On the relationship between generalised continued fractions and G-continued fractions

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Abstract: In this paper the connection between generalised continued fractions (de Bruin (1974)) and G-continued fractions (Levrie (1988)) is studied. This connection is used to prove a convergence theorem for generalised continued fractions and to accelerate the convergence of generalised continued fractions associated with a class of linear recurrence relations of Poincaré-type.

Keywords: Recurrence relation, generalised continued fraction, convergence acceleration.

1. Introduction

In this paper we discuss the relationship between two different generalisations of continued fractions, both associated with a linear recurrence relation of arbitrary order.

The first type of generalisation is the so-called n-fraction or generalised continued fraction (GCF). GCFs are used in the study of simultaneous rational approximation of functions using rational functions with a common denominator [1]. They can also be used for the computation of certain nondominant solutions of linear recurrence relations (see [3,7,8]).

The second type of generalisation of a continued fraction is the G-continued fraction introduced in [6,10]. G-continued fractions are used for the calculation of minimal solutions of linear recurrence relations (i.e., solutions for which the Miller algorithm converges [6]). The method is related to Gautschi’s continued fraction algorithm for second-order recurrence relations [2].

In 1976 it was shown by Zahar [11] that there exists a close connection between the convergence of the G-continued fraction associated with a linear recurrence relation and the convergence of the GCF associated with its adjoint equation. In this paper we take a closer look at this result and we will show that the value of a GCF may be calculated from the value of the G-continued fraction associated with the adjoint recurrence relation. Furthermore, it is shown that under some mild conditions the convergence of the G-continued fraction implies the convergence of the GCF. These results can, for instance, be used to derive new convergence
theorems for GCFs from known convergence theorems for G-continued fractions.

In the rest of this section we will give the necessary definitions and notations concerning GCFs and G-continued fractions. In Section 2 we prove that the approximants of a GCF associated with a linear recurrence relation may be calculated from the approximants of the G-continued fraction associated with the adjoint equation. We also prove a convergence result for GCFs. In Section 3 we look at some consequences of this relationship for GCFs. First of all we prove a convergence theorem for GCFs related to Pringsheim’s theorem for ordinary continued fractions. Then we look at how convergence acceleration for G-continued fractions can be used to accelerate the convergence of n-fractions.

We consider a \( p \)th order recurrence relation of the form

\[
\begin{align*}
&c_0(n)y_{n+p} + c_1(n)y_{n+p-1} + \cdots + c_{p-1}(n)y_{n+1} + c_p(n)y_n = 0, \quad n = 0, 1, \ldots, \\
&c_i(n) \in \mathbb{C}, \quad c_0(n)c_p(n) \neq 0 \text{ for all } i \text{ and } n.
\end{align*}
\]

with \( c_i(n) \in \mathbb{C} \), \( c_0(n)c_p(n) \neq 0 \) for all \( i \) and \( n \). If \( f_n^{(1)}, f_n^{(2)}, \ldots, f_n^{(p-1)} \) are solutions of this recurrence relation, we shall use the notation \( E_n(f^{(1)}, \ldots, f^{(p-1)}) \) for the determinant of the matrix

\[
\begin{pmatrix}
  f_n^{(1)} & f_n^{(2)} & \cdots & f_n^{(p-1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{n+p-2}^{(1)} & f_{n+p-2}^{(2)} & \cdots & f_{n+p-2}^{(p-1)}
\end{pmatrix}
\]

The \((p - 1)\)-fraction or generalised continued fraction (see [1]) associated with (1), denoted by

\[
\kappa^\infty_{n=0} \begin{pmatrix} -c_p(n)/c_0(n) \\ \vdots \\ -c_1(n)/c_0(n) \end{pmatrix},
\]

is given by the sequence of approximants \( \{ C_k^{(1)}/C_k^{(p)}, \ldots, C_k^{(p-1)}/C_k^{(p)} \}^\infty_{k=p-1} \) where the numerators and denominators satisfy the recurrence relation (1) with initial values

\[
C_n^{(i)} = \delta_{n,i-1}, \quad i = 1, \ldots, p, \quad n = 0, 1, \ldots, p - 1.
\]

The GCF is said to converge in \( \mathbb{C}^{p-1} \) if the following limit exists in \( \mathbb{C}^{p-1} \):

\[
\{ b_{p-1}(0), \ldots, b_1(0) \} = \lim_{k \to \infty} \{ C_k^{(1)}/C_k^{(p)}, \ldots, C_k^{(p-1)}/C_k^{(p)} \}.
\]

The \( N \)th approximant of the GCF, \( \{ C_{N+p}^{(1)}/C_{N+p}^{(p)}, \ldots, C_{N+p}^{(p-1)}/C_{N+p}^{(p)} \} \), may be calculated using the following algorithm: let

\[
\begin{align*}
b_1^N(N + 1) &= \cdots = b_{p-1}^N(N + 1) = 0, \\
b_h^N(k) &= \frac{c_{h+1}(k) - c_0(k)b_h^N(k + 1)}{c_1(k) - c_0(k)b_1^N(k + 1)}, \quad h = 1, \ldots, p - 2, \\
b_{p-1}^N(k) &= \frac{c_p(k)}{c_1(k) - c_0(k)b_1^N(k + 1)},
\end{align*}
\]

for \( k = N, \ N - 1, \ldots, 0 \). Then we have

\[
b_h^N(0) = \frac{C_{N+p}^{(p-h)}}{C_{N+p}^{(p)}}, \quad h = 1, \ldots, p - 1.
\]
The $G$-continued fraction associated with the recurrence relation (1) is given by the sequence of approximants \(-{T_k^{(2)}/T_k^{(1)}}\) where $T_n^{(h)}$, $h = 1, \ldots, p$, is defined by

\[ T_n^{(h)} = E_n(C^{(1)}, \ldots, C^{(h-1)}, C^{(h+1)}, \ldots, C^{(p)}) \]

(see [6]). The $G$-continued fraction converges if the following limit exists:

\[ B(0) = -\lim_{k \to \infty} \frac{T_k^{(2)}}{T_k^{(1)}}. \]  \hfill (6)

The $N$th approximant $B_N(0) = -T_N^{(2)}/T_N^{(1)}$ may be calculated using the algorithm

\[ B_N(N+i) = 0, \quad i = 1, \ldots, p-1, \]

\[ B_N(k) = -c_p(k) \sum_{i=1}^{p} c_{p-i}(k) \left( \prod_{h=1}^{k-N} B_N(k+h) \right), \quad k = N, \ldots, 0. \]  \hfill (7)

The $G$-continued fraction above is denoted by

\[ \sum_{n=0}^{\infty} \left( \begin{array}{c} -c_p(n)/c_0(n) \\ c_{p-1}(n)/c_0(n); \ldots; c_1(n)/c_0(n) \end{array} \right). \]  \hfill (8)

The $j$th tail of the $G$-continued fraction is then defined to be

\[ t^{(j)} = \sum_{n=j}^{\infty} \left( \begin{array}{c} -c_p(n)/c_0(n) \\ c_{p-1}(n)/c_0(n); \ldots; c_1(n)/c_0(n) \end{array} \right). \]

If the $j$th tail converges, then its value is denoted by $B(j)$.

2. Connection between the two types of generalised continued fractions

Let us consider the linear recurrence relation

\[ a_p(n+1) y_{n+p} + a_{p-1}(n) y_{n+p-1} + \cdots + a_1(n-p+2) y_{n+1} + y_n = 0, \quad n = 0, 1, \ldots, \]

with $a_i(n) \in \mathbb{C}$ for all $i$ and $n$, and $a_p(n) \neq 0$ for all $n$, and let $A_n^{(i)}$, $i = 1, \ldots, p$, be the solution of (9) with initial values given by

\[ A_n^{(i)} = \delta_{n,i-1}, \quad n = 0, 1, \ldots, p-1. \]  \hfill (10)

The adjoint equation of (9) is given by

\[ y_{n+p} + a_1(n) y_{n+p-1} + \cdots + a_{p-1}(n) y_{n+1} + a_p(n) y_n = 0, \quad n = 0, 1, \ldots. \]  \hfill (11)

Let $A_n^{(i)}$ be the corresponding solutions of (11). It is possible to construct a fundamental system of solutions for (11) from the $A_n^{(i)}$ [9]: let $U_n^{(h)}$ ($h = 1, \ldots, p$) be defined by

\[ U_n^{(h)} = E_n(A^{(1)}, \ldots, A^{(h-1)}, A^{(h+1)}, \ldots, A^{(p)}); \]

then $y_n^{(h)}$, $h = 1, \ldots, p$, defined by

\[ y_n^{(h)} = (-1)^p U_n^{(h)} \prod_{j=0}^{n-1} a_p(j), \]  \hfill (12)
form a fundamental system of solutions for the recurrence relation (11). From the initial values (10) we deduce that

\[ U_n^{(h)} = 0, \quad n = 0, \ldots, h - 1, \]
\[ U_n^{(h)} = (-1)^{\nu-1(h^{(h-1)} \prod_{j=1}^{h-1} a_p(j)^{h-1}}, \quad h = 1, \ldots, p - 1, \]
\[ U_0^{(\nu)} = 1, \quad U_n^{(\nu)} = 0, \quad n = 1, \ldots, p - 1. \]  

Furthermore, if for \( h = 1, \ldots, p - 1 \) we define \( y_n^{(h)} = 0, \quad n = h - p + 1, \ldots, -1 \), then, using (12), (13), we obtain \( p \) initial values for the \( y_n^{(h)} \):

\[ y_n^{(h)} = 0, \quad n = 0, \ldots, h - 1, \]
\[ y_n^{(h)} = (-1)^{p+h-1} a_p(0), \quad h = 1, \ldots, p - 1, \]
\[ y_0^{(p)} = 1, \quad y_n^{(p)} = 0, \quad n = 1, \ldots, p - 1, \]

and we can calculate them for \( n = h + 1, \ldots \) from

\[ y_{n+p} + a_{n+p}(n) y_{n+p-1} + \cdots + a_{p-1}(n) y_{n+1} + a_p(n) y_n = 0, \quad n = h - p + 1, \ldots. \]

Using this it is not difficult to prove that we have

\[ A_n^{(i)} = y_n^{(p)}, \]
\[ A_n^{(i)} = (-1)^{p+i} a_p(0) \left( y_n^{(i-1)} + \sum_{j=i}^{p-1} (-1)^{j-i+1} a_{j-i+1} (j - p) y_n^{(j)} \right), \quad i = 2, \ldots, p, \]

for all \( n \geq 0 \).

The \( N \)th approximant of the GCF associated with (11) is given by

\[ \left\{ b_{N+p-1}^N(0), \ldots, b_1^N(0) \right\} = \left\{ A_{N+p}^{(1)}, \ldots, A_{N+p}^{(p-1)}, A_{N+p}^{(p)} \right\}. \]

For the approximants of the G-continued fraction associated with (9) we have from [3]:

\[ \prod_{h=0}^{j} B^N_{N+p-2}(h) = (-1)^{j+1} \frac{y_{N+p}^{(j+2)}}{y_{N+p}^{(1)}}, \quad j = 0, \ldots, p - 2. \]

Hence, using (14) we find for \( h = 2, \ldots, p - 1 \):

\[ b_{N+h}^N(0) = \frac{A_{N+p}^{(h)}}{A_{N+p}^{(p)}} = \frac{(-1)^{p+h} \left( y_{N+p}^{(h-1)} + \sum_{j=h}^{p-1} (-1)^{j-h+1} a_{j-h+1}(j - p) y_{N+p}^{(j)} \right)}{(-1)^{p+p} y_{N+p}^{(p)}} \]
\[ \left( (-1)^{h} y_{N+p}^{(h-1)}/y_{N+p}^{(1)} + (-1)^{h} \sum_{j=h}^{p-1} (-1)^{j-h+1} a_{j-h+1}(j - p) y_{N+p}^{(j)}/y_{N+p}^{(1)} \right) \]
\[ = \frac{(-1)^{p} y_{N+p}^{(p-1)}/y_{N+p}^{(1)}}{(-1)^{p} y_{N+p}^{(p-1)}/y_{N+p}^{(1)}} \]
\[
(-1)^{h-2} \frac{y_{n+p}^{(h-1)}}{y_{n+p}} + \sum_{j=h}^{p-1} (-1)^{j-1} a_{j-h+1}(j-p) \frac{y_{n+p}^{(j)}}{y_{n+p}} = \frac{(-1)^{p-2} y_{n+p}^{(p-1)}}{y_{n+p}} \]
\[
\prod_{j=0}^{h-3} B^{N+p-2}(j) + \sum_{j=h}^{p-1} a_{j-h+1}(j-p) \prod_{k=0}^{j-2} B^{N+p-2}(k) \prod_{j=0}^{p-3} B^{N+p-2}(j)
\]
and
\[
b_{p-1}^{N}(0) = \frac{A_{N+p}^{(1)}}{A_{N+p}^{(p)}} = (-1)^{2p} a_{p}(0) \frac{y_{n+p}^{(p)}}{y_{n+p}^{(p-1)}} = -a_{p}(0) B^{N+p-2}(p-2).
\]

Hence we have proved the following theorem.

**Theorem 1.** The approximants of the GCF associated with (11) and the G-continued fraction associated with (9) are related by

\[
b_{p-h}(0) = \frac{\prod_{j=0}^{h-3} B^{N+p-2}(j) + \sum_{j=h}^{p-1} a_{j-h+1}(j-p) \prod_{k=0}^{j-2} B^{N+p-2}(k) \prod_{j=0}^{p-3} B^{N+p-2}(j)}{\prod_{j=0}^{p-3} B^{N+p-2}(j)}, \quad h = 2, \ldots, p-1,
\]

\[
b_{p-1}^{N}(0) = -a_{p}(0) B^{N+p-2}(p-2).
\]

We have the following convergence theorem.

**Theorem 2.** If the G-continued fraction associated with (9) and its tails \(t^{(1)}, \ldots, t^{(p-2)}\) converge, with \(B(0)B(1) \cdots B(p-3) \neq 0\), then the \((p-1)\)-fraction associated with (11) converges.

**Proof.** This theorem is a consequence of two other theorems, one of them due to Van der Cruyssen [8], the other one given in [6]. In [8] Van der Cruyssen proves the following result. The GCF associated with (11) converges if the equation

\[
a_{p}(n+p-1) y_{n+p} + a_{p-1}(n+p-2) y_{n+p-1} + \cdots + a_{1}(n) y_{n+1} + y_{n} = 0,
\]

\[
n = 0, 1, \ldots,
\]
has a fundamental system of solutions \(f_{n}^{(1)}, \ldots, f_{n}^{(p-1)}, g_{n}\) with \(g_{0} \neq 0\) and for which

\[
\lim_{n \to \infty} \frac{E_{n}(f^{(1)}, \ldots, f^{(h-1)}, g, f^{(h+1)}, \ldots, f^{(p-1)})}{E_{n}(f^{(1)}, \ldots, f^{(p-1)})} = 0
\]

for \(h = 1, \ldots, p-1\).

In [6] we stated the following result. If the G-continued fraction associated with (9) and its tails \(t^{(1)}, \ldots, t^{(p-2)}\) converge, then the recurrence relation (9) has a fundamental system of
solutions \( f_{n}^{(1)}, \ldots, f_{n}^{(p-1)}, g_{n} \) for which
\[
\lim_{n \to \infty} \frac{E_{n}(f_{n}^{(1)}, \ldots, f_{n}^{(h-1)}, g', f_{n}^{(h+1)}, \ldots, f_{n}^{(p-1)})}{E_{n}(f_{n}^{(1)}, \ldots, f_{n}^{(p-1)})} = 0
\]
and with \( g_{0} \neq 0 \). Hence if we choose
\[
f_{n}^{(1)} = f_{n+p-2}^{(1)}, \ldots, f_{n}^{(p-1)} = f_{n+p-2}^{(p-1)}, \quad g_{n} = g_{n+p-2}
\]
for all \( n \geq 0 \), then (16) is satisfied. Furthermore, from [6] we have that \( g_{0} = g_{p-2}^{'} = B(p-3) \cdot B(p-2) \cdot \ldots \cdot B(0) \cdot g_{0}^{'} \neq 0 \).

This proves the theorem. \( \square \)

**Remark.** We note that the condition \( B(0)B(1) \cdot \ldots \cdot B(p-3) \neq 0 \) is not a very strong one since there is a certain freedom of choice for the \( a_{i}(k) \) with \( k < 0 \).

3. Applications

3.1. A Pringsheim-like convergence theorem

From Theorem 2 and [4] we obtain the following variant of Pringsheim’s theorem.

**Theorem 3.** If the coefficients of the recurrence relation (11) satisfy the inequalities
\[
|a_{1}(n)| \geq 1 + |a_{2}(n+1)| + |a_{3}(n+2)| + \ldots + |a_{p}(n+p-1)|
\]
for all \( n \geq -p+2 \), where the \( a_{i}(k) \) with \( k < 0 \) may be chosen arbitrarily (but satisfying (17)), then the GCF associated with (11) converges.

**Proof.** In [4] we have proved that if the coefficients of the recurrence relation (9) satisfy the inequalities (17), then the \( G \)-continued fraction associated with (9) and all its tails converge. Furthermore, we have that \( |B^{N}(k)| < 1 \) for all \( n \) and for all \( k \leq N+p-1 \). Since the \( B^{N}(k) \) satisfy the nonlinear recurrence relation
\[
B^{N}(k) = -\frac{1}{\sum_{i=1}^{p} a_{i}(k-p+i+1) \left( \prod_{h=1}^{i-1} B^{N}(k+h) \right)}, \quad k = N, \ldots, 0,
\]

it is easy to see that
\[
|B^{N}(k)| \geq \frac{1}{\sum_{i=1}^{p} |a_{i}(k-p+i+1)|}
\]
for all \( N \) and hence
\[
|B(k)| \geq \frac{1}{\sum_{i=1}^{p} |a_{i}(k-p+i+1)|} > 0.
\]

The result then follows from Theorem 2. \( \square \)
3.2. Convergence acceleration for n-fractions

It is immediately clear that we can use convergence acceleration methods for G-continued fractions to calculate the value of GCFs. In [6] we have discussed a convergence acceleration method for G-continued fractions for which the associated recurrence relation is of Poincaré-type. Let us assume that the coefficients of (11) satisfy \( \lim_{k \to \infty} a_i(k) = a_i \) for all \( i \) with \( a_p \neq 0 \). Furthermore, let us assume that the roots \( w_1, w_2, \ldots, w_p \) of the characteristic polynomial associated with (11)

\[
x^p + a_1 x^{p-1} + a_2 x^{p-2} + \cdots + a_{p-1} x + a_p
\]

are all different in modulus:

\[
|w_1| > |w_2| > \cdots > |w_p|.
\]

In this case the recurrence relation (9) is also of Poincaré-type and its characteristic polynomial has the roots \( 1/w_1, 1/w_2, \ldots, 1/w_p \). It was shown in [6] that the following algorithm

\[
\tilde{B}^N(N+i) = 1/w_i, \quad i = 1, \ldots, p - 1,
\]

\[
\tilde{B}^N(k) = \frac{1}{\sum_{i=1}^{p} a_i(k-p+i+1) \left( \prod_{h=1}^{i-1} \tilde{B}^N(k+h) \right)}, \quad k = N, \ldots, 0,
\]

(19)

calculates approximations to \( B(0) \): \( \lim_{N \to \infty} \tilde{B}^N(0) = B(0) \). Furthermore, it converges faster than the algorithm (7) in the sense that

\[
\lim_{N \to \infty} \frac{B(0) - \tilde{B}^N(0)}{B(0) - B^N(0)} = 0.
\]

Let us consider an example: let the coefficients of the third-order recurrence relation (11) be given by:

\[
\begin{align*}
a_1(k) &= -6.0 + 0.8^{k+1}, \\
a_2(k) &= 11.75 + 0.8^{k+1}, \\
a_3(k) &= -7.5.
\end{align*}
\]

(20)

Its characteristic polynomial is given by

\[
x^3 - 6x^2 + 11.75 x - 7.5,
\]

and it has the zeros \( w_1 = 2.5, w_2 = 2.0, w_3 = 1.5 \). The two-fraction associated with this recurrence relation converges to the value \( \{ b_1(0) = -4.1856365261145, \ b_2(0) = 2.8278080774441 \} \). In this case we get for the expressions in Theorem 1:

\[
b_1^{N}(0) = \frac{1 + a_1(-1) B^{N+1}(0)}{B^{N+1}(0)} = \frac{1 - 5B^{N+1}(0)}{B^{N+1}(0)},
\]

\[
b_2^{N}(0) = -a_2(0) B^{N+1}(1) = 7.5 \ B^{N+1}(1).
\]

If instead of the \( B \)'s we use the \( \tilde{B} \)'s, we obtain approximations \( \tilde{b}_i^{N}(0) \) to \( b_i(0) \), \( i = 1, 2, \)

\[
\tilde{b}_1^{N}(0) = \frac{1 - 5\tilde{B}^{N+1}(0)}{\tilde{B}^{N+1}(0)}, \quad \tilde{b}_2^{N}(0) = 7.5 \ \tilde{B}^{N+1}(1),
\]

\[
\]
for which
\[ \lim_{N \to \infty} \frac{b_1(0) - \tilde{b}_1^N(0)}{b_1(0) - b_1^N(0)} = 0, \quad \lim_{N \to \infty} \frac{b_2(0) - \tilde{b}_2^N(0)}{b_2(0) - b_2^N(0)} = 0. \]

We shall briefly look at yet another convergence acceleration method for G-continued fractions. Let us assume that the third-order recurrence relation
\[ y_{n+3} + c_1(n)y_{n+2} + c_2(n)y_{n+1} + c_3(n)y_n = 0, \quad n = 0, 1, \ldots, \quad (21) \]
is of Poincaré-type and that the roots \( v_1, v_2, v_3 \) of its characteristic equation satisfy \(|v_1| > |v_2| > |v_3|\). Furthermore, let us assume that for the coefficients of the recurrence relation we have:
\[ \lim_{k \to \infty} \frac{v_3^3 + c_1(k)v_3^2 + c_2(k)v_3 + c_3(k)}{v_3^2 + c_2(k)v_3 + c_3(k)} = t. \]
Then the tails of the G-continued fraction associated with (21) satisfy:
\[ B(k) - v_3 \sim \epsilon_k = \frac{v_3^3 + c_1(k)v_3^2 + c_2(k)v_3 + c_3(k)}{c_2(k) + v_3(c_1(k) + v_3) + v_3t(c_1(k) + v_3 + v_3t)}, \quad k \to \infty, \]
(see [5]). This means that \( v_3 + \epsilon_k \) is a better approximation for the tail \( B(k) \) than \( v_3 \). (Note that since (21) corresponds to (9) we have for the coefficients
\[ c_1(n) = \frac{a_2(n)}{a_3(n+1)}, \quad c_2(n) = \frac{a_4(n-1)}{a_3(n+1)}, \quad c_3(n) = \frac{1}{a_3(n+1)}, \]
and also \( v_3 = 1/w_1 \).) So if instead of (19) we calculate
\[ \tilde{B}^N(N+i) = v_3 + \epsilon_{N+i} \]
\[ = -\frac{c_3(N+i) - v_3^2t(c_1(N+i) + v_3 + v_3t)}{c_2(N+i) + v_3(c_1(N+i) + v_3) + v_3t(c_1(N+i) + v_3 + v_3t)}, \quad i = 1, 2, \]
\[ \tilde{B}^N(k) = -\frac{c_3(k)}{c_2(k) + \tilde{B}^N(k+1)(c_1(k) + \tilde{B}^N(k+2))}, \quad k = N, \ldots, 0, \]

### Table 1a

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### Table 1b

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(with $v_3 = 0.4$ and $t = 0.8$) and

$$\hat{b}_1^N(0) = \frac{1 - 5\hat{b}_{N+1}^N(0)}{\hat{b}_{N+1}^N(0)}, \quad \hat{b}_2^N(0) = 7.5 \hat{b}_{N+1}^N(1),$$

we obtain approximations to the value of the GCF associated with (20). In Table 1 we have used these three methods to calculate the value of the GCF.

References