On inequalities of Lyapunov for linear Hamiltonian systems on time scales

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A R T I C L E   I N F O

Article history:
Received 16 January 2011
Available online 21 March 2011
Submitted by Steven G. Krantz

Keywords:
Hamiltonian system
Lyapunov inequality
Generalized zero
Time scales

A B S T R A C T

In this paper, we establish several new Lyapunov type inequalities for linear Hamiltonian systems on an arbitrary time scale \( T \) when the end-points are not necessarily usual zeroes, but rather, generalized zeroes, which generalize and improve all related existing ones including the continuous and discrete cases.

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1. Introduction

In recent years, the theory of time scales (or measure chains) has been developed by several authors with one goal being the unified treatment of differential equations (the continuous case) and difference equations (the discrete case). A time scale is an arbitrary nonempty closed subset of the real numbers \( \mathbb{R} \). Throughout this paper, we assume that \( T \) is a time scale and \( \mathbb{T} \) has the topology that it inherits from the standard topology on the real numbers \( \mathbb{R} \). The two most popular examples are \( T = \mathbb{R} \) and \( T = \mathbb{Z} \). In the next section, we’ll briefly introduce the time scale calculus and some related basic concepts of Hilger [8–10] and refer the reader to the books of Kaymakcalan et al. [12] and Bohner and Peterson [3] for further details.

Consider a linear Hamiltonian system

\[
\begin{align*}
x^\Delta(t) &= \alpha(t)x(\sigma(t)) + \beta(t)y(t), \\
y^\Delta(t) &= -\gamma(t)x(\sigma(t)) - \alpha(t)y(t),
\end{align*}
\]

on an arbitrary time scale \( T \), where \( \alpha(t), \beta(t) \) and \( \gamma(t) \) are real-valued rd-continuous functions defined on \( T \). Throughout this paper, we always assume that

\[
\beta(t) \geq 0, \quad \forall t \in T. \tag{1.2}
\]

For the second-order linear dynamic equation

\[
[p(t)x^\Delta(t)]^\Delta + q(t)x(\sigma(t)) = 0, \quad t \in \mathbb{T}, \tag{1.3}
\]
where $p(t) > 0$, and $p(t), q(t)$ are real-valued rd-continuous functions defined on $\mathbb{T}$. If we let $y(t) = p(t)x^\Delta(t)$, then (1.3) can be written as an equivalent Hamiltonian system of type (1.1):

$$x^\Delta(t) = \frac{1}{p(t)} y(t), \quad y^\Delta(t) = -q(t)x(\sigma(t)), \quad (1.4)$$

where

$$\alpha(t) = 0, \quad \beta(t) = \frac{1}{p(t)}, \quad \gamma(t) = q(t).$$

It is obvious that system (1.1) covers the continuous Hamiltonian system and discrete Hamiltonian system respectively when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, i.e.,

$$x'(t) = \alpha(t)x(t) + \beta(t)y(t), \quad y'(t) = -\gamma(t)x(t) - \alpha(t)y(t), \quad t \in \mathbb{R},$$

$$\Delta x(n) = \alpha(n)x(n+1) + \beta(n)y(n), \quad \Delta y(n) = -\gamma(n)x(n+1) - \alpha(n)y(n), \quad n \in \mathbb{Z}.$$ Furthermore, system (1.1) extends the above classical cases to some cases in between as well, such as the so-called $q$-difference equations, where

$$\mathbb{T} = q^Z := \{ q^k \mid k \in \mathbb{Z} \} \cup \{ 0 \}$$

for some $q > 1$, and difference equations with constant step size, where

$$\mathbb{T} = h\mathbb{Z} := \{ hk \mid k \in \mathbb{Z} \}$$

for some $h > 0$. Particularly useful for the discretization aspect are time scales of the form

$$\mathbb{T} = \{ tk \in \mathbb{R} \mid t_k < t_{k+1}, \ k \in \mathbb{Z} \}.$$

It is a classical topic for us to study Lyapunov type inequalities which have proved to be very useful in oscillation theory, disconjugacy, eigenvalue problems and numerous other applications in the theory of differential and difference equations. There are many literatures which improved and extended the classical Lyapunov inequality for the Hamiltonian systems including continuous and discrete cases. We refer to [2,4–6,13]. Recently, there has been much attention paid to Lyapunov-type inequality for linear Hamiltonian systems on time scales and some authors including Agarwal [1], He [7], Jiang [11] and Saker [14] have contributed the above results. Our motivation comes from the recent papers by Guseinov and Kaymakcalan [6] and Jiang and Zhou [11]. In paper [11], Jiang has obtained some interesting Lyapunov-type inequalities and these results have almost covered the corresponding continuous and discrete versions that may be found in [6].

**Theorem 1.1.** (See [11].) Suppose

$$1 - \mu(t)\alpha(t) > 0, \quad \beta(t) > 0, \quad \gamma(t) > 0, \quad \forall t \in \mathbb{T}, \quad (1.5)$$

and let $a, b \in \mathbb{T}$ with $\sigma(a) < b$. Assume that (1.1) has a real solution $(x(t), y(t))$ such that $x(a)x(\sigma(a)) < 0, x(b)x(\sigma(b)) < 0$. Then the inequality

$$\int_a^b |\alpha(t)|\Delta(t) + \left[ \int_a^b \beta(t)\Delta(t) \int_a^b \gamma(t)\Delta(t) \right]^{1/2} > 1 \quad (1.6)$$

holds, where and in the sequel

$$\gamma^+(t) = \max\{ \gamma(t), 0 \}. \quad (1.7)$$

**Theorem 1.2.** (See [11].) Suppose

$$1 - \mu(t)\alpha(t) > 0, \quad \beta(t) > 0, \quad \forall t \in \mathbb{T}, \quad (1.8)$$

and let $a, b \in \mathbb{T}$ with $\sigma(a) < b$. Assume that (1.1) has a real solution $(x(t), y(t))$ such that $x(a)x(\sigma(a)) < 0, x(\sigma(b)) = 0$. Then the inequality

$$\int_{\sigma(a)}^b |\alpha(t)|\Delta(t) + \left[ \int_{\sigma(a)}^b \beta(t)\Delta(t) \int_a^b \gamma^+(t)\Delta(t) \right]^{1/2} > 1 \quad (1.9)$$

holds.
In this paper, by using some simpler methods different from [11], we obtain several Lyapunov-type inequalities than (1.6) and (1.9)
\[
\int_a^b \left| \alpha(t) \right| \Delta(t) + \left[ \int_a^b \beta(t) \Delta(t) \int_a^b \gamma^+(t) \Delta(t) \right]^{1/2} \geq 2,
\]
(1.10)
and
\[
\int_a^b \left| \alpha(t) \right| \Delta(t) + \left[ \int_a^b \beta(t) \Delta(t) \int_a^b \gamma^+(t) \Delta(t) \right]^{1/2} \geq 2,
\]
(1.11)
only under the assumption
\[
1 - \mu(t) \alpha(t) > 0, \quad \forall t \in \mathbb{T}.
\]
(1.12)
Our results not only cover the corresponding continuous versions, but also improve greatly discrete versions that may be found in [6]. In addition, when the end-point \( b \) satisfies some general conditions (see Theorem 3.5 in Section 3), it is not necessarily a generalized zero, we also establish a better Lyapunov-type inequality than (1.10)
\[
\int_a^b \left| \alpha(t) \right| \Delta(t) + \left[ \int_a^b \beta(t) \Delta(t) \int_a^b \gamma^+(t) \Delta(t) \right]^{1/2} \geq 2.
\]
(1.13)
Instead of the usual zero, we adopt the following concept of generalized zero on time scales.

**Definition 1.3.** A function \( f : \mathbb{T} \to \mathbb{R} \) is said to have a generalized zero at \( t_0 \in \mathbb{T} \) provided either \( f(t_0) = 0 \) or \( f(t_0) f(\sigma(t_0)) < 0 \).

**2. Preliminaries about the time scales calculus**

Now, we introduce the basic notions connected to time scales. We start by defining the forward and backward jump operators.

**Definition 2.1.** (See [3].) Let \( t \in \mathbb{T} \). We define the forward jump operator \( \sigma : \mathbb{T} \to \mathbb{T} \) by
\[
\sigma(t) := \inf \{ s \in \mathbb{T} : s > t \} \quad \text{for all } t \in \mathbb{T},
\]
while the backward jump operator \( \rho : \mathbb{T} \to \mathbb{T} \) by
\[
\rho(t) := \sup \{ s \in \mathbb{T} : s < t \} \quad \text{for all } t \in \mathbb{T}.
\]
In this definition we put \( \inf \emptyset = \sup \mathbb{T} \) (i.e., \( \sigma(M) = M \) if \( \mathbb{T} \) has a maximum \( M \)) and \( \sup \emptyset = \inf \mathbb{T} \) (i.e., \( \rho(m) = m \) if \( \mathbb{T} \) has a minimum \( m \)), where \( \emptyset \) denotes the empty set. If \( \sigma(t) > t \), we say that \( t \) is right-scattered, while if \( \rho(t) < t \), we say that \( t \) is left-scattered. Also, if \( t < \sup \mathbb{T} \) and \( \sigma(t) = t \), then \( t \) is called right-dense, and if \( t > \inf \mathbb{T} \) and \( \rho(t) = t \), then \( t \) is called left-dense. Points that are right-scattered and left-scattered at the same time are called isolated. Points that are right-dense and left-dense at the same time are called dense. If \( \mathbb{T} \) has a left-scattered maximum \( M \), then we define \( \mathbb{T}^k = \mathbb{T} - \{ M \} \), otherwise \( \mathbb{T}^k = \mathbb{T} \). The graininess function \( \mu : \mathbb{T} \to [0, \infty) \) is defined by
\[
\mu(t) := \sigma(t) - t, \quad \forall t \in \mathbb{T}.
\]
We consider a function \( f : \mathbb{T} \to \mathbb{R} \) and define so-called delta (or Hilger) derivative of \( f \) at a point \( t \in \mathbb{T}^k \).

**Definition 2.2.** (See [3].) Assume \( f : \mathbb{T} \to \mathbb{R} \) is a function and let \( t \in \mathbb{T}^k \). Then we define \( f^\Delta(t) \) to be the number (provided it exists) with the property that given any \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) (i.e., \( U = (t - \delta, t + \delta) \cap \mathbb{T} \) for some \( \delta > 0 \)) such that
\[
| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) | \leq \varepsilon | \sigma(t) - s |, \quad \forall s \in U.
\]
We call \( f^\Delta(t) \) the delta (or Hilger) derivative of \( f \) at \( t \).
Lemma 2.3. (See [3].) Assume \( f, g : \mathbb{T} \to \mathbb{R} \) are differential at \( t \in \mathbb{T}^k \). Then

(i) For any constant \( a \) and \( b \), the sum \( af + bg : \mathbb{T} \to \mathbb{R} \) is differential at \( t \) with

\[
(af + bg)^\Delta(t) = af^\Delta(t) + bg^\Delta(t).
\]

(ii) If \( f^\Delta(t) \) exists, then \( f \) is continuous at \( t \).

(iii) If \( f^\Delta(t) \) exists, then \( f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t) \).

(iv) The product \( fg : \mathbb{T} \to \mathbb{R} \) is differential at \( t \) with

\[
(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).
\]

(v) If \( g(t)g(\sigma(t)) \neq 0 \), then \( f/g \) is differential at \( t \) and

\[
\left( \frac{f}{g} \right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.
\]

Definition 2.4. (See [3].) A function \( f : \mathbb{T} \to \mathbb{R} \) is called rd-continuous provided it is continuous at right-dense points in \( \mathbb{T} \) and left-sided limits exist (finite) at left-dense points in \( \mathbb{T} \) and denotes by \( C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}) \).

Definition 2.5. (See [3].) A function \( F : \mathbb{T} \to \mathbb{R} \) is called an antiderivative of \( f : \mathbb{T} \to \mathbb{R} \) provided \( F^\Delta(t) = f(t) \) holds for all \( t \in \mathbb{T}^k \). We define the Cauchy integral by

\[
\int_{\tau}^{s} f(t) \Delta t = F(s) - F(\tau), \quad \forall s, \tau \in \mathbb{T}.
\]

The following lemma gives several elementary properties of the delta integral.

Lemma 2.6. (See [3].) If \( a, b, c, t \in \mathbb{T}, k \in \mathbb{R} \) and \( f, g \in C_{rd} \), then

(i) \( \int_{a}^{b} [f(t) + g(t)]\Delta(t) = \int_{a}^{b} f(t)\Delta(t) + \int_{a}^{b} g(t)\Delta(t) \);

(ii) \( \int_{a}^{b} (kf(t))\Delta(t) = k\int_{a}^{b} f(t)\Delta(t) \);

(iii) \( \int_{a}^{b} f(t)\Delta(t) = \int_{c}^{b} f(t)\Delta(t) + \int_{a}^{c} f(t)\Delta(t) \);

(iv) \( \int_{a}^{b} f(\sigma(t))g^\Delta(t)\Delta(t) = (fg)(b) - (fg)(a) - \int_{a}^{b} f^\Delta(t)g(t)\Delta(t) \);

(v) \( \int_{a}^{b} f(s)\Delta(s) = \mu(t)f(t) \) for \( t \in \mathbb{T}^k \);

(vi) if \( |f(t)| \leq g(t) \) on \( [a, b] \), then

\[
\left| \int_{a}^{b} f(t)\Delta(t) \right| \leq \int_{a}^{b} g(t)\Delta(t).
\]

The notation \( [a, b], (a, b] \) and \( [a, +\infty) \) will denote time scales intervals. For example, \( [a, b] = \{ t \in \mathbb{T} : a \leq t < b \} \). To prove our results, we present the following lemma.

Lemma 2.7 (Cauchy–Schwarz inequality). (See [3].) Let \( a, b \in \mathbb{T} \). For \( f, g \in C_{rd} \) we have

\[
\left( \int_{a}^{b} |f(t)g(t)|\Delta(t) \right)^{\frac{1}{2}} \leq \left( \int_{a}^{b} f^{2}(t)\Delta(t) \right)^{\frac{1}{2}} \cdot \left( \int_{a}^{b} g^{2}(t)\Delta(t) \right)^{\frac{1}{2}}.
\]

Lemma 2.8. (See [3].) Let

\[
A = \{ t \in \mathbb{T} : t \text{ is left-dense and right-scattered} \}, \quad B = \{ t \in \mathbb{T} : t \text{ is right-dense and left-scattered} \}.
\]

Then

\[
\sigma(\rho(t)) = t, \quad \forall t \in \mathbb{T} \setminus A; \quad \rho(\sigma(t)) = t, \quad \forall t \in \mathbb{T} \setminus B.
\]
3. Lyapunov type inequalities

In this section, we establish some new Lyapunov type inequalities on time scales \( \mathbb{T} \).

**Theorem 3.1.** Suppose that (1.12) holds and let \( a, b \in \mathbb{T}^k \) with \( \sigma(a) \leq b \). Assume (1.1) has a real solution \((x(t), y(t))\) such that \( x(t) \) has generalized zeroes at end-points \( a \) and \( b \) and \( x(t) \) is not identically zero on \([a, b]\), i.e.,

\[
\begin{align*}
  x(a) &= 0 \quad \text{or} \quad x(a)x(\sigma(a)) < 0; \quad x(b) &= 0 \quad \text{or} \quad x(b)x(\sigma(b)) < 0;
  \max_{a \leq t \leq b} |x(t)| &> 0.
\end{align*}
\]

Then one has the following inequality

\[
\begin{align*}
  \int_a^b |\alpha(t)| \Delta(t) + \left[ \int_a^b \beta(t) \Delta(t) \int_a^b \gamma^+(t) \Delta(t) \right]^{1/2} \geq 2.
\end{align*}
\]

**Proof.** It follows from (3.1) that there exist \( \xi, \eta \in [0, 1) \) such that

\[
(1 - \xi)x(a) + \xi x(\sigma(a)) = 0,
\]

and

\[
(1 - \eta)x(b) + \eta x(\sigma(b)) = 0.
\]

Multiplying the first equation of (1.1) by \( y(t) \) and the second one by \( x(\sigma(t)) \), and then adding, we get

\[
\begin{align*}
  [x(t)y(t)]^\Delta &= \beta(t)y^2(t) - \gamma(t)x^2(\sigma(t)).
\end{align*}
\]

Integrating Eq. (3.5) from \( a \) to \( b \), we can obtain

\[
\begin{align*}
  x(b)y(b) - x(a)y(a) &= \int_a^b \beta(t)y^2(t)\Delta t - \int_a^b \gamma(t)x^2(\sigma(t))\Delta t.
\end{align*}
\]

From the first equation of (1.1) and using Lemma 2.3(iii), we have

\[
\begin{align*}
  \left[ 1 - \mu(t)\alpha(t) \right] x(\sigma(t)) &= x(t) + \mu(t)\beta(t)y(t).
\end{align*}
\]

Combining (3.7) with (3.3), we have

\[
\begin{align*}
  x(a) = -\frac{\xi \mu(a)\beta(a)}{1 - (1 - \xi)\mu(a)\alpha(a)} y(a).
\end{align*}
\]

Similarly, it follows from (3.7) and (3.4) that

\[
\begin{align*}
  x(b) = -\frac{\eta \mu(b)\beta(b)}{1 - (1 - \eta)\mu(b)\alpha(b)} y(b).
\end{align*}
\]

Substituting (3.8) and (3.9) into (3.6), we have

\[
\begin{align*}
  \int_a^b \beta(t)y^2(t)\Delta t - \int_a^b \gamma(t)x^2(\sigma(t))\Delta t &= -\frac{\eta \mu(b)\beta(b)}{1 - (1 - \eta)\mu(b)\alpha(b)} y^2(b) + \frac{\xi \mu(a)\beta(a)}{1 - (1 - \xi)\mu(a)\alpha(a)} y^2(a),
\end{align*}
\]

by using Lemma 2.6(v), we get

\[
\begin{align*}
  \frac{(1 - \xi)[1 - \mu(a)\alpha(a)]}{1 - (1 - \xi)\mu(a)\alpha(a)} \mu(a)\beta(a)y^2(a) + \int_{\sigma(a)}^b \beta(t)y^2(t)\Delta t + \frac{\eta \mu(b)\beta(b)}{1 - (1 - \eta)\mu(b)\alpha(b)} y^2(b) = \int_a^b \gamma(t)x^2(\sigma(t))\Delta t.
\end{align*}
\]

Denote that

\[
\begin{align*}
  \tilde{\beta}(a) &= \frac{(1 - \xi)[1 - \mu(a)\alpha(a)]}{1 - (1 - \xi)\mu(a)\alpha(a)} \beta(a),
  \tilde{\beta}(b) &= \frac{\eta}{1 - (1 - \eta)\mu(b)\alpha(b)} \beta(b).
\end{align*}
\]
and
\[ \hat{\beta}(t) = \beta(t), \quad \sigma(a) \leq t \leq \rho(b). \] (3.13)

Then we can rewrite (3.10) as
\[
\int_{a}^{b} \hat{\beta}(t)y^2(t)\Delta t = \int_{a}^{b} \gamma(t)y^2(\sigma(t))\Delta t.
\] (3.14)

On the other hand, integrating the first equation of (1.1) from \(a\) to \(\tau\) and using (3.8), (3.11), (3.13) and Lemma 2.6(v), we obtain
\[
\begin{align*}
x(\tau) &= x(a) + \int_{a}^{\tau} \alpha(t)x(\sigma(t))\Delta t + \int_{a}^{\tau} \beta(t)y(t)\Delta t \\
&= -\frac{\xi(\mu(a))\beta(a)}{1 - (1 - \xi(\mu(a)))} y(a) + \int_{a}^{\tau} \alpha(t)x(\sigma(t))\Delta t + \int_{a}^{\tau} \beta(t)y(t)\Delta t \\
&= \int_{a}^{\tau} \alpha(t)x(\sigma(t))\Delta t - \int_{a}^{\tau} \beta(t)y(t)\Delta t, \quad \sigma(a) \leq \tau \leq b.
\end{align*}
\] (3.15)

Similarly, integrating the first equation of (1.1) from \(\tau\) to \(b\) and using (3.9), (3.12), (3.13) and Lemma 2.6(v), we have
\[
\begin{align*}
x(\tau) &= x(b) - \int_{\tau}^{b} \alpha(t)x(\sigma(t))\Delta t - \int_{\tau}^{b} \beta(t)y(t)\Delta t \\
&= -\frac{\eta(\mu(b))\beta(b)}{1 - (1 - \eta(\mu(b)))} y(b) - \int_{\tau}^{b} \alpha(t)x(\sigma(t))\Delta t - \int_{\tau}^{b} \beta(t)y(t)\Delta t \\
&= -\int_{\tau}^{b} \alpha(t)x(\sigma(t))\Delta t + \int_{\tau}^{\sigma(b)} \hat{\beta}(t)y(t)\Delta t, \quad \sigma(a) \leq \tau \leq b.
\end{align*}
\] (3.16)

It follows from (3.15), (3.16) and Lemma 2.6 that
\[
\begin{align*}
|\chi(\tau)| &\leq \int_{a}^{\tau} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{a}^{\hat{\beta}(t)} |y(t)|\Delta t, \quad \sigma(a) \leq \tau \leq b,
\end{align*}
\]
and
\[
\begin{align*}
|\chi(\tau)| &\leq \int_{\tau}^{b} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{\tau}^{\sigma(b)} |\hat{\beta}(t)||y(t)|\Delta t, \quad \sigma(a) \leq \tau \leq b.
\end{align*}
\]

Adding the above two inequalities, we have
\[
2|\chi(\tau)| \leq \int_{a}^{b} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{a}^{\sigma(b)} |\hat{\beta}(t)||y(t)|\Delta t, \quad \sigma(a) \leq \tau \leq b.
\] (3.17)

Let \(|\chi(\tau^*)| = \max_{\sigma(a) \leq t \leq b} |\chi(t)|\). There are two possible cases:

(1) end-point \(b\) is left-scattered;
(2) end-point \(b\) is left-dense.
Case (1). In this case, it follows from Lemma 2.8 that $\sigma(\rho(b)) = b$. Hence,

$$
\int_a^b |\alpha(t)||x(\sigma(t))|\Delta t = \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t + \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t
$$

$$
= \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t + \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t
$$

$$
= \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t + \mu(\rho(b))|\alpha(\rho(b))||x(\sigma(\rho(b)))|
$$

$$
\leq |x(\tau^*)| \left[ \int_a^b |\alpha(t)|\Delta t + \mu(\rho(b))|\alpha(\rho(b))| \right]
$$

$$
\leq |x(\tau^*)| \int_a^b |\alpha(t)|\Delta t.
$$

That is

$$
\int_a^b |\alpha(t)||x(\sigma(t))|\Delta t \leq |x(\tau^*)| \int_a^b |\alpha(t)|\Delta t.
$$

(3.18)

Similarly, we have

$$
\int_a^b \gamma^+(t)x^2(\sigma(t))\Delta t \leq |x(\tau^*)|^2 \int_a^b \gamma^+(t)\Delta t.
$$

(3.20)

Case (2). In this case, there exists a sequence $\{bn\}$ of $T$ such that

$$
a < b_1 < b_2 < b_3 < \cdots < b_n < \cdots < b, \quad \lim_{n \to \infty} b_n = b.
$$

Hence

$$
\int_a^b |\alpha(t)||x(\sigma(t))|\Delta t = \int_a^{b_1} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{b_1}^{b_2} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{b_2}^{b_3} |\alpha(t)||x(\sigma(t))|\Delta t + \cdots
$$

$$
\leq |x(\tau^*)| \int_a^{b_1} |\alpha(t)|\Delta t + \int_{b_1}^{b_2} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{b_2}^{b_3} |\alpha(t)||x(\sigma(t))|\Delta t + \cdots
$$

$$
\leq |x(\tau^*)| \int_a^{b} |\alpha(t)|\Delta t, \quad n \to \infty,
$$

which implies that (3.19) holds. Similarly, we can prove that (3.20) holds as well. Applying Lemma 2.7 and using (3.14), (3.19) and (3.20), we have
Thus, it follows from Lemma 2.6(iii) and (v) and the assumption that the ones of the latter. Furthermore, the assumptions of the former is weaker than those of (3.14) and (3.17), respectively. It is easy to see that (3.23) holds because

\[
2|x(\tau^*)| \leq \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t + \int_a^b \tilde{\beta}(t)|y(t)|\Delta t
\]

\[
\leq |x(\tau^*)| \int_a^b |\alpha(t)|\Delta t + \left[ \int_a^b \hat{\beta}(t)\Delta t + \int_a^b \tilde{\beta}(t)\tilde{y}^2(t)\Delta t \right]^{1/2}
\]

\[
= |x(\tau^*)| \int_a^b |\alpha(t)|\Delta t + \left[ \int_a^b \hat{\beta}(t)\Delta t \int_a^b \gamma(t)x^2(\sigma(t))\Delta t \right]^{1/2}
\]

\[
\leq |x(\tau^*)| \left\{ \int_a^b |\alpha(t)|\Delta t + \left[ \int_a^b \hat{\beta}(t)\Delta t \int_a^b \gamma^+(t)\Delta t \right]^{1/2} \right\}.
\]

(3.21)

Dividing the latter inequality of (3.21) by \(|x(\tau^*)|\), we obtain

\[
\int_a^b |\alpha(t)|\Delta t + \left[ \int_a^b \tilde{\beta}(t)\Delta t \int_a^b \gamma^+(t)\Delta t \right]^{1/2} \geq 2.
\]

(3.22)

Since \(\mu(t) \geq 0, 1 - \mu(t)\alpha(t) > 0\) and \(\xi, \eta \in [0, 1]\), we have

\[
\tilde{\beta}(t) \leq \beta(t), \quad a \leq t < \sigma(b),
\]

then it follows from (3.22) that (3.2). \(\square\)

**Remark 3.2.** It is obvious that the Lyapunov type inequality (3.2) of Theorem 3.1 is better than (1.6) of Theorem 1.1 for the bound 2 in the right side of (3.2) is better than that of (1.6). Furthermore, the assumptions of the former is weaker than the ones of the latter.

In case \(x(b) = 0\), i.e. \(\eta = 0\), then we have the following equation

\[
\int_a^b \tilde{\beta}(t)y^2(t)\Delta t = \int_a^b \gamma(t)x^2(\sigma(t))\Delta t
\]

(3.23)

and inequality

\[
2|x(\tau)| \leq \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t + \int_a^b \tilde{\beta}(t)|y(t)|\Delta t, \quad \sigma(a) \leq \tau \leq b.
\]

(3.24)

instead of (3.14) and (3.17), respectively. It is easy to see that (3.23) holds because \(\tilde{\beta}(b) = 0\). Next, we prove (3.24) is true. If \(b\) is left-dense, then \(\rho(b) = b\), and so (3.24) holds. If \(b\) is left-scattered, then it follows from Lemma 2.8 that \(\sigma(\rho(b)) = b\). Thus, it follows from Lemma 2.6(iii) and (v) and the assumption \(x(b) = 0\) that

\[
\int_a^b |\alpha(t)||x(\sigma(t))|\Delta t = \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t + \int_a^b \tilde{\beta}(t)|y(t)|\Delta t
\]

\[
\leq \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t + \mu(\rho(b))|\alpha(\rho(b))||x(b)| = \int_a^b |\alpha(t)||x(\sigma(t))|\Delta t,
\]

which, together with (3.17) and the fact that \(\tilde{\beta}(b) = 0\), implies that (3.24) holds. Similar to the proof of (3.22), we have

\[
\int_a^b |\alpha(t)|\Delta t + \left[ \int_a^b \tilde{\beta}(t)\Delta t \int_a^b \gamma^+(t)\Delta t \right]^{1/2} \geq 2.
\]

(3.25)
Since \( \beta(t) \leq \beta(t) \) for \( a \leq t \leq b \), it follows that
\[
\int_a^b |\alpha(t)| \Delta(t) + \left[ \int_a^b \beta(t) \Delta(t) \int_a^b \gamma^+(t) \Delta(t) \right]^{1/2} \geq 2. \tag{3.26}
\]

Therefore, we can obtain the following theorem.

**Theorem 3.3.** Suppose that (1.12) holds and let \( a, b \in \mathbb{T}^k \) with \( \sigma(a) \leq \rho(b) \). Assume (1.1) has a real solution \((x(t), y(t))\) such that \( x(t) \) has a generalized zero at end-point \( a \) but a usual zero at end-point \( b \) and \( x(t) \) is not identically zero on \([a, b] \), i.e.,
\[
x(a) = 0 \quad \text{or} \quad x(a) \sigma(\alpha(a)) < 0; \quad x(b) = 0; \quad \max_{a \leq t \leq b} |x(t)| > 0.
\]

Then inequality (3.26) holds.

**Remark 3.4.** In view of the proof of Theorem 3.3, in case both end-points \( a \) and \( b \) are usual zeros, i.e. \( x(a) = x(b) = 0 \), then assumption (1.12) can be dropped in Theorem 3.3.

While the end-point \( b \) is not necessarily a generalized zero of \( x(t) \), we still can establish the following more general theorem.

**Theorem 3.5.** Suppose that (1.12) holds and let \( a, b \in \mathbb{T}^k \) with \( \sigma(a) \leq b \). Assume (1.1) has a real solution \((x(t), y(t))\) such that \( x(t) \) has a generalized zero at end-point \( a \) and \((x(b), y(b)) = (\lambda_1 x(a), \lambda_2 y(a))\) with \( 0 < \lambda_2^2 \leq \lambda_1 \lambda_2 \leq 1 \) and \( x(t) \) is not identically zero on \([a, b]\). Then one has the following inequality
\[
\int_a^b |\alpha(t)| \Delta(t) + \left[ \int_a^b \beta(t) \Delta(t) \int_a^b \gamma^+(t) \Delta(t) \right]^{1/2} \geq 2. \tag{3.27}
\]

**Proof.** It follows from the assumption \( x(t) = 0 \) or \( x(t) \sigma(\sigma(t)) < 0 \) that there exists \( \xi \in [0, 1) \) such that (3.3) holds. Further, by the proof of Theorem 3.1, (3.5)–(3.8) hold. Since \((x(b), y(b)) = (\lambda_1 x(a), \lambda_2 y(a))\), then by (3.6), we have
\[
(\lambda_1 \lambda_2 - 1)x(a)y(a) = \int_a^b \beta(t) y^2(t) \Delta t - \int_a^b \gamma(t) x^2(\sigma(t)) \Delta t. \tag{3.28}
\]

Substituting (3.8) into (3.28), we have
\[
\int_a^b \beta(t) y^2(t) \Delta t - \int_a^b \gamma(t) x^2(\sigma(t)) \Delta t = \frac{(1 - \lambda_1 \lambda_2) \xi \mu(a) \beta(a)}{1 - (1 - \xi) \mu(a) \alpha(a)} y^2(a),
\]
which implies that
\[
k_1 \mu(a) \beta(a) y^2(a) + \int_{\sigma(a)}^b \beta(t) y^2(t) \Delta t = \int_a^b \gamma(t) x^2(\sigma(t)) \Delta t, \tag{3.29}
\]
where
\[
k_1 = \frac{(1 - \xi)(1 - \mu(a) \alpha(a)) + \lambda_1 \lambda_2 \xi}{1 - (1 - \xi) \mu(a) \alpha(a)}. \tag{3.30}
\]

On the other hand, integrating the first equation of (1.1) from \( a \) to \( \tau \) and using (3.8) and Lemma 2.6(v), we can obtain the following inequality which is similar to (3.15):
\[
x(\tau) = \frac{(1 - \xi)[1 - \mu(a) \alpha(a)]}{1 - (1 - \xi) \mu(a) \alpha(a)} \mu(a) \beta(a) y(a) + \int_a^r \alpha(t) x(\sigma(t)) \Delta t + \int_a^r \beta(t) y(t) \Delta t, \quad \sigma(a) \leq \tau \leq b. \tag{3.31}
\]

Similarly, integrating the first equation of (1.1) from \( \tau \) to \( b \) and using (3.8), Lemma 2.6(v) and the fact that \( x(b) = \lambda_1 x(a) \), we have
\[ x(\tau) = x(b) - \int_{\tau}^{b} \alpha(t)x(\sigma(t))\Delta t - \int_{\tau}^{b} \beta(t)y(t)\Delta t \]
\[ = \lambda_1 x(a) - \int_{\tau}^{b} \alpha(t)x(\sigma(t))\Delta t - \int_{\tau}^{b} \beta(t)y(t)\Delta t \]
\[ = -\frac{\lambda_1 \mu(a)}{1 - (1 - \xi) \mu(a) \alpha(a)} \beta(a)y(a) - \int_{\tau}^{b} \alpha(t)x(\sigma(t))\Delta t - \int_{\tau}^{b} \beta(t)y(t)\Delta t, \quad \sigma(a) \leq \tau \leq b. \quad (3.32) \]

From (3.31) and (3.32), we obtain
\[ |x(\tau)| \leq \frac{(1 - \xi)[1 - \mu(a)\alpha(a)]}{1 - (1 - \xi) \mu(a) \alpha(a)} \mu(a) \beta(a) |y(a)| + \int_{\alpha(a)}^{\tau} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{\sigma(a)}^{\tau} \beta(t)|y(t)|\Delta t, \quad \sigma(a) \leq \tau \leq b, \]
and
\[ |x(\tau)| \leq \frac{\lambda_1 |\xi|}{1 - (1 - \xi) \mu(a) \alpha(a)} \mu(a) \beta(a) |y(a)| + \int_{\tau}^{b} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{\tau}^{b} \beta(t)|y(t)|\Delta t, \quad \sigma(a) \leq \tau \leq b. \]

Adding the above two inequalities, we have
\[ 2 |x(\tau)| \leq \kappa_2 \mu(a) \beta(a) |y(a)| + \int_{\alpha(a)}^{\tau} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{\sigma(a)}^{\tau} \beta(t)|y(t)|\Delta t, \quad \sigma(a) \leq \tau \leq b, \quad (3.33) \]

where
\[ \kappa_2 = \frac{(1 - \xi)[1 - \mu(a)\alpha(a)] + |\lambda_1| \xi}{1 - (1 - \xi) \mu(a) \alpha(a)}. \quad (3.34) \]

Let \( |x(\tau^*)| = \max_{\sigma(a) \leq \tau \leq b} |x(\tau)| \). Applying Lemmas 2.6 and 2.7 and using (3.19), (3.20), (3.29) and (3.33), we have
\[ 2 |x(\tau^*)| \leq \kappa_2 \mu(a) \beta(a) |y(a)| + \int_{\alpha(a)}^{\tau} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{\sigma(a)}^{\tau} \beta(t)|y(t)|\Delta t \]
\[ \leq |x(\tau^*)| \left[ \int_{\alpha(a)}^{\tau} |\alpha(t)|\Delta t + \left[ \frac{\kappa_2}{K_1} \mu(a) \beta(a) + \int_{\sigma(a)}^{\tau} \beta(t)\Delta t \right] \right]^{1/2} \]
\[ = |x(\tau^*)| \left[ \int_{\alpha(a)}^{\tau} |\alpha(t)|\Delta t + \left[ \frac{\kappa_2}{K_1} \mu(a) \beta(a) + \int_{\sigma(a)}^{\tau} \beta(t)\Delta t \right] \right]^{1/2} \]
\[ \leq \kappa_2 \mu(a) \beta(a) |y(a)| + \int_{\alpha(a)}^{\tau} |\alpha(t)||x(\sigma(t))|\Delta t + \int_{\sigma(a)}^{\tau} \beta(t)|y(t)|\Delta t \]
\[ \leq |x(\tau^*)| \left[ \int_{\alpha(a)}^{\tau} |\alpha(t)|\Delta t + \left[ \frac{\kappa_2}{K_1} \mu(a) \beta(a) + \int_{\sigma(a)}^{\tau} \beta(t)\Delta t \right] \right]^{1/2}. \quad (3.35) \]

Dividing the latter inequality of (3.35) by \(|x(\tau^*)|\), we obtain
\[ \int_{\alpha(a)}^{\tau} |\alpha(t)|\Delta t + \left[ \frac{\kappa_2}{K_1} \mu(a) \beta(a) + \int_{\sigma(a)}^{\tau} \beta(t)\Delta t \right]^{1/2} \geq 2. \quad (3.36) \]

Set \( d = 1 - (1 - \xi) \mu(a) \alpha(a) \). Since \((1 - \xi)(1 - \mu(a)\alpha(a)) > 0\), it follows that \( d > \xi \geq 0 \), and so
\[ [d - (1 - |\lambda_1|)\xi]^2 \leq d[d - (1 - \lambda_1\lambda_2)\xi] \].
This, together with (3.30) and (3.34), implies that

\[
\frac{\kappa_2^2}{\kappa_1} = \frac{(1-\xi)(1-\mu(a)\alpha(\xi)) + |\lambda_1|^2}{d - (1 - |\lambda_1|)|\xi|^2} \leq 1.
\]

Substituting this into (3.36), we obtain (3.27). □

References