Irredundance, secure domination and maximum degree in trees

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Abstract

It is shown that the lower irredundance number and secure domination number of an \(n\) vertex tree \(T\) with maximum degree \(\Delta \geq 3\), are bounded below by \(2(n + 1)/(2\Delta + 3)\) (\(T \neq K_{1,\Delta}\)) and \((\Delta n + \Delta - 1)/(3\Delta - 1)\), respectively. The bounds are sharp and extremal trees are exhibited.

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1. Introduction

Let \(G = (V, E)\) be a simple graph and \(v \in X \subseteq V\). Vertex \(u \in V - X\) is an \(X\)-external private neighbour (abbreviated \(X\)-epn) of \(v\) if \(N(u) \cap X = \{v\}\). The set of all \(X\)-epns of \(v\) is denoted by \(\text{EPN}(v, X)\) and

\[
\text{PN}(v, X) = \begin{cases} 
\text{EPN}(v, X) \cup \{v\} & \text{if } v \text{ is isolated in } G[X], \\
\text{EPN}(v, X) & \text{otherwise}.
\end{cases}
\]

A subset \(X \subseteq V\) is irredundant if for all \(v \in X\), \(\text{PN}(v, X) \neq \emptyset\).

Irredundance is a property which makes a dominating set minimal and hence has been well-studied (see [8]). In particular, Cockayne and Mynhardt [6] showed that for a graph \(G\) with order \(n\) and maximum degree \(\Delta (\geq 2)\), the lower irredundance number \(\text{ir}(G)\) (i.e. the smallest cardinality of a maximal irredundant set) is at least \(2n/3\). Grobler [7] has provided a simpler proof of this bound. The bound is sharp but the only trees which attain it are paths with order divisible by three. Hence, the bound may be improved for other trees and in this work we show that for a tree \(T\) with \(\Delta \geq 3\) which is not a star, \(\text{ir}(T) \geq 2(n + 1)/(2\Delta + 3)\) and exhibit extremal trees.

In [4], four strategies for the protection of a graph by placing guards at vertices, were discussed. The minimum number of guards required under one of these strategies is called the secure domination number and is denoted by \(\gamma_s(G)\). We now give a formal definition.

The set \(X\) is a secure dominating set (SDS) if for each \(u \in V - X\), there exists \(v\) such that

\[
v \in N(u) \cap X \quad \text{and} \quad (X - \{v\}) \cup \{u\} \text{ is dominating.}
\]

The parameter \(\gamma_s(G)\) is the minimum cardinality of an SDS of \(G\).
Secure domination has also been studied in [1–3,9]. In particular, in [4] it was proved that
\[ \gamma_s(G) \geq \frac{n(2\Delta - 1)}{(\Delta^2 + 2\Delta - 1)} \]
for triangle-free graphs with maximum degree \( \Delta \geq 3 \). An improvement is possible for trees and we show that for a tree \( T \) with \( \Delta \geq 3 \), \( \gamma_s(T) \geq \frac{(\Delta n + \Delta - 1)}{(3\Delta - 1)} \).

The following partition of \( V \) induced by the vertex subset \( X \) will be involved in the proof of each bound:
\[ V = X \cup B \cup C \cup R \] (disjoint union),
where
\[ B = \{ u \in V - X \mid |N(u) \cap X| = 1 \}, \]
\[ C = \{ u \in V - X \mid |N(u) \cap X| \geq 2 \}, \]
\[ R = V - N[X]. \]

In the following sections the cardinality of any set (except \( V \)) denoted by an upper case letter will be denoted by the corresponding lower case letter, i.e. \(|B| = b, |C| = c\), etc.

2. Lower irredundance

We will need the following characterisation of maximality of an irredundant set which involves the partition of \( V \) defined in Section 1.

**Theorem 1** (Cockayne et al. [5]). The irredundant set \( X \) is maximal if and only if for each \( u \in N[R] \), there exists \( v \in X \) such that
\[ P_N(v, X) \subseteq N[u]. \] (2)

If (2) is satisfied we say that \( u \) annihilates \( v \). For the remainder of this section, \( X \) will denote a maximal irredundant set of a forest \( G \). We need to refine the partition of Section 1. Let
\[ Z = \{ v \in X \mid v \text{ is isolated in } G[X] \}, \]
\[ Y = X - Z, \]
\[ Y_1 = \{ v \in Y \mid |EPN(v, X)| = 1 \} \]
and
\[ Y_2 = \{ v \in Y \mid |EPN(v, X)| \geq 2 \}. \]

Observe that \( X = Z \cup Y_1 \cup Y_2 \) (disjoint union) and finally define
\[ E_1 = B \cap N(Y_1). \]

The proof of the bound will use the following two preliminary results.

**Lemma 2.** Each \( u \in R \) annihilates some \( v \in Y_1 \).

**Proof.** For \( u \in R \), \( u \) does not annihilate any \( v \in Z \) since \( u \) and \( v \) are not adjacent and if \( u \) annihilated \( v \in Y_2 \), then \( G[N[u] \cup N[v]] \) would contain a cycle (in fact a \( C_4 \)). The result follows from Theorem 1. \( \square \)

Suppose that \( \mathcal{F}(x, \Delta) \) is the set of forests of maximum order which have maximum degree \( \Delta \) and a maximal irredundant set of size \( x \). (Note that the maximum order does exist since the bound for general graphs gives an upper bound of \( 3\Delta x/2 \) for the number of vertices.)

**Lemma 3.** Let \( X \) be a maximal irredundant set of size \( x \) of the forest \( G \in \mathcal{F}(x, \Delta) \). Then
(i) Each \( u \in E_1 \) has degree \( \Delta \).
(ii) For each \( u \in R \), \( N(u) \cap (B \cup C) = \{ w \} \), where \( w \in E_1 \).
Theorem 4. If $x = \phi$ (i.e. $Y = Y_1$).

Proof. In each argument, we will negate the assertion and construct a new forest $G_1$ of maximum degree $A$ with more vertices than $G$. It will be routine in each case to show that $X$ is irredundant in $G_1$ and then to show maximality by Theorem 1. These details will be omitted. Thus, in each case $G$ contradicts the definition of $\mathcal{F}(x, A)$ and the assertion is thus established.

(i) Suppose that $u \in E_1$ and $\deg(u) < A$. Form $G_1$ by joining $u$ to a new leaf.

(ii) By Lemma 2, each $u \in R$ is adjacent to $w \in E_1$. Suppose that $\{w, w'\} \subseteq N(u) \cap (B \cup C)$. Form $G_1$ by deleting $uw'$ and adding an edge from $w'$ to a new leaf.

(iii) By (ii) and Lemma 2, in order to satisfy the maximality condition (Theorem 1), the only vertices of $X$ are the vertices of $N(R) \cap E_1$ and each such vertex annihilates its neighbour in $Y_1$. Suppose $u \in E_1$ and $w \in N(u) \cap (B \cup C)$. Then by the above paragraph, $w$ need not annihilate $u$. Form $G_1$ by removing $uw$ and joining a new leaf to $u$. The contradiction and (i), prove the assertion.

(iv) Suppose that $v \in Y_2$ and $\text{EPN}(v, X) = B_v$. By (ii) no vertex of $B_v$ is adjacent to $R$ and so is not required to be an annihilator for maximality. Form $G_1$ by identifying a vertex of degree one in a star $K_{1,4}$ with the vertex $v$ in $G - B_v$. □

We now state and prove the principal result of this section.

Theorem 4. If $T(\neq K_{1,4})$ is a tree with $n$ vertices and maximum degree $A \geq 3$, then

$$\text{iR}(T) \geq \frac{2(n + 1)}{2A + 3}.$$  

Proof. Note that for $T \neq K_{1,4}$, $\text{iR}(T) > 2$. To prove the theorem, we determine upper bounds on the number of vertices of forests $G \in \mathcal{F}(x, A)$ when a maximal irredundant set $X$ of size $x \geq 2$ has (a) $y = 0$ and (b) $y \geq 2$. (Note $y = 1$ is impossible.) We emphasize that $G$, $X$ satisfy Lemma 3.

(a) If $y = 0$, then by Lemma 2, $R = \phi$ and $Z$ is independent dominating. Therefore,

$$n \leq (A + 1)z = (A + 1)x. \quad (3)$$

(b) If $y \geq 2$, let $b_z$, $c_z$, $c_y$ be the number of edges in $G$ from $B$ to $Z$, $C$ to $Z$, $C$ to $Y = Y_1$, respectively. Then,

$$b_z + c_z \leq Az. \quad (4)$$

Since $G[C \cup Y]$ is acyclic and $G[Y]$ has at least $\lceil y/2 \rceil$ edges,

$$\left\lceil \frac{y}{2} \right\rceil + c_y \leq c + y - 1. \quad (5)$$

Moreover, each vertex of $C$ is adjacent to at least two vertices of $Y \cup Z$. Hence,

$$2c \leq c_y + c_z \leq c + y - \left\lceil \frac{y}{2} \right\rceil - 1 + c_z \quad \text{(by (5))}. \quad (6)$$

Therefore,

$$c \leq c_z + y - \left\lceil \frac{y}{2} \right\rceil - 1. \quad (6)$$

Using (6) we obtain

$$n = x + b + c + r \leq (y + z) + (b_z + y) + \left( c_z + y - \left\lceil \frac{y}{2} \right\rceil - 1 \right) + (A - 1)y. \quad (7)$$
Then, (4) and (7) give
\[
\begin{align*}
n \leq & \ (A + 1)z + (A + 2)y - \left\lceil \frac{y}{2} \right\rceil - 1 \\
\leq & \ (A + 1)z + \left( A + \frac{3}{2} \right) y - 1 \\
\leq & \ \left( A + \frac{3}{2} \right) x - 1.
\end{align*}
\]  

(8)

We observe that the right-hand side of (8) is at least as large as that of (3) for \( x \geq 2 \) and that equality occurs for \( x = 2 \). Hence from (8), if a forest \( G \neq K_1 \) has \( n \) vertices and maximum degree \( A \), then
\[
\ir(G) \geq \frac{2(n + 1)}{2A + 3},
\]
as required. □

The bound is sharp. We characterise the extremal trees \( T \) for which \( \ir(T) \) is even. Let \( X \) be maximal irredundant of size \( \ir(T) \) for which \( Y = \{v_1, \ldots, v_y\} \), where \( y \geq 2 \).

By Lemma 3, \( T \) contains \( y \) disjoint copies \( S_1, \ldots, S_y \) of \( K_{1, A} \), where for \( i = 1, \ldots, y \), \( v_i \) is a degree one vertex of \( S_i \). Note that \( \bigcup_{i=1}^{y} V(S_i) = Y \cup E_1 \cup R \).

Since \( T \) is extremal, we have equality in (4)–(8). In particular by (8), \( Z = \phi \). The only additional vertices are those of \( C \) and by (5) and (6), \( c \) attains its maximum value of \( y/2 - 1 \) when \( T[C \cup Y] \) is any tree \( T^* \) in which each vertex of \( C \) sends exactly two edges to \( Y \), \( C \) is independent and \( T^*[Y] = (y/2)K_2 \).

Since \( T \) is a tree, there are no further edges. In Fig. 1 we show extremal trees where \( T^* \) is a path.

3. Secure domination

Let \( X \subseteq V(G) \) for any graph \( G \). If \( u \in V - X \) and \( v \in X \) satisfy (1), then we say that \( v \) X-defends \( u \). We will need the following result proved in [4].

Proposition 5 (Cockayne et al. [4]). Let \( v \in X \) and \( u \in V - X \). Then, \( v \) X-defends \( u \) if and only if \( u \) is adjacent to each vertex of \( \{v\} \cup (\text{EPN}(v, X) - \{u\}) \).

Theorem 6. For a tree \( T \) with \( n \) vertices and maximum degree \( A \geq 3 \),
\[
\gamma_s(T) \geq \frac{(\Delta n + A - 1)}{(3\Delta - 1)}.
\]

Proof. Let \( X \) be an SDS of \( T \). Suppose that \( \{u, w\} \subseteq \text{EPN}(v, X) \) for some \( v \in X \). Since \( u \) can only be X-defended by \( v \), Proposition 5 implies that \( \{u, v, w\} \) induces a \( K_3 \). This contradiction allows us to write \( X = X_0 \cup X_1 \) (disjoint union) where each \( v \in X_0 \) (resp. \( X_1 \)) has precisely 0 (resp. 1) X-epn.

Consider the partition of \( V \) induced by \( X \) defined in the Introduction and note that \( R = \phi \), since \( X \) is dominating.
The number of edges from \( X \) to \( C \) is at least \( 2c \) by definition of \( C \) and at most \( x_0 + x_1 + c - 1 \), since \( T[X \cup C] \) is acyclic. Hence,

\[
2c \leq c + x_0 + x_1 - 1.
\]  

(9)

From (9)

\[
n = x_0 + 2x_1 + c \\
\leq x_0 + 2x_1 + (x_0 + x_1 - 1).
\]  

(10)

Therefore,

\[
2x_0 + 3x_1 \geq n + 1.
\]  

(11)

No \( u \in C \) may be defended by \( v \in X_1 \), for otherwise by Proposition 5, there exists a \( K_3 \). Hence, each \( u \in C \) is adjacent to a vertex in \( X_0 \) and so

\[
c \leq \Delta x_0.
\]  

(12)

By (10) and (12)

\[
(A + 1)x_0 + 2x_1 \geq n.
\]  

(13)

The minimum value of \( x = x_0 + x_1 \) subject to the constraints (11) and (13), is

\[
x = (\Delta n + A - 1)/(3A - 1)
\]

taken when \( x_0 = (n - 2)/(3A - 1) \) and \( x_1 = (\Delta n - n + A + 1)/(3A - 1) \) as required. \( \square \)

For \( A \geq 3 \) and \( k \geq 1 \), construct disjoint trees \( T_1, \ldots, T_k \) as follows. Let

\[
V(T_i) = \{u_i\} \cup Q_i \cup C_i \cup B_i \quad \text{(disjoint union),}
\]

where \( q_i = b_i = A - 1 \) and \( c_i = A \). Join \( u_i \) to each vertex of \( C_i \). Add a matching between \( A - 1 \) vertices of \( C_i \) to \( Q_i \) and a further matching from \( Q_i \) to \( B_i \). Let \( v_i \) be the unmatched vertex of \( C_i \) and \( w_i \) be chosen arbitrarily in \( Q_i \). Now, let

\[
V(T) = \bigcup_{i=1}^k V(T_i) \cup \{w_0, w^*\}
\]

and

\[
E(T) = \bigcup_{i=1}^k E(T_i) \cup \{w_i v_{i+1} | i = 0, \ldots, k - 1\} \cup \{w_0 w^*\}.
\]

It is easy to check that \( T \) has SDS \( X = X_0 \cup X_1 \), where \( X_0 = \{u_1, \ldots, u_k\} \) and \( X_1 = \{w_0\} \cup \bigcup_{i=1}^k Q_i \) and that the bound of Theorem 6 is attained.

We illustrate an extremal tree for \( A = 4 \) and \( k = 3 \) in Fig. 2.

![Fig. 2. Extremal tree for Theorem 6.](image-url)
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