\nu \leqslant |\arg z| \leqslant \pi$, ν is an arbitrary number in $(\pi\gamma/2, \pi\gamma)$. For $b < a^2$ the coefficient of t^γ in (23) is negative, while for $b \geqslant a^2$ it has nonpositive real part. In all cases of interest for us, (25) applies.

Note that in the case γ is rational, the summation in (25) has repeated vanishing terms, namely those corresponding to k such that $1 - k\gamma$ is nonpositive integer. This is due to the fact that the Gamma function has poles at $0, -1, -2, \ldots$. For example, when $\gamma = 1/2$, the summation contains only odd k's (see Section 2.4). Also, for $\gamma = 1$, all the terms in the summation are zero.

Now take $N = 1$ in (25). Thus the solution u of (24) satisfies

$$\begin{aligned} u(t) &= \sum_{j=1}^{2} Q_j E_{\gamma,1}\big(\big(-a + (-1)^{j+1}\sqrt{a^2-b}\big)t^\gamma\big) \\ &= \frac{1}{\Gamma(1-\gamma)}\left(\frac{Q_1}{a-\sqrt{a^2-b}} + \frac{Q_2}{a+\sqrt{a^2-b}}\right)\frac{1}{t^\gamma} + O\left(\frac{1}{t^{2\gamma}}\right) \\ &= \frac{1}{\Gamma(1-\gamma)}\left(\frac{2af_1+f_2}{b}\right)\frac{1}{t^\gamma} + O\left(\frac{1}{t^{2\gamma}}\right) \end{aligned}$$

as $t \to \infty$.

Let us return to (24) with $\gamma = 1$. The unique solution is (see Section 2)

$$u(t) = g_1 e^{(-a+\sqrt{a^2-b})t} + g_2 e^{(-a-\sqrt{a^2-b})t}.$$

For $b < a^2$ and $g_1 \neq 0$,

$$u(t) = g_1 e^{(-a+\sqrt{a^2-b})t} + w,$$

where w is an error term which is (relatively) negligible for large t. Using $\sqrt{1-x} \approx 1 - x/2$ for $0 < x \ll 1$ by the Taylor series approximation, we have

$$u(t) = e^{-a(1-(1-b/a^2)^{1/2})t} g_1 + w = e^{-\frac{b}{2a}t} g_1 + \widetilde{w},$$

where $\widetilde{w}$ is similar to w. This principal term $v(t) = e^{-\frac{b}{2a}t} g_1$ solves

$$2aDv + bv = 0,$$

which can be obtained from the ODE

$$D^2v + 2aDv + bv = 0$$

by dropping the highest order term. In [2] it was shown (for a positive self-adjoint operator A) that the solution of

$$D^2u + 2aDu + Au = 0, \qquad u(0) = f_1, \qquad Du(0) = f_2,$$

asymptotically equals a solution of

$$2aDv + Av = 0,$$

which (viewing $A = -\Delta$) explains the term asymptotic analyticity.

Now return to the fractional telegraph equation with $\gamma < 1$.

$$\left(D^\gamma\right)^2 u + 2aD^\gamma u + bu = 0.$$

If we drop the term $(D^\gamma)^2 u$, we are left with the equation

$$2aD^\gamma v + bv = 0. \tag{26}$$

The Laplace transform satisfies

$$V(\lambda) = \frac{\lambda^{\gamma-1} v(0)}{2a\lambda^\gamma + b} = \frac{\lambda^{\gamma-1} Q}{\lambda^\gamma + \delta},$$

where $Q = \frac{v(0)}{2a}$ and $\delta = \frac{b}{2a}$. Consequently,

$$\begin{aligned} v(t) = QE_{\gamma,1}(-\delta t^\gamma) &= \frac{v(0)}{2a} \frac{1}{\Gamma(1-\gamma)} \frac{1}{\delta t^\gamma} + O\left(\frac{1}{t^{2\gamma}}\right) \\ &= \frac{v(0)}{b\Gamma(1-\gamma)} \frac{1}{t^\gamma} + O\left(\frac{1}{t^{2\gamma}}\right). \end{aligned}$$

In order to match the first term in the asymptotic expansions for $u(t)$ and $v(t)$, it is enough to choose, for the initial value problem (26), the initial data

$$v(0) = 2af_1 + f_2.$$

In short,

$$u(t) - v(t) = O\left(\frac{1}{t^{2\gamma}}\right).$$

Thus we can say that $v(t)$ approximates "asymptotically" $u(t)$. To be more precise, in what follows we estimate the relative error

$$\varepsilon(t) = \frac{|u(t) - v(t)|}{v(t)}.$$

From (24) with $N = 2$ we obtain

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(1-\gamma)}\left(\frac{Q_1}{\mu_+} + \frac{Q_2}{\mu_-}\right)\frac{1}{t^\gamma} - \frac{1}{\Gamma(1-2\gamma)}\left(\frac{Q_1}{\mu_+^2} + \frac{Q_2}{\mu_-^2}\right)\frac{1}{t^{2\gamma}} \\ &+ O\left(\frac{1}{t^{3\gamma}}\right), \end{aligned}$$

or,

$$u(t) = \frac{2af_1+f_2}{b}\left(\frac{1}{\Gamma(1-\gamma)t^{\gamma}} - \frac{2a}{b}\frac{1}{\Gamma(1-2\gamma)t^{2\gamma}}\right) + \frac{f_1}{b}\frac{1}{\Gamma(1-2\gamma)t^{2\gamma}} + O\left(\frac{1}{t^{3\gamma}}\right)$$

and

$$v(t) = \frac{2af_1+f_2}{b}\left(\frac{1}{\Gamma(1-\gamma)t^{\gamma}} - \frac{2a}{b}\frac{1}{\Gamma(1-2\gamma)t^{2\gamma}}\right) + O\left(\frac{1}{t^{3\gamma}}\right).$$

The relative error in approximating u by v is, in this case,

$$\varepsilon(t) = \frac{f_1}{2af_1+f_2}\frac{\Gamma(1-\gamma)}{\Gamma(1-2\gamma)}\frac{1}{t^{\gamma}} + O\left(\frac{1}{t^{2\gamma}}\right). \tag{27}$$

When $\gamma = 1/2$ one can obtain a slightly better estimate (see also Section 2.4). Using (24) with $N = 3$,

$$u(t) = -\frac{1}{\sqrt{\pi}}\left(\frac{Q_1}{\mu_+} + \frac{Q_2}{\mu_-}\right)\frac{1}{t^{1/2}} + \frac{1}{2\sqrt{\pi}}\left(\frac{Q_1}{\mu_+^3} + \frac{Q_2}{\mu_-^3}\right)\frac{1}{t^{3/2}} + O\left(\frac{1}{t^{5/2}}\right),$$

or, using the formulas (13) for Q_1 and Q_2,

$$\begin{aligned} u(t) &= \frac{2af_1+f_2}{b}\frac{1}{\sqrt{\pi}t^{1/2}} - \frac{(2af_1+f_2)(4a^2-2b)+f_2b}{b^3}\frac{1}{2\sqrt{\pi}t^{3/2}} \\ &\quad + O\left(\frac{1}{t^{5/2}}\right) \\ &= \frac{2af_1+f_2}{b}\left(\frac{1}{\sqrt{\pi}t^{1/2}} - \frac{4a^2}{b^2}\frac{1}{2\sqrt{\pi}t^{3/2}}\right) + \frac{4af_1+f_2}{b^2}\frac{1}{2\sqrt{\pi}t^{3/2}} \\ &\quad + O\left(\frac{1}{t^{5/2}}\right). \end{aligned}$$

On the other hand,

$$v(t) = \frac{2af_1+f_2}{b}\left(\frac{1}{\sqrt{\pi}t^{1/2}} - \frac{4a^2}{b^2}\frac{1}{2\sqrt{\pi}t^{3/2}}\right) + O\left(\frac{1}{t^{5/2}}\right).$$

Thus,

$$\varepsilon(t) = \frac{4af_1+f_2}{4af_1+2f_2}\frac{1}{bt} + O\left(\frac{1}{t^2}\right). \tag{28}$$

We can now state the main result, which generalizes Theorem 5.1 in [2].

Theorem. *Let $A = A^*$ be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ and let a be a positive constant. Let f_1 and f_2 be arbitrary and let $u = u(t)$ be the unique solution of the initial value problem*

$$\left(D^{\gamma}\right)^2 u + 2aD^{\gamma}u + Au = 0, \qquad u(0) = f_1, \qquad D^{\gamma}u(0) = f_2. \tag{29}$$

Then there exists $v = v(t)$, a solution of

$$2aD^{\gamma} v + Av = 0, \tag{30}$$

which has the same asymptotic behavior as u, in the sense that

$$u(t) = v(t) + o\big(v(t)\big) \quad \text{as } t \to +\infty.$$

Here the convergence is in a weak sense which is explained in the proof.

Proof. Recall from Section 3 that we can regard the term A in the fractional telegraph equation as being constant. Then $u(t) = \mathcal{U}^{-1}(\tilde{u}(t,\cdot))$ where $\tilde{u}(t,\cdot) \in L^2(\Omega, \Sigma, \mu)$ and $\tilde{u}$ satisfies

$$\big(D^{\gamma}\big)^2 \tilde{u}(t,\omega) + 2aD^{\gamma}\tilde{u}(t,\omega) + m(\omega)\tilde{u}(t,\omega) = 0, \tag{31}$$

$$\tilde{u}(0,\omega) = \tilde{f}_1(\omega), \qquad D^{\gamma}\tilde{u}(0,\omega) = \tilde{f}_2(\omega), \tag{32}$$

for all $\omega \in \Omega$. Here Ω is the transform space, which comes from the spectral theorem applied to the operator A. The convergence referred to in the last sentence of the theorem is pointwise for each ω in the transform space Ω.

Fix $\omega \in \Omega$. Using the estimates above with $b = m(\omega)$, we conclude that there exists $\tilde{v} = \tilde{v}(t,\omega)$, a solution of

$$2aD^{\gamma}\tilde{v}(t,\omega) + m(\omega)\tilde{v}(t,\omega) = 0,$$

such that

$$\tilde{u}(t,\omega) = \tilde{v}(t,\omega) + \tilde{\delta}(t,\omega),$$

where $\tilde{\delta}(t,\omega) = O(1/(t^{2\gamma}))$ as $t \to \infty$. In the case $\gamma = 1/2$ the approximation is even better, in the sense that $\tilde{\delta} = O(1/(t^{3/2}))$.

Note that the rate of decay of $\tilde{u}(t,\omega) - \tilde{v}(t,\omega)$, as $t \to \infty$, is estimated for each ω and it may, in principle, depend on $m(\omega)$ (see (28)). Therefore, when returning to the x-space by $\mathcal{U}^{-1}$, one cannot guarantee a uniform decay rate. In the case when A has a bounded inverse A^{-1} (which does not hold for the Laplacian appearing in the Kac model [6]), this issue is resolved by the fact that $m(\omega) \geqslant m_0 > 0$ for all ω. For the general case of a positive self-adjoint operator A this asymptotic analyticity should be understood in the sense given in the proof of the theorem, that is pointwise convergence in the Fourier space. The asymptotic behavior and the limited ability to perform inverse operations may be crucial to modeling particle motions in suspension flows, which appear to involve abrupt, erratic changes in physical space, but would be expected to be well described by the states of an energy space.

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