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Incomparability graphs of (3 + 1)-free posets are s-positive

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Abstract

In [5] Stanley associated to a (finite) graph G a symmetric function X_G generalizing the chromatic polynomial of G. Using an involution on a special type of arrays constructed by Gessel and Viennot [1], we show that if G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset, then X_G is a nonnegative linear combination of Schur functions. Since the elementary symmetric functions are nonnegative linear combinations of Schur functions, this result gives supportive evidence for a conjecture of Stanley and Stembridge ([5, Conjecture 5.1] or [6, Conjecture 5.5]).

Résumé

Dans [5], Stanley associe à tout graphe (fini) G une fonction symétrique X_G qui généralise le polynôme chromatique de G. En utilisant une involution sur certains tableaux construits par Gessel et Viennot [1], nous démontrons que si G est le graphe de la relation d'incomparabilité d'un ensemble partiellement ordoné qui ne contient pas $(\mathbf{3}+\mathbf{1})$, alors X_G est une combinaison linéaire de fonctions de Schur dont les coefficients sont positifs. Puisque les fonctions symétriques élémentaires sont des combinaisons linéaires de fonctions de Schur dont les coefficients sont positifs, notre résultat confirme une conjecture de Stanley et Stembridge ([5, Conjecture 5.1] ou [6, Conjecture 5.5]).

1. Introduction

Let G be a (finite) graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_d\}$. A *coloring* of G is a function $\kappa : V \to \boldsymbol{P}$, where $\boldsymbol{P} = \{1, 2, \ldots\}$ is the set of colors. A coloring κ is called *proper* if $\kappa(u) \neq \kappa(v)$ whenever u and v are the vertices of an edge of G. The *chromatic polynomial* of G is the function $\chi_G : \boldsymbol{P} \to \boldsymbol{P}$ such that $\chi_G(n)$ is the number of proper colorings of G with n colors. (It is not difficult to see that χ_G is indeed a polynomial of degree d.) In [5] Stanley introduced and studied a symmetric

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function X_G, generalizing χ_G. It is defined as follows. Let $x_1, x_2, \ldots$ be commuting indeterminates. Then

$$X_G = X_G(x) = X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \ldots x_{\kappa(v_d)},$$

where the sum ranges over all proper colorings of G. It is immediate from the definition that $X_G(1^n) = \chi_G(n)$, where $X_G(1^n)$ is the specialization of X_G obtained by letting $x_1 = x_2 = \cdots = x_n = 1$ and $x_{n+1} = x_{n+2} = \cdots = 0$. One very interesting question is to study the coefficients that arise in the expansion of X_G in terms of the 'natural' bases for the vector space of symmetric functions Λ. (In [5] many results related to this question are proved.) In particular, we may ask whether these coefficients are nonnegative. Following [5] we say that a symmetric function f is *u-positive*, where $\{u_\lambda\}$ is a basis for Λ, if the coefficients d_λ in the expansion $f = \sum_\lambda d_\lambda u_\lambda$ are all nonnegative. A graph G is said to be u-positive if X_G is u-positive. Let s and e stand for the Schur functions and the elementary symmetric functions, respectively. A poset P is called $(\boldsymbol{a}+\boldsymbol{b})$*-free* if P does not have an induced subposet isomorphic to the direct sum $(\boldsymbol{a}+\boldsymbol{b})$ of an a-element chain and a b-element chain. The *incomparability graph* of a finite poset P, inc(P), is the graph with vertex set P and edge set $E = \{(u,v) \in P^2 \mid u \text{ and } v \text{ are incomparable in } P\}$. Stanley stated the following conjecture [5, Conjecture 5.1], which as he mentions is equivalent to [6, Conjecture 5.5].

Conjecture 1 (Stanley–Stembridge). If P is $(\mathbf{3}+\mathbf{1})$-free, then inc(P) is e-positive.

This conjecture has been verified for all posets with at most 7 elements by Stanley and Stembridge [6, pp. 277–278] and for all 8-element posets by Stembridge.

Since each e_λ is s-positive, Conjecture 1 implies that the incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets are s-positive. Further evidence in support of Conjecture 1 is Theorem 1 below, which follows easily from results of Haiman [2, Theorem 1.4] and Stembridge [7, Corollary 3.3]. An *indifference graph* (or *unit interval graph*) is an incomparability graph of a poset which is both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free.

Theorem 1. *Let G be an indifference graph. Then G is s-positive.*

In this paper we prove (a generalization of) the following theorem.

Theorem 2. *If G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset, then G is s-positive.*

This theorem provides new evidence in support of Conjecture 1. Its proof is relatively short and uses only standard facts from the theory of symmetric functions.

2. The main result

Definition. Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_d\}$. A *multicoloring* of G is a function $\kappa: V \to 2^{\boldsymbol{P}}$, where $2^{\boldsymbol{P}}$ is the set of all finite subsets of $\boldsymbol{P} = \{1, 2, \ldots\}$,

including the empty set. The multicoloring is *proper* if $\kappa(u) \cap \kappa(v) = \emptyset$ whenever u and v are the vertices of an edge of G. If $\boldsymbol{m} = (m_1, m_2, \ldots, m_d)$ is a sequence of nonnegative integers, then an ***m*-multicoloring** is a multicoloring κ such that $|\kappa(v_i)| = m_i$ for $i = 1, 2, \ldots, d$. For a finite subset $S = \{s_1, s_2, \ldots\}$ of $\boldsymbol{P}$ we define $x_S = x_{s_1} x_{s_2} \cdots$.

Definition. Let $\boldsymbol{m} = (m_1, m_2, \ldots, m_d)$ be as above. Define

$$\tilde{X}_G^{\boldsymbol{m}} = \tilde{X}_G^{\boldsymbol{m}}(x) = \tilde{X}_G^{\boldsymbol{m}}(x_1, x_2, \cdots) = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)},$$

where the sum ranges over all proper $\boldsymbol{m}$-multicolorings $\kappa : V \to 2^{\boldsymbol{P}}$.

It is clear that $\tilde{X}_G^{\boldsymbol{m}}$ is a symmetric function which generalizes X_G in the sense that $\tilde{X}_G^{(1,1,\ldots,1)} = X_G$. Then Theorem 2 is a special case of the following theorem.

Theorem 3. *If G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset with d elements, and $\boldsymbol{m} = (m_1, m_2, \ldots, m_d)$ is any sequence of nonnegative integers, then $\tilde{X}_G^{\boldsymbol{m}}$ is s-positive.*

Proof. Let $P = (P, \prec)$ be a partially ordered set. We define a *P-array* to be an array

$$\begin{array}{llll} a_{11} & a_{12} & \ldots \\ a_{21} & a_{22} & \ldots \\ \ldots \end{array}$$

of elements in P, arranged in left-justified rows, and satisfying the following condition:

$$a_{ij} \prec a_{i,j+1} \text{ whenever } a_{i,j} \text{ and } a_{i,j+1} \text{ are defined.} \tag{2.1}$$

(Note that a P-array is allowed to have empty rows in the middle of non-empty rows.) A *P-tableau* is a P-array satisfying the additional condition:

$$\text{If } i, j \geqslant 1 \text{ and } a_{i+1,j} \text{ is defined, then } a_{ij} \text{ is defined and } a_{i+1,j} \not\prec a_{ij}. \tag{2.2}$$

Such arrays were first considered by Gessel and Viennot [1]. The *weight* of an array T with entries in P is the sequence $\mathrm{wt}(T) = (n_1, n_2, \ldots)$, where n_i is the number of occurrences of v_i in T.

Let G be the incomparability graph of P. To each proper multicoloring κ of G, we can associate a P-array T_κ in the following way. For any $i \geqslant 1$, let $\{v_1^{(i)}, v_2^{(i)}, \ldots\} = \kappa^{-1}(i)$. Since κ is proper, it follows that $\kappa^{-1}(i)$ is a stable subset of V, i.e., no two vertices in $\kappa^{-1}(i)$ are connected by an edge in G. This implies that $\kappa^{-1}(i)$ is a chain in P, so we may assume that $v_1^{(i)} \prec v_2^{(i)} \prec \ldots$. Then T_κ is the array

$$\begin{array}{llll} v_1^{(1)} & v_2^{(1)} & \ldots \\ v_1^{(2)} & v_2^{(2)} & \ldots \\ \ldots \end{array}$$

It is clear that for any P-array T, there is a unique multicoloring κ of G such that $T = T_\kappa$. For a partition λ, let m_λ and h_λ denote, respectively, the monomial symmetric function and the complete symmetric function indexed by λ. It is well known (see [3, Ch. I, Section 4] for example) that the s_λ form an orthonormal basis for Λ with the inner product defined by $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$. Therefore, if $\tilde{X}_G^{\boldsymbol{m}} = \sum_\lambda c_\lambda s_\lambda$, then $c_\lambda = \langle \tilde{X}_G^{\boldsymbol{m}}, s_\lambda \rangle$. Let $l(\lambda)$ be the length of λ. The Jacobi–Trudi identity (see e.g. [3, Ch. I, (3.4)] or [4, Theorem 4.5.1]) is the following:

$$s_\lambda = \det(h_{\lambda_i - i+j})_{1 \leqslant i,j \leqslant l(\lambda)}. \tag{2.3}$$

Let S_l denote the group of permutations of $\{1,2,\ldots,l\}$. If $\lambda = (\lambda_1,\ldots,\lambda_l)$ is a partition of length l, and $\pi \in S_l$, then we denote by $\pi(\lambda)$ the sequence $\{\lambda_{\pi(j)} - \pi(j) + j\}_{j=1}^{l}$. Expanding the determinant on the right-hand side of (2.3) we get that

$$s_\lambda = \sum_{\pi \in S_l} \operatorname{sgn}(\pi) h_{\pi(\lambda)},$$

where, for any integer sequence $\alpha = (\alpha_1,\ldots,\alpha_l)$, $h_\alpha = h_{\alpha_1} \ldots h_{\alpha_l}$. (We set $h_r = 0$ if $r < 0$.) Thus $c_\lambda = \sum_{\pi \in S_l} \operatorname{sgn}(\pi) \langle \tilde{X}_G^{\boldsymbol{m}}, h_{\pi(\lambda)} \rangle$, where $\langle \tilde{X}_G^{\boldsymbol{m}}, h_{\pi(\lambda)} \rangle$ is, by the definition of the inner product $\langle -, - \rangle$, the coefficient of $m_{\pi(\lambda)}$ in $\tilde{X}_G^{\boldsymbol{m}}$, i.e., the coefficient of $x^{\pi(\lambda)}$ in $\tilde{X}_G^{\boldsymbol{m}}$, which in turn is the number of proper $\boldsymbol{m}$-multicolorings of G with weight $\pi(\lambda)$. (The *weight* $\operatorname{wt}(\kappa)$ of a multicoloring κ is the sequence $(|\kappa^{-1}(1)|, |\kappa^{-1}(2)|, \ldots)$.) Since the shape of T_κ is $\operatorname{wt}(\kappa)$, we get that $\langle \tilde{X}_G^{\boldsymbol{m}}, h_{\pi(\lambda)} \rangle$ is the number of P-arrays of shape $\pi(\lambda)$ and weight $\boldsymbol{m}$. Let

$$A = \{(\pi, T) \mid \pi \in S_l \text{ and } T \text{ is a } P\text{-array of shape } \pi(\lambda) \text{ and weight } \boldsymbol{m}\}.$$

Then $c_\lambda = \sum_{(\pi,T) \in A} \operatorname{sgn}(\pi)$. Let

$$B = \{(\pi, T) \in A \mid T \text{ is not a } P\text{-tableau}\}$$

and note that if T is a P-tableau, then $\pi(\lambda)_1 \geqslant \pi(\lambda)_2 \geqslant \ldots$, hence $\pi = id$. Thus to prove that $c_\lambda \geqslant 0$ it will be enough to find an involution $\varphi : B \to B$ such that if $(\sigma, T') = \varphi(\pi, T)$, then $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\pi)$. One such involution is constructed in [1, Proof of Theorem 11]. We describe a slight modification of it below. Let

$$T = \begin{matrix} a_{11} & a_{12} & \ldots \\ a_{21} & a_{22} & \ldots \\ \ldots \end{matrix}$$

and let $c = c(T)$ be the smallest positive integer such that (2.2) fails for some i and $j = c$. Let $r = r(T)$ be the largest i such that (2.2) fails for i and $j = c$. Then define $\sigma = \pi \circ (r, r+1)$, where $(r, r+1)$ is the permutation that interchanges r and $r+1$. Define T' to be the array obtained from T by switching the elements in row r from column c on with those in row $r+1$ after column c, so

$$T' = \begin{matrix} b_{11} & b_{12} & \ldots \\ b_{21} & b_{22} & \ldots \\ \ldots, \end{matrix}$$

where

$$b_{ij} = a_{ij} \quad \text{if } i \neq r, r+1 \quad \text{or } i = r \text{ and } j \leqslant c-1 \quad \text{or } i = r+1 \text{ and } j \leqslant c;$$

$$b_{rj} = a_{r+1,j+1} \quad \text{if } j \geqslant c \text{ and } a_{r+1,j+1} \text{ is defined;}$$
$$b_{r+1,j} = a_{r,j-1} \quad \text{if } j \geqslant c+1 \text{ and } a_{r,j-1} \text{ is defined.}$$

Note that rows r and $r+1$ in T' have lengths $\pi(\lambda)_{r+1} - 1 = \lambda_{\pi(r+1)} - \pi(r+1) + r = \lambda_{\sigma(r)} - \sigma(r) + r = \sigma(\lambda)_r$ and $\pi(\lambda)_r + 1 = \lambda_{\pi(r)} - \pi(r) + r + 1 = \lambda_{\sigma(r+1)} - \sigma(r+1) + r + 1 = \sigma(\lambda)_{r+1}$, respectively, so T' has shape $\sigma(\lambda)$. Since by assumption a_{rc} is not defined or $a_{r+1,c} \prec a_{rc}$, it follows that T' satisfies (2.1) for $i = r+1$. So to show that T' is a P-array, it suffices to show that T' satisfies (2.1) for $i = r$, i.e., $a_{r,c-1} \prec a_{r+1,c+1}$. But $a_{r+1,c-1} \prec a_{r+1,c} \prec a_{r+1,c+1}$ is a 3-element chain in P and $a_{r+1,c-1} \not\prec a_{r,c-1}$, so by the assumption that P is $(\mathbf{3}+\mathbf{1})$-free, it follows that $a_{r,c-1} \prec a_{r+1,c+1}$. Thus T' is a P-array which is not a tableau ($b_{r+1,c} = a_{r+1,c} \prec b_{rc} = a_{r+1,c+1}$ if b_{rc} is defined) and clearly $c(T') = c(T)$ and $r(T') = r(T)$. This shows that $\varphi(\sigma, T') = (\pi, T)$, so φ is an involution. Moreover, $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\pi)$. □

From the proof of Theorem 3 we get the following combinatorial interpretation of the coefficients c_λ.

Theorem 4. *If $\tilde{X}_G^{\mathbf{m}} = \sum_\lambda c_\lambda s_\lambda$, then c_λ is the number of P-tableaux of shape λ and weight $\mathbf{m}$.*

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