Computers and Mathematics with Applications 59 (2010) 612–621

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

He's homotopy perturbation method for continuous population models for single and interacting species

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ARTICLE INFO

Article history:
Received 4 June 2009
Received in revised form 28 October 2009
Accepted 28 October 2009

Keywords:
Analytical solution
Nonlinear differential equations
Logistic growth
Predator–prey models
He's homotopy perturbation method

ABSTRACT

He's homotopy perturbation method is applied for obtaining approximate analytical solutions of continuous population models for single and interacting species. In comparison with existing techniques, this method is very straightforward, and the solution procedure is very simple. Also, it is highly effective in terms of accuracy and rapid convergence. Analytical and numerical studies are presented.

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1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and science. Therefore, over the last ten years or so many mathematical methods that are aimed at obtaining approximate analytical solutions of non-linear differential equations arising in various fields of science and engineering have appeared in the research literature [1–7]. However, most of them require a tedious analysis or a large computer memory to handle these problems.

The main aim of this paper is to present applications of He's homotopy perturbation method (HPM) to four non-linear biological problems. The first problem is a logistic growth model in a population whereas the second one is a prey–predator model: Lotka–Volterra system. The third problem is a simple 2-species Lotka–Volterra competition model, and the fourth one is a prey–predator model with limit cycle periodic behavior.

HPM was proposed by Ji-Huan He [3] in 1999. According to this method the solution is obtained as the summation of an infinite series, which converges to exact solution. Using the homotopy technique from topology, a homotopy is constructed with an imbedding parameter $p \in [0, 1]$, which is considered as a "small parameter". The approximations obtained by the HPM are uniformly valid not only for small parameters, but also for very large parameters.

First, we consider the *logistic growth* in a population as a single species model to be governed by [8]

$$\frac{dN}{dt} = rN(1 - N/K), \tag{1}$$

where r and K are positive constants. Here $N = N(t)$ represents the population of the species at time t, and $r(1 - N/K)$ is the per capita growth rate, and K is the carrying capacity of the environment. We non-dimensionalize Eq. (1) by setting

$$u(\tau) = \frac{N(t)}{K}, \quad \tau = rt,$$

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doi:10.1016/j.camwa.2009.10.031

and it becomes

$$\frac{du}{d\tau} = u(1-u). \tag{2}$$

If $N(0) = N_0$, then $u(0) = N_0/K$. Therefore, the analytical solution of Eq. (2) is easily obtained

$$u(\tau) = \frac{1}{1 + (K/N_0 - 1)e^{-\tau}}.$$

For numerical purposes we take $N_0 = 2$ and $K = 1$, therefore the last equation reads

$$u(\tau) = \frac{2}{2 - e^{-\tau}}. \tag{3}$$

Second, we consider the Predator–Prey Models: Lotka–Volterra systems as an interacting species model to be governed by [8]

$$\begin{aligned} \frac{dN}{dt} &= N(a - bP), \\ \frac{dP}{dt} &= P(cN - d), \end{aligned} \tag{4}$$

where a, b, c and d are constants. Here $N = N(t)$ is the prey population and $P = P(t)$ that of the predator at time t. We non-dimensionalize the system (4) [8] by setting

$$u(\tau) = \frac{cN(t)}{d}, \qquad v(\tau) = \frac{bP(t)}{a}, \qquad \tau = at, \quad \alpha = d/a,$$

and it becomes

$$\begin{aligned} \frac{du}{d\tau} &= u(1 - v), \\ \frac{dv}{d\tau} &= \alpha v(u - 1), \end{aligned} \tag{5}$$

where we take $\alpha = 1$ for our computations below.

Third, we consider the simple 2-species Lotka–Volterra competition model with each species N_1 and N_2 having logistic growth in the absence of the other. Inclusion of logistic growth in the Lotka–Volterra systems makes them much more realistic but to highlight the principle we consider the simpler model which nevertheless reflects many of the properties of more complicated models, particularly as regards stability. We thus consider the system [8]

$$\begin{aligned} \frac{dN_1}{dt} &= r_1 N_1 \left[1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right], \\ \frac{dN_2}{dt} &= r_2 N_2 \left[1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}\right], \end{aligned} \tag{6}$$

where $r_1, K_1, r_2, K_2, b_{12}$ and b_{21} are all positive constants and the r's are the linear birth rates and the K's are the carrying capacities. The b_{12} and b_{21} measure the competitive effect of N_2 on N_1 and N_1 on N_2 respectively: they are generally not equal. If we non-dimensionalize this model by writing [8]

$$u = \frac{N_1}{K_1}, \qquad v = \frac{N_2}{K_2}, \qquad \tau = r_1 t, \qquad \rho = \frac{r_2}{r_1}, \qquad a = b_{12}\frac{K_2}{K_1}, \qquad b = b_{21}\frac{K_1}{K_2},$$

the system given by Eq. (6) becomes

$$\begin{aligned} \frac{du}{d\tau} &= u(1 - u - av), \\ \frac{dv}{d\tau} &= \rho v(1 - v - bu), \end{aligned} \tag{7}$$

where we take $a = 1$, $b = 0.8$, $\rho = 1$ for our computations.

Last, we consider a prey–predator model with limit cycle periodic behavior [8]:

$$\begin{aligned} \frac{dN}{dt} &= N\left[r\left(1 - \frac{N}{K}\right) - \frac{kP}{N + D}\right], \\ \frac{dP}{dt} &= P\left[s\left(1 - \frac{hP}{N}\right)\right], \end{aligned} \tag{8}$$

where r, K, k, D, s and h are all positive constants. We non-dimensionalize this model by writing [8]

$$u = \frac{N}{K}, \qquad v = \frac{hP}{K}, \qquad \tau = rt, \qquad a = \frac{k}{hr}, \qquad b = \frac{s}{r}, \qquad d = \frac{D}{K},$$

and the system in Eq. (8) becomes

$$\begin{aligned} \frac{du}{d\tau} &= u(1-u) - \frac{auv}{u+d}, \\ \frac{dv}{d\tau} &= bv\left(1 - \frac{v}{u}\right), \end{aligned} \tag{9}$$

where we take $a = 1$, $d = 10$, $b = 5$ for our computations.

2. The idea of HPM

The basic idea of the HPM can be illustrated as follows [3,4]: we consider the nonlinear differential equation

$$A(u) - f(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega, \tag{10}$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad \mathbf{r} \in \Gamma, \tag{11}$$

where A is a general differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytic function, Γ is the boundary of the domain Ω.

In general, one divides the operator A into two parts L and N, where L is linear, while N is nonlinear. Therefore, Eq. (10) is written as follows

$$L(u) + N(u) - f(\mathbf{r}) = 0. \tag{12}$$

By the homotopy technique [3], one constructs a homotopy $v(\mathbf{r}, p) : \Omega \times [0, 1] \to \mathbf{R}$ which satisfies

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(\mathbf{r})] = 0, \quad p \in [0, 1], \mathbf{r} \in \Omega, \tag{13}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(\mathbf{r})] = 0, \tag{14}$$

where $p \in [0, 1]$ is an imbedding parameter, u_0 is an initial approximation of Eq. (10), which satisfies the boundary conditions. It is clear that

$$H(v, 0) = L(v) - L(u_0) = 0,$$

and

$$H(v, 1) = A(v) - f(\mathbf{r}) = 0,$$

the changing process of p from zero to unity is just that of $v(\mathbf{r}, p)$ from $u_0(\mathbf{r})$ to $u(\mathbf{r})$.

According to the HPM, we can first use the imbedding parameter p as a "small parameter", and assume that the solution of Eqs. (13) and (14) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{15}$$

Setting $p = 1$ results in the approximate solution of Eq. (10):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{16}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated limitations of the traditional perturbation methods.

The series in Eq. (16) is convergent for most cases, however, the convergent rate depends on the nonlinear operator $A(v)$ (the following opinions are suggested by Ji-Huan He [3,4]):

(1) The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e., $p \to 1$.

(2) The norm of $L^{-1}\partial N/\partial v$ must be smaller than one so that the series converges.

3. Applications of HPM

Example 1. We solve Eq. (2) using HPM with the initial condition $u(0) = 2$ [9]. We rewrite Eq. (2) in the form

$$\frac{du}{d\tau} = pu(1-u),$$
$$u(0) = 2, \tag{17}$$

where $p \in [0, 1]$ is an imbedding parameter. As in He's HPM, it is clear that when $p = 0$, the initial value problem in Eq. (17) becomes linear; when $p = 1$, it becomes the original nonlinear one. We consider the imbedding parameter p as a "small parameter". We assume the solution of the problem given by Eq. (17) is expressed as a power series given in Eq. (15). Substituting Eq. (15) into Eq. (17), and equating coefficients of like p, we obtain the following differential equations:

$$\begin{aligned}
&p^0 : \{v_0' = 0, \quad v_0(0) = 2,\\
&p^1 : \{v_1' = v_0 - v_0^2, \quad v_1(0) = 0,\\
&p^2 : \{v_2' = v_1 - 2v_0v_1, \quad v_2(0) = 0,\\
&p^3 : \{v_3' = v_2 - v_1^2 - 2v_0v_2, \quad v_3(0) = 0,\\
&p^4 : \{v_4' = v_3 - 2v_0v_3 - 2v_1v_2, \quad v_4(0) = 0,\\
&p^5 : \{v_5' = v_4 - v_2^2 - 2v_1v_3 - 2v_0v_4, \quad v_5(0) = 0,\\
&p^6 : \{v_6' = v_5 - 2v_5v_0 - 2v_1v_4 - 2v_2v_3, \quad v_6(0) = 0,\\
&p^7 : \{v_7' = v_6 - v_3^2 - 2v_0v_6 - 2v_1v_5 - 2v_2v_4, \quad v_7(0) = 0,\\
&p^8 : \{v_8' = v_7 - 2v_7v_0 - 2v_1v_6 - 2v_2v_5 - 2v_3v_4, v_8(0) = 0,\\
&\vdots
\end{aligned}$$

where "primes" denote differentiation with respect to τ. Thus, solving the equations above yields

$$\begin{aligned}
&v_0 = 2,\\
&v_1 = -2\tau,\\
&v_2 = 3\tau^2,\\
&v_3 = -13/3\tau^3,\\
&v_4 = 25/4\tau^4,\\
&v_5 = -541/60\tau^5,\\
&v_6 = 1561/120\tau^6,\\
&v_7 = -47293/2520\tau^7,\\
&v_8 = 36389/1344\tau^8,\\
&\vdots
\end{aligned}$$

Substituting these in Eq. (15) gives

$$\begin{aligned}
v = {} & 2 - 2p\tau + 3p^2\tau^2 - 13/3p^3\tau^3 + 25/4p^4\tau^4 - 541/60p^5\tau^5 + 1561/120p^6\tau^6\\
& - 47293/2520p^7\tau^7 + 36389/1344p^8\tau^8 - \cdots.
\end{aligned}$$

Hence, by Eq. (16) one has

$$u = 2 - 2\tau + 3\tau^2 - 13/3\tau^3 + 25/4\tau^4 - 541/60\tau^5 + 1561/120\tau^6 - 47293/2520\tau^7 + 36389/1344\tau^8 - \cdots, \tag{18}$$

which is the expansion of

$$u = \frac{2}{2 - e^{-\tau}}. \tag{19}$$

This solution is exactly the same as in Eq. (3).

Example 2. We now solve the system in Eq. (5) using HPM with $\alpha = 1$, $u(0) = 1.3$, $v(0) = 0.6$ [9]. Therefore, we rewrite it in the form

$$\frac{du}{d\tau} = pu(1-v),$$

$$\frac{dv}{d\tau} = pv(u-1),$$
$$u(0) = 1.3, \qquad v(0) = 0.6, \tag{20}$$

where $p \in [0, 1]$ is an embedding parameter. We assume the solutions of Eq. (20), (u, v) are expressed as power series

$$u = u_0 + pu_1 + p^2u_2 + \cdots, \tag{21}$$
$$v = v_0 + pv_1 + p^2v_2 + \cdots, \tag{22}$$

respectively. Substituting Eqs. (21) and (22) into the system in Eq. (20), and equating coefficients of like p, we obtain the following systems of differential equations:

$$p^0 : \begin{cases} u_0' = 0, \\ v_0' = 0, \\ u_0(0) = 1.3, \qquad v_0(0) = 0.6, \end{cases}$$
$$p^1 : \begin{cases} u_1' = u_0(1 - v_0), \\ v_1' = v_0(u_0 - 1), \\ u_1(0) = 0, \qquad v_1(0) = 0, \end{cases}$$
$$p^2 : \begin{cases} u_2' = u_1 - (u_0v_1 + u_1v_0), \\ v_2' = u_0v_1 + u_1v_0 - v_1, \\ u_2(0) = 0, \qquad v_2(0) = 0, \end{cases}$$
$$p^3 : \begin{cases} u_3' = u_2 - (u_0v_2 + u_1v_1 + u_2v_0), \\ v_3' = u_0v_2 + u_1v_1 + u_2v_0 - v_2, \\ u_3(0) = 0, \qquad v_3(0) = 0, \end{cases}$$
$$p^4 : \begin{cases} u_4' = u_3 - (u_0v_3 + u_1v_2 + u_2v_1 + u_3v_0), \\ v_4' = u_0v_3 + u_1v_2 + u_2v_1 + u_3v_0 - v_3, \\ u_4(0) = 0, \qquad v_4(0) = 0, \end{cases}$$
$$\vdots$$

where "primes" denote differentiation with respect to τ. Thus, solving the above systems of equations yield

$$u_0 = 1.3, \qquad v_0 = 0.6,$$
$$u_1 = 0.52\tau, \qquad v_1 = 0.18\tau,$$
$$u_2 = -0.013\tau^2, \qquad v_2 = 0.183\tau^2,$$
$$u_3 = -0.1122\tau^3, \qquad v_3 = 0.0469\tau^3,$$
$$u_4 = -0.0497\tau^4, \qquad v_4 = 0.0099\tau^4,$$
$$\vdots$$

Substituting these u_n, v_n , $n \geq 0$ into Eqs. (21) and (22), respectively we have

$$u = 1.3 + 0.52p\tau - 0.013p^2\tau^2 - 0.1122p^3\tau^3 - 0.0497p^4\tau^4 - \cdots,$$
$$v = 0.6 + 0.18p\tau + 0.183p^2\tau^2 + 0.0469p^3\tau^3 + 0.0099p^4\tau^4 + \cdots.$$

Letting $p \to 1$ one obtains

$$u = 1.3 + 0.52\tau - 0.013\tau^2 - 0.1122\tau^3 - 0.0497\tau^4 - \cdots, \tag{23}$$
$$v = 0.6 + 0.18\tau + 0.183\tau^2 + 0.0469\tau^3 + 0.0099\tau^4 + \cdots \tag{24}$$

Example 3. In this example we solve the system in Eq. (7) using HPM with $u(0) = 1$, $v(0) = 1$. Therefore, we rewrite it in the form

$$\frac{du}{d\tau} = pu(1 - u - av),$$
$$\frac{dv}{d\tau} = p\rho v(1 - v - bu),$$
$$u(0) = 1, \qquad v(0) = 1, \tag{25}$$

where a, ρ, b are some positive constants, and $p \in [0, 1]$ is an embedding parameter. We assume the solutions of Eq. (25), (u, v) are expressed as power series given by Eqs. (21)–(22) respectively. Substituting Eqs. (21)–(22) into the system in Eq. (25), and equating coefficients of like p, we obtain the following systems of differential equations:

$$
\begin{aligned}
p^0 &: \begin{cases} u_0' = 0, \\ v_0' = 0, \\ u_0(0) = 1, \qquad v_0(0) = 1, \end{cases} \\
p^1 &: \begin{cases} u_1' = u_0 - u_0^2 - a u_0 v_0, \\ v_1' = \rho(v_0 - v_0^2 - b u_0 v_0), \\ u_1(0) = 0, \qquad v_1(0) = 0, \end{cases} \\
p^2 &: \begin{cases} u_2' = u_1 - 2u_0 u_1 - a(u_0 v_1 + u_1 v_0), \\ v_2' = \rho(v_1 - 2v_0 v_1 - b(u_0 v_1 + u_1 v_0)), \\ u_2(0) = 0, \qquad v_2(0) = 0, \end{cases} \\
p^3 &: \begin{cases} u_3' = u_2 - 2u_0 u_2 - u_1^2 - a(u_0 v_2 + u_1 v_1 + u_2 v_0), \\ v_3' = \rho(v_2 - 2v_0 v_2 - v_1^2 - b(u_0 v_2 + u_1 v_1 + u_2 v_0)), \\ u_3(0) = 0, \qquad v_3(0) = 0, \end{cases} \\
p^4 &: \begin{cases} u_4' = u_3 - 2u_1 u_2 - 2u_0 u_3 - a(u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3 v_0), \\ v_4' = \rho(v_3 - 2v_1 v_2 - 2v_0 v_3 - b(u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3 v_0)), \\ u_4(0) = 0, \qquad v_4(0) = 0, \end{cases} \\
&\vdots
\end{aligned}
$$

where "primes" denote differentiation with respect to τ. We take $a = 1$, $\rho = 1$ and $b = 0.8$ for numerical purposes. Thus, solving the above systems of equations yield

$$
\begin{aligned}
&u_0 = 1, \qquad v_0 = 1, \\
&u_1 = -\tau, \qquad v_1 = -0.8\tau, \\
&u_2 = 1.4\tau^2, \qquad v_2 = 1.12\tau^2, \\
&u_3 = -1.9067\tau^3, \qquad v_3 = -1.4720\tau^3, \\
&u_4 = 2.5814\tau^4, \qquad v_4 = 1.9397\tau^4, \\
&\vdots
\end{aligned}
$$

Substituting these u_n, v_n, $n \geq 0$ into Eqs. (21) and (22), respectively we have

$$
\begin{aligned}
u &= 1 - p\tau + 1.4p^2\tau^2 - 1.9067p^3\tau^3 + 2.5814p^4\tau^4 - \cdots, \\
v &= 1 - 0.8p\tau + 1.12p^2\tau^2 - 1.4720p^3\tau^3 + 1.9397p^4\tau^4 + \cdots.
\end{aligned}
$$

Letting $p \to 1$ one obtains

$$u = 1 - \tau + 1.4\tau^2 - 1.9067\tau^3 + 2.5814\tau^4 - \cdots, \tag{26}$$

$$v = 1 - 0.8\tau + 1.12\tau^2 - 1.4720\tau^3 + 1.9397\tau^4 + \cdots. \tag{27}$$

Example 4. We finally solve the system in Eq. (9) using HPM with $u(0) = 1.3$, $v(0) = 1.2$. Therefore, we rewrite it in the form

$$
\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}\tau} = p\left(u(1-u) - \frac{auv}{u+d}\right), \\
&\frac{\mathrm{d}v}{\mathrm{d}\tau} = pbv\left(1 - \frac{v}{u}\right), \\
&u(0) = 1.3, \qquad v(0) = 1.2,
\end{aligned} \tag{28}
$$

where a, d, b are some positive constants, and $p \in [0, 1]$ is an embedding parameter. We assume the solutions of Eq. (28), (u, v) are expressed as power series given by Eqs. (21)–(22) respectively. Substituting Eqs. (21)–(22) into the system in

Eq. (28), and equating coefficients of like p, we obtain the following systems of differential equations:

$$p^0: \begin{cases} u_0' = 0, \\ v_0' = 0, \\ u_0(0) = 1.3, \qquad v_0(0) = 1.2, \end{cases}$$

$$p^1: \begin{cases} u_1' = u_0 - u_0^2 - au_0v_0/(d+u_0), \\ v_1' = bv_0(1 - v_0/u_0), \\ u_1(0) = 0, \qquad v_1(0) = 0, \end{cases}$$

$$p^2: \begin{cases} u_2' = 2u_1 - 4u_0u_1 - 2a(u_1v_0 + u_0v_1)/(d+u_0) + 2au_0v_0u_1/(d+u_0)^2, \\ v_2' = 2bv_1 - 4bv_0v_1/u_0 + 2bv_0^2u_1/u_0^2, \\ u_2(0) = 0, \qquad v_2(0) = 0, \end{cases}$$

$$p^3: \begin{cases} u_3' = 6u_2 - 6u_1^2 - 12u_0u_2 - 6a(u_2v_0 + u_1v_1 + u_0v_2)/(d+u_0) \\ \qquad + 6a(u_1^2v_0 + u_0v_1u_1 + u_0v_0u_2)/(d+u_0)^2 - 6au_0v_0u_1^2/(d+u_0)^3, \\ v_3' = 6bv_2 - 6b(v_1^2 + 2v_0v_2)/u_0 + 6b(v_0^2u_2 + 2v_0v_1u_1)/u_0^2 - 6bv_0^2u_1^2/u_0^3, \\ u_3(0) = 0, \qquad v_3(0) = 0, \end{cases}$$

$$p^4: \begin{cases} u_4' = 24u_3 - 48u_1u_2 - 48u_0u_3 - 24a(u_3v_0 + u_2v_1 + u_1v_2 + u_0v_3)/(d+u_0) \\ \qquad + 24a(2u_2v_0u_1 + u_1^2v_1 + u_0v_2u_1 + u_0v_1u_2 + u_0v_0u_3)/(d+u_0)^2 \\ \qquad - 24a(u_1^3v_0 + u_0v_1u_1^2 + 2u_0v_0u_1u_2)/(d+u_0)^3 + 24au_0v_0u_1^3/(d+u_0)^4, \\ v_4' = 24bv_3 - 48b(v_0v_3 + v_1v_2)/u_0 + 24b(v_1^2u_1 + v_0^2u_3 + 2v_0v_2u_1 + 2v_0v_1u_2)/u_0^2 \\ \qquad - 48b(u_1^2v_0v_1 + v_0^2u_1u_2)/u_0^3 + 24bv_0^2u_1^3/u_0^4, \\ u_4(0) = 0, \qquad v_4(0) = 0, \end{cases}$$

$$\vdots$$

where "primes" denote differentiation with respect to τ. We take $a = 1$, $b = 5$ and $d = 10$ for numerical purposes. Thus, solving the above systems of equations yield

$$\begin{aligned} &u_0 = 1.3, \qquad v_0 = 1.2, \\ &u_1 = -0.5281\tau, \qquad v_1 = 0.4615\tau, \\ &u_2 = 0.8415\tau^2, \qquad v_2 = -4.2024\tau^2, \\ &u_3 = -2.3990\tau^3, \qquad v_3 = 35.8016\tau^3, \\ &u_4 = 3.7396\tau^4, \qquad v_4 = -760.493\tau^4, \\ &\vdots \end{aligned}$$

Substituting these u_n, v_n, $n \geq 0$ into Eqs. (21) and (22), respectively we have

$$\begin{aligned} u &= 1.3 - 0.5281p\tau + 0.8415p^2\tau^2 - 2.3990p^3\tau^3 + 3.7396p^4\tau^4 - \cdots, \\ v &= 1.2 + 0.4615p\tau - 4.2024p^2\tau^2 + 35.8016p^3\tau^3 - 760.493p^4\tau^4 + \cdots. \end{aligned}$$

Letting $p \to 1$ one obtains

$$u = 1.3 - 0.5281\tau + 0.8415\tau^2 - 2.3990\tau^3 + 3.7396\tau^4 - \cdots, \tag{29}$$

$$v = 1.2 + 0.4615\tau - 4.2024\tau^2 + 35.8016\tau^3 - 760.493\tau^4 + \cdots. \tag{30}$$

4. Conclusion and results

In this paper we obtain the approximate analytical solutions of continuous population models for single and interacting species using He's homotopy perturbation method. Fig. 1 shows a very good approximation to the analytical solution of logistic growth model in the time interval [0, 0.4] by using only 9 terms of the series given by Eq. (16), which indicates that the speed of convergence of HPM is very fast. In addition, a better approximation to the exact solution for $\tau \geq 0.375$ can be achieved by adding new terms to this series.

Figs. 2 and 3 show the comparison between the five-term HPM solutions of the system in Eq. (5) and the numerical solutions with $\alpha = 1$, $u(0) = 1.3$, $v(0) = 0.6$. We obtain these numerical solutions using ode23, an ordinary differential equation solver found in the *Matlab* package. It is clear from both of the figures that there is a very close agreement between the solutions for u (prey population) and v (predator population) in the time interval [0, 1.1]. As mentioned above for the logistic growth model, a very good approximation to the approximate analytical solution for $\tau \geq 1.1$ can be achieved by adding new terms to the series in Eq. (16).

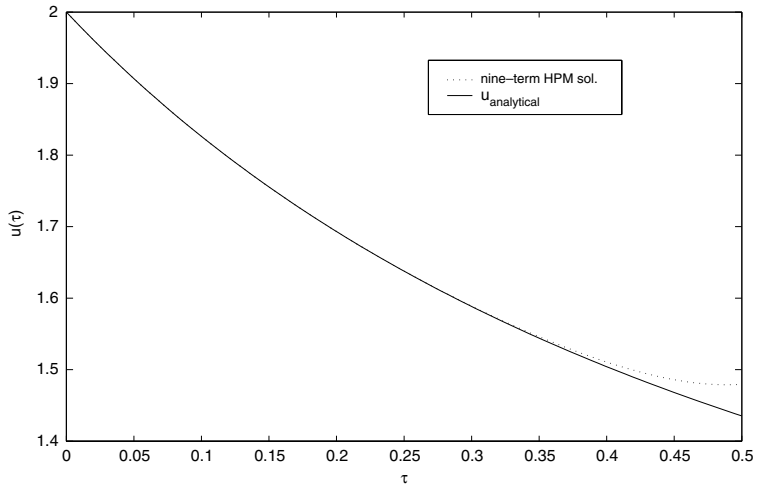

Fig. 1. Comparison between the nine-term HPM solution of the logistic growth model in Eq. (2) and the analytical solution with $u(0) = 2$.

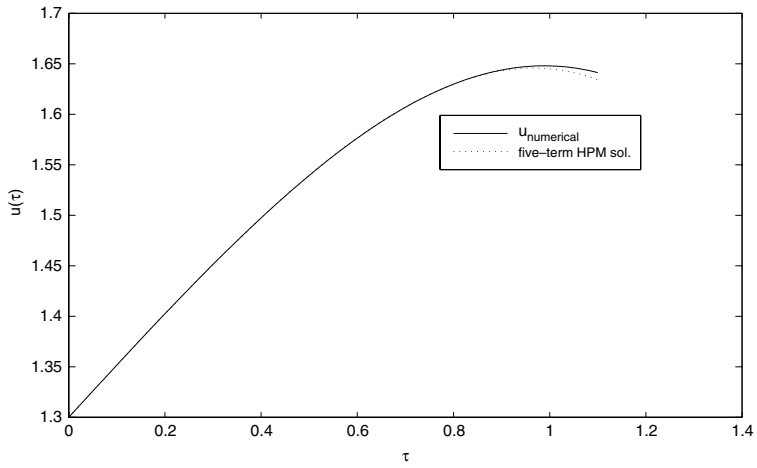

Fig. 2. Comparison between the five-term HPM solution of the system in Eq. (5) and the numerical solution with $\alpha = 1$, $u(0) = 1.3$, $v(0) = 0.6$.

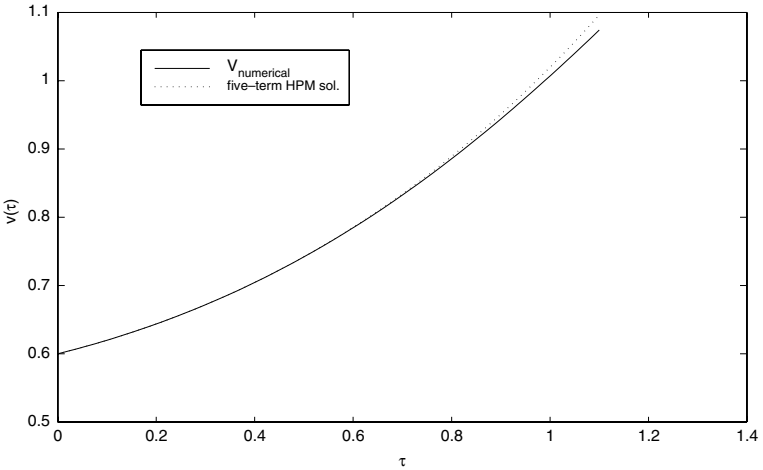

Fig. 3. Comparison between the five-term HPM solution of the system in Eq. (5) and the numerical solution with $\alpha = 1$, $u(0) = 1.3$, $v(0) = 0.6$.

Figs. 4 and 5 show the comparison between the five-term HPM solutions of the system in Eq. (7) and the numerical solutions with $a = 1$, $\rho = 1$, $b = 0.8$, $u(0) = 1$, $v(0) = 1$. It seems that the solutions for u and v look almost identical

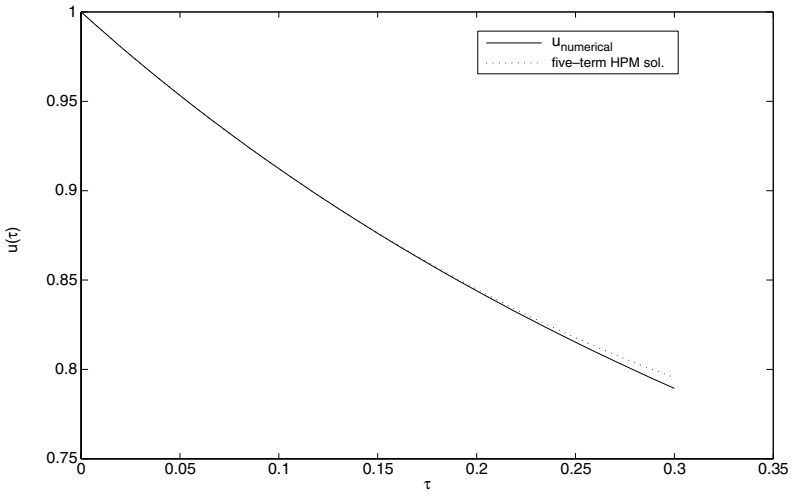

Fig. 4. Comparison between the five-term HPM solution of the system in Eq. (7) and the numerical solution with $a = 1, \rho = 1, b = 0.8, u(0) = 1, v(0) = 1$.

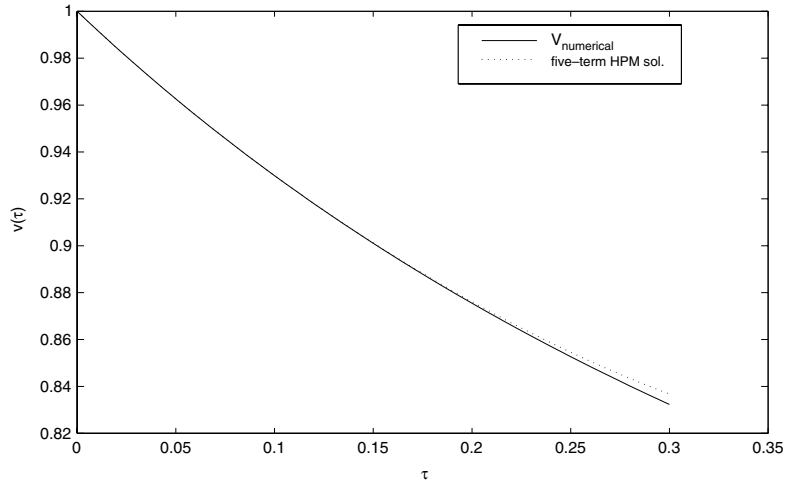

Fig. 5. Comparison between the five-term HPM solution of the system in Eq. (7) and the numerical solution with $a = 1, \rho = 1, b = 0.8, u(0) = 1, v(0) = 1$.

in the time interval [0, 0.25]. One can obtain a better approximation to the numerical solutions by adding new terms to the series in Eq. (16).

Figs. 6 and 7 show the comparison between the five-term HPM solutions of the system in Eq. (9) and the numerical solutions with $a = 1, d = 10, b = 5, u(0) = 1.3, v(0) = 1.2$. It is again clear that the two figures for the solutions (u, v) look identical in the time interval [0, 0.01]. Each equation in Eq. (9) is strongly non-linear. That's why we had to solve it in such a small time interval to obtain a five-term HPM solution. A better approximation for larger time interval can be achieved by adding new terms to the series in Eq. (16).

On the other hand, in [5] we have solved the problems we present in Examples 1 and 2 using Adomian decomposition method (ADM) [10,11], an iterative method which provides approximate analytical solutions in the form of an infinite power series for nonlinear equations. Also, ADM is used by many scientists, e.g., [6,7,9,12,13]. Although HPM and ADM give the same results for both of the problems (as the authors of Ref. [10] have faced for their problems), the HPM needs not to calculate Adomian polynomials, and it is very straightforward, and the solution procedure is very simple [14,15].

Even though the examples in this paper are non-linear ordinary differential equations, HPM is, of course, applicable to non-linear partial differential equations (as stated in [2]). One of the major lack of ADM is that it could not always satisfy all the boundary conditions of the nonlinear problems, leading to an error at the boundary of the domain in which the problem is solved [1].

Therefore, one clearly can conclude that HPM and He polynomials can completely replace the Adomian method and Adomian polynomials.

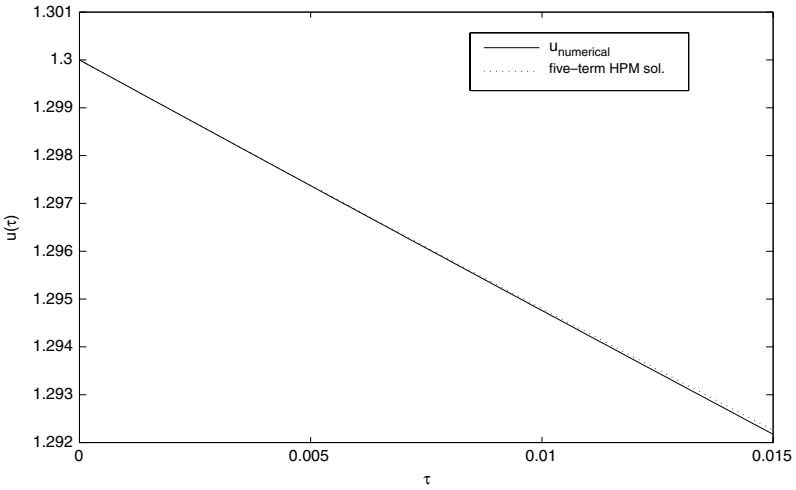

Fig. 6. Comparison between the five-term HPM solution of the system in Eq. (9) and the numerical solution with $a = 1, d = 10, b = 5, u(0) = 1.3$, $v(0) = 1.2$.

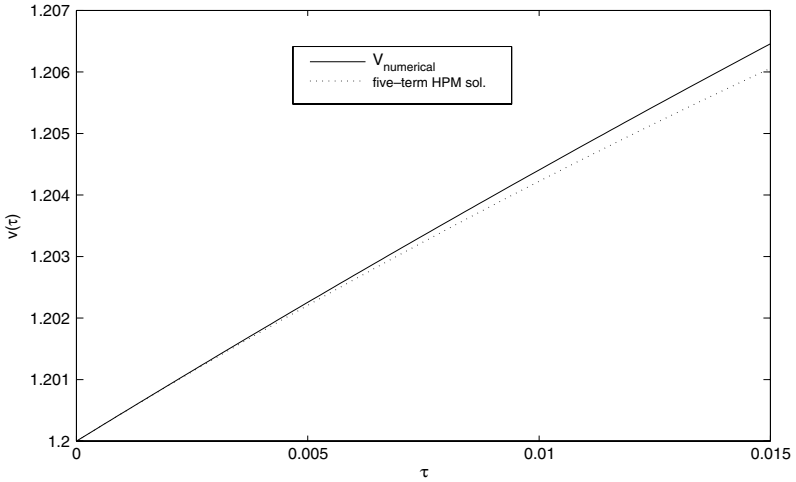

Fig. 7. Comparison between the five-term HPM solution of the system in Eq. (9) and the numerical solution with $a = 1, d = 10, b = 5, u(0) = 1.3$, $v(0) = 1.2$.

References

[1] J.H. He, Homotopy perturbation method for solving boundary value problems, Physics Letters A 350 (2006) 87–88.
[2] JH. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, International Journal of Non-Linear Mechanics 35 (2000) 37–43.
[3] JH. He, Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering 178 (1999) 257–262.
[4] JH. He, Homotopy perturbation method: A new nonlinear analytical technique, Applied Mathematics and Computation 135 (2003) 73–79.
[5] S. Pamuk, The decomposition method for continuous population models for single and interacting species, Applied Mathematics and Computation 163 (2005) 79–88.
[6] S. Pamuk, An application for linear and nonlinear heat equations by Adomian's decomposition method, Applied Mathematics and Computation 163 (2005) 89–96.
[7] S. Pamuk, Solution of the Porous media equation by adomian's decomposition method, Physics Letters A 344 (2005) 184–188.
[8] J.D. Murray, Mathematical Biology, Springer, Berlin, 1993.
[9] S. Pamuk, A review of some recent results for the approximate analytical solutions of non-linear differential equations, Mathematical Problems in Engineering 2009 (2009) 34 pp. Article ID 202307.
[10] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, 1994.
[11] G. Adomian, Solving frontier problems modelled by nonlinear partial differential equations, Computers & Mathematics with Applications 22 (8) (1991) 91–94.
[12] A.M. Wazwaz, A. Gorguis, Exact solutions for heat-like and wave-like equations with variable coefficients, Applied Mathematics and Computation 149 (1) (2004) 15–29.
[13] A.M. Wazwaz, A reliable technique for solving the wave equation in an infinite one-dimensional medium, Applied Mathematics and Computation 92 (1998) 1–7.
[14] A. Ghorbani, Beyond Adomian polynomials: He polynomials, Chaos, Solitons and Fractals 39 (3) (2009) 1486–1492.
[15] A. Ghorbani, J. Saberi-Nadjafi, He's homotopy perturbation method for calculating adomian polynomials, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 229–232.