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Permutation Complexes and Modular Representation Theory

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INTRODUCTION

Let G be a finite group, and R be a commutative ring with identity. We denote by $\mathcal{M}(RG)$ the category of RG-modules. For any subgroup $H \subseteq G$, one has two basic functors $\mathcal{M}(RG) \xrightarrow{\mathrm{Res}_H^G} \mathcal{M}(RH)$ and $\mathcal{M}(RH) \xrightarrow{\mathrm{Ind}_H^G} \mathcal{M}(RG)$, given by restriction and induction, which play an essential role in representation theory. An important and elementary class of RG-representations is permutation modules which are direct sums of modules $\mathrm{Ind}_H^G(R)$ obtained by induction from the trivial RH-module R for various $H \subseteq G$. In another extreme, one has RG-modules which arise by induction from RH-projective modules, leading to the concept of relative projectivity and Green's theory of vertices and sources [CR], [GR]. The value of these subcategories of modules in representation theory and related areas is well known. In a different direction (influenced by algebraic geometry and topology), one considers not only module categories, but various categories of chain complexes of modules and their cohomologies. This culminates in the more recent approaches to representation theory through the theory of derived categories. See [Sc], [CPS] and their many references.

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A natural problem is to develop and study generalizations of induction–restriction theories in the set-up of derived categories. Of course, one has the various generalizations of the restriction and induction functors to the categories of chain complexes. However, most natural examples of RG-chain complexes which arise in applications are complexes whose constituent chain modules only happen to be permutation modules. This leads to the study of complexes of permutation modules and the representations afforded by their homologies. On the other hand such RG-complexes are far too general for the purposes of induction–restriction theory. For example, an RG-free resolution C_* of an arbitrary RG-module M may be thought of as a complex of permutation modules whose only non-vanishing homology $H_0(C_*) = M$. Moreover, the usual finiteness conditions in the derived category lead to undue restriction. For example, if we require further that C_* above be quasi-isomorphic to a bounded RG-free chain complex, then M will be very close to being RG-projective. For instance, if R is a field of characteristic p and G is a p-group, then RG is a local ring, and M is necessarily RG-free. Thus the familiar conditions in the derived category lead to either severe restrictions or unmanageable generality.

A middle ground is provided by "permutation complexes" which form a restricted and proper subcategory of the complexes of permutation modules. See Section 1 for exact definitions. In particular, permutation complexes which are quasi-isomorphic to bounded permutation complexes form a distinguished and suitably large subcategory with a rich structure. Homology representations afforded by bounded permutation complexes demonstrate remarkable properties which make them desirable objects of study. In practice, such complexes arise naturally in the combinatorial approach to group theory, topology, and algebraic geometry (see Section 1).

The theme of the present paper is a preliminary study of the deep relationship between the representation-theoretic and homological properties of permutation complexes and their homology representations from a local-to-global point of view. In particular, we prove a localization theorem (Theorem 2.1) which is an elementary but basic tool. A projectivity criterion (Theorem 3.3) is applied to relate the present subject to more familiar constructions in group theory (Theorem 3.4). We introduce and study a Hermitian analogue of the theory in Section 4 which is applied to some well-known and classical topics in fixed point theory of topological transformation groups (Theorem 4.5 and Corollary 4.13). In Section 5 we study the so-called invertible elements of the stable Green ring and endo-trivial homology representations.

1. Permutation Complexes

Let S be a G-set, i.e., a disjoint union of left cosets G/H for various $H \subseteq G$. The free R-module whose basis is given by S is denoted by $R[S]$. The trivial G-action on R and the left action of G on S give $R[S]$ the structure of an RG-module. $R[S]$ is called the permutation module with permutation basis S. If $S = \emptyset$, $R[S] = 0$. A *complex of permutation modules* is a chain (cochain) RG-complex C_* such that each C_i is a permutation module. A special case occurs in the following:

1.1. Definition. Let $\mathscr{S} = \bigsqcup_{i \in \mathbb{Z}} S_i$ be a disjoint union of G-sets. An RG-complex X_* is called a permutation complex with permutation basis $\mathscr{S}$ if

(1) each $X_i = R[S_i]$ is a permutation module with basis S_i:

(2) the boundary homomorphisms $\partial_i : C_i \to C_{i-1}$ is RG-linear and satisfies $\partial_i(S_i^H) \subset R[S_{i-1}^H]$ for each $H \subseteq G$.

It follows that $\bigoplus_{i \in \mathbb{Z}} R[S_i^H] \subseteq X_*$ is a subcomplex, which we denote by $X_*(H)$. It is clear that condition (2) of 1.1 is equivalent to the following:

(2)′ For each $H \subseteq G$, the graded submodule $X_*(H)$ is a subcomplex of X_*. We call $X_*(H)$ the subcomplex of H-fixed points of X_*. The equivalent properties (2) and (2)′ tie the local and global structures of X_* together and impose non-trivial restrictions on the homology representations of bounded permutaion complexes. The isotropy (or stability) subgroups of $\mathscr{S}$ are called also the isotropy subgroups of X^*. With respect to the natural action of $N_G(H)/H$ on S_i^H, $X^*(H)$ becomes an $R[N_G(H)/H]$-permutation complex, and restricting actions to $N_G(H)$ yields a pair of $N_G(H)$-permutation complexes $(X_*, X_*(H))$. Let $\mathscr{C}(RG)$ be the category of RG-complexes and RG-chain maps. There are two subcategories of $\mathscr{C}(RG)$ whose objects consist of permutation complexes. The first one is $\mathscr{P}(RG)$, where the morphisms are the chain maps $X_* \to Y_*$ which are induced from the G-maps of the permutation bases (as G-sets) of X_* and Y_*. The second category is $\hat{\mathscr{P}}(RG)$, which is the full subcategory of $\mathscr{C}(RG)$ whose objects are the same as the objects of $\mathscr{P}(RG)$. $\mathscr{P}(RG)$ is closed under most of the familiar constructions: quotient complexes, mapping cylinders, mapping cones, push-outs, etc.

1.2. Definition. Let X_* be a positive permutation complex, and let $\underline{R}$ be concentrated in degree zero. X_* is called *based* if there is a split augmentation in $\mathscr{P}(RG)$ $X_* \underset{\varepsilon}{\leftrightarrows} \underline{R}$, so that $X_* \cong \sigma(\underline{R}) \oplus \mathrm{Ker}(\varepsilon)$ in $\mathscr{P}(RG)$. Based complexes and based chain homomorphisms form a subcategory $\mathscr{P}_0(RG)$.

1.3. *Constructions on permutation complexes.* Let X_* and Y_* be permutation complexes with permutation bases $A=\bigsqcup_{n\in\mathbb{Z}} A_n$, $B=\bigsqcup_{n\in\mathbb{Z}} B_n$, and let X'_* and Y'_* be based permutation complexes with split augmentations

$$X'_0 \underset{\varepsilon_1}{\overset{\sigma_1}{\leftrightarrows}} \underline{R} \quad \text{and} \quad Y'_0 \underset{\varepsilon_2}{\overset{\sigma_2}{\leftrightarrows}} \underline{R}.$$

We have the following constructions in $\mathscr{P}(RG)$:

(i) Direct sum $X_* \oplus Y_*$ corresponding to the disjoint union $A \sqcup B$.

(ii) Tensor product $X_* \otimes Y_*$ corresponding to the cartesian product $A \times B$.

(iii) m-fold shift for $m \in \mathbb{Z}$ by shifting the grading of the basis, or equivalently, $(X_*[m])_i = X_{i-m}$.

(iv) Wedge $X'_* \vee Y'_* = Z'_*$ in the subcategory of based complexes $\mathscr{P}_0(RG)$ is defined by $Z'_i = X_i \oplus Y_i$ for $i \geqslant 1$, and Z'_0 is the push-out

$$\begin{tikzcd} R \arrow[r, "\sigma_1"] \arrow[d, "\sigma_2"'] & X' \arrow[d] \\ Y'_0 \arrow[r] & Z'_0 \end{tikzcd}$$

together with the induced split augmentation $Z'_0 \leftrightarrows R$ from this square. One may think of $X'_* \vee Y'_*$ as "sum" in $\mathscr{P}_0(RG)$.

(v) Product in $\mathscr{P}_0(RG)$ is the smash-product $X'_* \wedge Y'_*$ defined as the pull-back

$$\begin{tikzcd} X'_* \wedge Y'_* \arrow[r] \arrow[d] & X'_* \arrow[d, "\varepsilon_1"] \\ Y'_* \arrow[r, "\varepsilon_2"] & \underline{R} \end{tikzcd}$$

Equivalently, let $X'_* \vee Y'_* \equiv X'_* \otimes \sigma_2(\underline{R}) \vee \sigma_1(\underline{R}) \otimes Y'_*$ and $(X'_* \wedge Y'_*)_i = ((X'_* \otimes Y'_*)/(X'_* \vee Y'_*))_i$ for $i \geqslant 1$ and for $i=0$ the pull-back diagram of RG-modules:

$$\begin{tikzcd} (X'_* \wedge Y'_*)_0 \arrow[r] \arrow[d] & X'_0 \arrow[d, "\varepsilon_1"] \\ Y'_0 \arrow[r] & R \end{tikzcd}$$

(vi) Reduced suspension in $\mathscr{P}_0(RG)$ of X'_* is the based complex $\Sigma X'_*$ defined by $(\Sigma X_*)_{i+1} = X_i$ for $i \geqslant 0$, and $(\Sigma X_*)_0 = R \oplus R$ with $(\Sigma\partial)_{i+1} = \partial_i$ and $\Sigma\partial_0 : (\Sigma X_*)_1 \to (\Sigma X_*)$ given by $\varepsilon : X_0 \to (R)_1 =$ first factor in $(\Sigma X_*)_0$. The split augmentation is provided by the projection onto the second factor of $(\Sigma X_*)_0$. The iteration of suspension for each $n \geqslant 1$ is denoted by $\Sigma^n X_*$. This is the analogue of the shift in (iii) for $\mathscr{P}_0(RG)$.

(vii) In addition, there are other constructions suggested by their analogues for topological spaces, e.g., the join $X_* \circ Y_*$, the cone on X_* denoted by cX_*, or unreduced suspension in $\mathscr{P}(RG)$. We leave these, and the verification of the fact that most of the other familiar constructions for chain complexes (e.g., mapping cylinders, mapping cones) can be performed in $\mathscr{P}(RG)$ or $\mathscr{P}_0(RG)$, to the reader. The proof of this lemma follows from definitions and is omitted.

1.4. Lemma. *The above constructions are functorial in $\mathscr{P}(RG)$ and $\mathscr{P}_0(RG)$. In particular, they commute with the formation of "subcomplexes of fixed points," e.g., $(X_* \wedge Y_*)(H) = X_*(H) \wedge Y_*(H)$.*

1.5. *Important Remark.* In literature, the terminology "permutation complex" occurs in various contexts with different meanings. Often, what we refer to as "a complex of permutation modules" (i.e., only condition (1) of Definition 1.1 above) is called "a permutation complex" and condition (2) is not imposed. See, e.g., Arnold [Ar1], [Ar2], Adem [Ad1], [Ad2], and Justin Smith [Sm1]. See [A1, Chap. VIII] for further references.

1.6. Examples. (1) It is obvious from the definition that a complex of permutation modules need not satisfy condition (2) of Definition 1.1. For instance, let $C_0 = \mathbb{Z}G$, $C_1 = \mathbb{Z}$, and $\partial : C_1 \to C_0$ be the norm map $\partial(1) = \sum_{g \in G} g$.

(2) Permutation complexes arise naturally in the combinatorial approach to finite group theory, e.g., as in Ken Brown [B1], [B2], Quillen [Q2], Webb [W1], [W2], S. D. Smith [Sd1], and their references. One considers a partially ordered set of subgroups of G and chooses the permutation basis in dimension n to be the chains of length n. The G-action is induced from the conjugation by elements of G.

(3) If X is a simplicial complex and elements of G act on X by simplicial maps, then the simplicial chains of the second barycentric subdivision of X yield a permutation complex. See Bredon [Bdn, Chap. 2].

(4) More generally, if X is a G–CW complex (see Bredon [Bdn, Chap. 2], also Illmann [I] for related discussion of G–CW complexes), then the complex $C_*(X)$ of cellular chains of X is a permutation complex.

If $X^G \neq \emptyset$, then $C_*(X)$ will be a based permutation complex if we choose a base point in X^G. In (3) and (4), $C_*(H)$ corresponds to the simplicial and cellular chains of X^H.

(5) Smooth G-manifolds as well as complex algebraic subsets of $\mathbb{C}^n$ and $\mathbb{C}P^n$ with algebraic G-actions also admit triangulations with simplicial G-actions. See Illmann [I] and Hironaka [Hir]. Thus, (3) above applies. For instance, one concludes that their homology representations arise as the homology of a permutation complex.

(6) For more general G-spaces (e.g., paracompact ones), it is possible to use suitable Čech coverings as in Bredon [Bdn, Chap. 2] to obtain a permutation complex whose cohomology computes the cohomology of the space.

(7) It is easy to see that $\mathscr{P}(RG)$ contains many permutation complexes which do not arise from topological situations of (3)–(6). Even for RG-complexes C_* whose underlying R-complex is the complex of cellular chains of a CW-complex X, it happens (more often than not) that C_* is not even RG-chain homotopy equivalent to a permutation complex of a G–CW complex as in (4) above. See Justin Smith [Sm1] and Quinn [Qf] for obstruction theories which analyze the homological obstructions for topological realization of chain complexes.

2. Localization and Varieties

In this section we discuss localization and its consequences in the theory of module varieties.

Let X_* be a permutation complex, and let W_* be a projective resolution of R over RG. The homology and cohomology of the total complexes associated to the double complexes $W_* \otimes_G X_*$ and $\operatorname{Hom}_G(W_*, X^*)$ are called the hyperhomology and the hypercohomology of X_*, and they are denoted by $\mathbb{H}_*(G; X_*)$ and $\mathbb{H}^*(G; X^*)$. The topological analogue of the above construction for topological transformation groups is the Borel equivariant homology $H_*^G(X; R)$ and $H_G^*(X; R)$ defined for a G-space X, using the twisted product (or the Borel construction) $E_G \times_G X \xrightarrow{\pi} BG$ associated to the universal principal bundle $E_G \to BG$. See Bredon [Bdn], W. Y. Hsiang [Hsg], Borel [Bor], or Quillen [Q1] for the topological theory, and Ken Brown [B3] and Cartan and Eilenberg [CE], as well as Swan [Sw1], for an algebraic discussion.

Let $R = \mathbb{F}_p$ or any other field of characteristic p (e.g., $\bar{\mathbb{F}}_p$), and let $G = (\mathbb{Z}_p)^n$. Then for $p = 2$, $H^*(BG; \mathbb{F}_p) \equiv H^*(G; \mathbb{F}_p) = \mathbb{F}_p[t_1, ..., t_n]$ with $t_i \in H^1(G; \mathbb{F}_p)$. For $p > 2$, let $\Lambda(u_1, ..., u_n)$ be the exterior algebra generated by $H^1(G; \mathbb{F}_p) \cong \operatorname{Hom}_{\mathbb{F}_p}((\mathbb{F}_p)^n, \mathbb{F}_p)$ and let $t_i \in H^2(G; \mathbb{F}_p)$ be the image of the Bockstein $\beta: H^1(G; \mathbb{F}_p) \to H^2(G; \mathbb{F}_p)$. Then $H^*(G; \mathbb{F}) = \Lambda(u_1, ..., u_n) \otimes \mathbb{F}_p[t_1, ..., t_n]$. Similar formulas hold for R replacing $\mathbb{F}_p$. If X is a finite-dimensional paracompact G-space, and $j: X^G \to X$ is the inclusion, then the induced homomorphism in equivariant cohomology $j_G^*: H_G^*(X; R) \to H_G^*(X^G; R)$ is $H^*(G; R)$-linear. Let $S \subset H^*(G; R)$ be the multiplicatively closed subset generated by the non-zero $\mathbb{F}_p$-linear combinations of the polynomial generators $\{t_1, ..., t_n\}$. The localization theorem in equivariant cohomology (originally due to Borel [Bor] and further generalized by W. Y. Hsiang [Hsg] and Quillen [Q1]) states that the localized homomorphism $S^{-1}j_G^*: S^{-1}H_G^*(X; R) \to S^{-1}H_G^*(X^G; R)$ is an isomorphism. This theorem and its ramifications have been at the heart of the developments in the cohomology theory of transformation groups since the 1950s. See Borel [Bor], Bredon [Bdn], W. Y. Hsiang [Hsg], and Quillen [Q1] for examples and applications.

We have the following generalization of the above localization theorem which will be one of the main technical tools in the homological study of permutation complexes.

2.1. THEOREM (Localization theorem for permutation complexes). *Let C_* be a bounded RG-permutation complex. Assume that $G = (\mathbb{Z}_p)^n$, R is a field of characteristic p, and $S \subset H^*(G; R)$ is as in the above. Then, the inclusion $\rho: C_*(G) \to C_*$ induces an isomorphism $S^{-1}\rho^*: S^{-1}\mathbb{H}^*(G; C^*) \to S^{-1}\mathbb{H}^*(G; C^*(G))$.*

Proof. Consider the exact sequence of RG-chain complexes: $0 \to C_*(G) \xrightarrow{p} C_* \xrightarrow{q} Q_* \to 0$. Consider the long exact sequence in hypercohomology: $\cdots \to \mathbb{H}^i(G; Q^*) \xrightarrow{q_G} \mathbb{H}^i(G; C^*) \xrightarrow{p^*} \mathbb{H}^i(G; C^*(G)) \xrightarrow{\delta} \cdots$ in which all homomorphisms are $H^*(G; R)$-linear. Since localization is an exact functor, the theorem will follow from the statement $S^{-1}\mathbb{H}^*(G; Q^*) = 0$. Note that Q_* is a permutation complex for which $Q_*(G) = 0$. Therefore, the following lemma will comlete the proof of the above theorem. ∎

2.2. LEMMA. *Let $G = (\mathbb{Z}_p)^n$ and R be a commutative ring. Suppose Q_* is a bounded complex of permutation modules with basis Σ_i such that $\Sigma_i^G = \phi$. Then $\mathbb{H}^*(G; Q^*)$ is an $H^*(G; R)$-torsion module. Therefore, if P_* is an RG-complex RG-chain homotopic to Q_*, then $\mathbb{H}^*(G; P^*)$ is also $H^*(G; R)$-torsion.*

Proof. Since Q^* is a bounded complex, the second spectral sequence of the double complex $\mathrm{Hom}(W_*, Q^*)$ is convergent. (See Cartan and Eilenberg [CE] for more details.) The E_2-term of this spectral sequence has a filtration by $H^*(G; R)$-torsion modules, since $H^*(G; Q^i) \cong \bigoplus_j H^*(K_j; R)$, where $Q^i = \bigoplus_j R[G/K_j]$ and $K_j \neq G$ by hypothesis. Thereore, the E_∞-term is $H^*(G; R)$-torsion, which implies that $\mathbb{H}^*(G; Q^*)$ is $H^*(G; R)$-torsion. ∎

2.2A. *Remark.* The referee has pointed out that an alternative proof of Theorem 2.1 is possible by adapting the proof of J. Smith [Sm, Proposition 2.3] to the above context. We refer the reader to [Sm] for further details and other aspects of localization.

2.3. COROLLARY. *Keep the notation and hypotheses of above theorem. Let D_* be an RG-chain complex which is RG-chain homotopic to a permutation subcomplex $C'_* \subset C_*$ and assume that $C_*(G) \subseteq C'_*$. Then $S^{-1}\mathbb{H}^*(G; D^*) \cong S^{-1}\mathbb{H}^*(G; C^*)$.*

Proof. The hypotheses and the above localization theorem imply that $S^{-1}\mathbb{H}^*(G; D^*) \cong S^{-1}\mathbb{H}^*(G; C'^*) \cong S^{-1}\mathbb{H}^*(G; C^*(G)) \cong S^{-1}\mathbb{H}^*(G; C^*)$. ∎

Next, we study the varieties for homology representations of permutation complexes. The localization process in cohomology is closely related to the notions of support and rank varieites for modules, introduced by J. Carlson [C1], [C2] and developed further by Avrunin and Scott [AS] and others. For simplicity, let $E = (\mathbb{Z}/p)^n$ be generated by $\langle x_1, ..., x_n \rangle$, and consider the reduced cohomology ring $H_E = H^*(E; k)/\text{Radical} \cong k[t_1, ..., t_n]$. Any kE-module M gives rise to an H_E-module $H^*(E; M)$, and as such, it has a support in Spec H_E. For many purposes, it suffices to consider the subspace of closed points in Spec H_E, namely Max H_E consisting of maximal ideals. Let $I(M) \subset H_E$ denote the annihilating ideal of the H_E-module $H^*(E; M)$. The cohomological support variety $V_E(M) \subset \text{Max}\, H_E$ is nothing but the variety defined by $I(M)$: $V_E(M) = \{\mathfrak{m} \in \text{Max}\, H_E : \mathfrak{m} \supseteq I(M)\}$. This definition generalizes directly to any p-group G, and with a slight modification to the case of general finite groups (see Avrunin and Scott [AS] for details, and Carlson [C1], [C2] for details of what follows). Note that Max $H_E \cong k^n$ = the affine k-space of dimension n. There is another n-dimensional affine space associated to $E = (\mathbb{Z}/p)^n$. Namely, let $J_E \subset kE$ be the usual augmentation ideal, and observe that $J_E/J_E^2 \cong H_1(E; k) \cong k^n$. By choosing a basis for J_E/J_E^2 and a splitting σ of the projection $\pi: J_E \overset{\sigma}{\leftrightarrows} J_E/J_E^2$, we obtain an n-dimensional k-subspace of kE, which is denoted by L. For example, for $E = \langle x_1, ..., x_n \rangle$, let a basis of L be $\{x_1 - 1, ..., x_n - 1\}$. To an n-tuple $(\alpha_1, ..., \alpha_n) \in k^n$, there corresponds the element $u_\alpha = 1 + \sum_{i=1}^n \alpha_i(x_i - 1) \in 1 + L$, which is a unit, and it generates a subgroup $\langle u_\alpha \rangle \cong \mathbb{Z}/p \subset kE$. $\langle u_\alpha \rangle$ is called a *shifted cyclic subgroup* of kE, and it was introduced by E. Dade [D] to study endo-trivial modules. Using shifted cyclic subgroups, Jon Carlson defined the subset $V_E^r(M) \subset L \cong k^n$ via $V_E^r(M) = \{(\alpha_1, ..., \alpha_n) \in k^n : M|_{k\langle u_\alpha \rangle}$ is not

$k\langle u_\alpha\rangle$-free$\} \cup \{0\}$ (called the rank variety of M). Indeed $V_E^r(M)$ is a well-defined subset of $J_E/J_E^2 = k^n$ independent of the choice of L, and it is a homogeneous affine subvariety of k^n. There is a natural identification $J_E/J_E^2 \xrightarrow{\cong} \operatorname{Max} H_E$, and this results in a map $V_E^r(M) \to V_E(M)$, which was shown to be an isomorphism of sets by Avrunin and Scott [AS], thus proving a conjecture of Carlson; see also [C2]. This isomorphism is natural and compatible with respect to the inclusion of subgroups, in particular, products of shifted cyclic subgroups $S = \langle u_\alpha\rangle \times \langle u_\beta\rangle \times \cdots \times \langle u_\zeta\rangle$ (the so-called shifted subgroups of kE which have ranks $\leqslant \operatorname{rank}(E)$).

The theory of varieties for modules has proved to be extremely valuable, not only in representation theory and finite group theory, but in the context of restricted Lie algebras (cf. Friedlander and Parshall [FP] and Jantzen [J]) and topological transformation groups and homotopy-theoretic aspects of geometric topology (e.g., Adem [Ad2], Assadi [A2], [A5], Benson and Carlson [BC], and many other references).

We use the theory of varieties in the following sections, and for future reference, we discuss briefly how this theory generalizes to the context of permutation complexes. The motivation and many of the details may be found in Assadi [A2] and further applications in [A5].

First suppose that C_* is any kG-complex such that $\bigoplus_{i\in\mathbb{Z}} H_i(C_*)$ is a finitely generated kG-module. For simplicity of exposition, assume that G is a p-group, so that the kG-module k (with trivial G-action necessarily) is the only simple kG-module. Following [A2], the idea is to modify C_* in the category of kG-complexes so as "to simplify" its cohomological structure without changing its hypercohomology $\mathbb{H}^*(G; C^*)$ locally. Namely, call C_* *freely equivalent* to a kG-chain complex D_* if there is a kG-chain complex K_* such that $C_* \subset K_*$ and $D_* \subset K_*$ are kG-subcomplexes and K_*/C_* and K_*/D_* are both kG-free, and bounded with finitely generated homology. This notion was introduced in Assadi [A1] in order to study combinatorial properties of permutation complexes. As in [A2] (compare with [A1]) it is easy to see that free equivalence is an equivalence relation, and the equivalence class of C_* has a representative $\hat{C}_*$ such that $H_i(\hat{C}_*) = 0$ for $i \neq l$ and $H_l(\hat{C}_*) = M$ is a finitely generated kG-module. Call $\hat{C}_*$ a *resolvent* for C_*.

2.4. Definition–Proposition. Let G be a p-elementary abelian group. The rank variety and support variety of C_* are defined by $V_G^r(C_*) \equiv V_G^r(H_*(\hat{C}_*)) \equiv V_G^r(M)$ and $V_G(C_*) = V_G(M)$, where $\hat{C}_*$ is any resolvent of C_* defined as above. $V_G(C_*)$ and $V_G^r(C_*)$ are independent of the choice of the resolvent $\hat{C}_*$.

Remark. The above definitions certainly make sense for any finite group G with the appropriately defined varieties, e.g., as in Avrunin and Scott [AS] and Assadi [A5].

When dealing with based kG-complexes, it is possible to choose the resolvent $\hat{C}_*$ also in the category of based complexes, hence $l \geqslant 0$. In this case, the sensible definition is to let $\tilde{M} = \tilde{H}_*(\hat{C}_*) \equiv$ the reduced homology and define $V^r_G(C_*, \underline{k}) = V^r_G(\tilde{M})$ and $V_G(C_*, \underline{k}) = V_G(\tilde{M})$. Clearly $V^r_G(C_*, \underline{k}) = V^r_G(\hat{C}_*/\underline{k}) = V^r_G(C_*/k)$ and similarly for V_G.

It is useful to generalize some of the properties of module varieties to kG-complexes before specializing to the case of permutation complexes.

2.5. PROPOSITION. *Let X_*, X'_*, Y_*, be kG-complexes with finitely generated total homology, and let X'_* and Y'_* be based. Then the following hold*:

(a) $V^r_G(X_*)$, $V_G(X_*)$, *and their based versions are unchanged under*:

(i) *free equivalence*;

(ii) *iterated shifts and iterated suspensions of* 1.3;

(iii) *taking duals* $X^* = \mathrm{Hom}(X_*, k) \equiv X_{-*}$;

(iv) *chain homotopy equivalence, or more generally kG chain maps of any degree inducing a homology isomorphism.*

(b) $V^r_G(X_*) \cong V_G(X_*)$

(c) $V^r_G(X_* \otimes Y_*) = V^r_G(X_*) \cap V^r_G(Y_*)$.

(d) *Similarly for the based version* $V^r_G(X'_* \vee Y'_*, \underline{k}) = V^r_G(X'_*, \underline{k}) \cup V^r_G(Y'_*, \underline{k})$ *and* $V^r_G(X'_* \wedge Y'_*, k) = V^r_G(X'_*, k) \cap V^r_G(Y'_*, k)$.

(e) *If X_* is bounded and kG-free then* $V^r_G(X_*) = 0$.

(f) *If X_* is kG-chain homotopy equivalent to a non-negative kG-complex, then $V_G(X_*)$ is the variety defined by the annihilating ideal of the H_G-module* $\mathbb{H}^*(G; X^*)$.

Proof. Most of the above follow from the definitions and elementary observations. Part (b) is essentially the Avrunin–Scott theorem [AS] mentioned above. In (c) and (d), we may first take resolvents having their non-trivial homologies in the same dimension (reduced homology for based complexes). In (e) the resolvent $\hat{X}_*$ is seen to have a kG-free homology since X_* is bounded and kG-free. (f) From the hypercohomology exact sequence of the short exact sequence $0 \to X_* \xrightarrow{j} \hat{X}_* \to \hat{X}_*/X_* \to 0$ that $j^*: \mathbb{H}^i(G; \hat{X}^*) \to \mathbb{H}^i(G; X^*)$ is an isomorphism for all sufficiently large i (since $\hat{X}_*/X_*$ is kG-free and bounded, hence with bounded hypercohomology). Therefore, the annihilating ideals of $\mathbb{H}^*(G; X^*)$ and $\mathbb{H}^*(G; \hat{X}^*)$ have the same radical. Similarly, $H^*(G; H^*(\hat{X}^*))$ and $\mathbb{H}^*(G; \hat{X}^*)$ define the same varieties and (f) follows. ∎

Next, we specialize to the case of permutation complexes. It is convenient to think of all varieties defined for complexes or modules over

kG as homogeneous affine subvarieties of $V_G^r(k)=k^n$ for $G=(\mathbb{Z}/p)^n$. In particular, for each subgroup $K\subseteq G$, $V_G^r(\mathrm{Ind}_K^G(k))$ is a linear subspace of $V_G^r(k)$ defined with $\mathbb{F}_p$-coefficients and it is isomorphic to $V_K^r(k)$. The cohomological analogue is the restriction of $\mathrm{Spec}\, H_K\to \mathrm{Spec}\, H_G$ induced by the restriction homomorphism $\rho_K^G\colon H^*(G;k)\to H^*(K;k)$ to the subspace of closed points. In this way, we establish a one-to-one correspondence between $\mathbb{F}_p$-rational linear subspaces of $V_G^r(k)$ (or equivalently $V_G(k)$) and subgroups of G itself. In particular, cyclic subgroups of G and $\mathbb{F}_p$-rational lines of J_G/J_G^2 correspond under the above. An important property of shifted cyclic subgroups $\langle u_\alpha\rangle\subset kG$ (corresponding to $\alpha=(\alpha_1,...,\alpha_n)\in k^n$ as above) is that kG is $k\langle u_\alpha\rangle$-free. Moreover the usual apparatus of induction and restriction of representations (e.g., Mackey's formula) and their homological consequences hold for shifted subgroups. See Carlson [C2] and Kroll [K] for justification and details. In particular, $k[G/H]=\mathrm{Ind}_H^G(k)$ is a $k\langle u_\alpha\rangle$-free module if $k\langle u_\alpha\rangle\cap kH=k\langle 1\rangle\cong k$ by Mackey's formula. Thus, if we choose α such that the line $k\{\alpha\}$ is not $\mathbb{F}_p$-rational in $J_G/J_G^2=k^n$, then $k[G/H]$ are $k\langle u_\alpha\rangle$-free for all proper subgroups $H\subsetneq G$. Suppose that X_* is a permutation complex with permutation basis $\mathscr{S}=\bigsqcup_{i\in\mathbb{Z}} S_i$. For the above choice of α, the only elements of $S_i\subset X_i$ which are left fixed by $\langle u_\alpha\rangle$ are those with isotropy group G. This suggests the slight abuse of notation $X_*(\langle u_\alpha\rangle)$ indicating the fact $\mathscr{S}^{\langle u_\alpha\rangle}=\mathscr{S}^G$. Since kG is $k\langle u_\alpha\rangle$-free and $X_*(\langle u_\alpha\rangle)=X_*(G)$, $X_*|k\langle u_\alpha\rangle$ is a $k\langle u_\alpha\rangle$-permutation complex and we can apply our machinery and results on $k[\mathbb{Z}/p]$-permutation complexes as before. The following summarizes these observations with a slight useful generalization.

2.6. Proposition. *Let X_* be a permutation kG-complex, where G is any finite group, and let $H\subseteq G$, $H=(\mathbb{Z}/p)^n$. Then for a suitable choice of a shifted cyclic subgroup $\langle u_\alpha\rangle\subset kH$, $X_*|k\langle u_\alpha\rangle$ will have a natural structure of a $k\langle u_\alpha\rangle$-permutation complex such that $X_*(k\langle u_\alpha\rangle)=X_*(H)$ and $X_*/X_*(H)$ is $k\langle u_\alpha\rangle$-free.*

Remark. Clearly the set of $\alpha\in V_H^r(k)$ for which $\langle u_\alpha\rangle$ has the above property forms a Zariski open dense subset. A useful application of the above discussion is a simplified calculation of fixed subcomplexes.

2.7. Proposition. *Suppose X_* is a bounded permutation kG-complex, and $(\mathbb{Z}/p)^n\cong H\subseteq G$ is a subgroup.*

(a) *For any shifted subgroup $\langle u_\alpha\rangle\subset kH$ as in Proposition 2.6 above,*

$$H_*(X_*(H))\cong(\mathbb{H}^*(\langle u_\alpha\rangle;X^*|k\langle u_\alpha\rangle)\left[\frac{1}{t_\alpha}\right]\otimes_A k,$$

where $A = \hat{H}^*(\langle u_\alpha \rangle; k) \cong H^*(\langle u_\alpha \rangle; k)[1/t_\alpha]$ *and* $t_\alpha \in H^i(\langle u_\alpha \rangle; k)$ *is the polynomial generator and* $i = 1$ *for* $p = 2$ *and* $i = 2$ *for* $p > 2$.

(b) *If* $\hat{X}_*$ *is a resolvent for* X_* *and* $H^*(\hat{X}^*) = M$, *then* $H_*(X_*(H)) \cong \hat{H}^*(\langle u_\alpha \rangle; M) \otimes_A k$ *(ungraded).*

Proof. Consider the short exact sequence $0 \to X_*(H) \xrightarrow{j} X_* \to X_*/X_*(H) \to 0$ and the corresponding long exact sequence in hypercohomology $\cdots \mathbb{H}^*(\langle u_\alpha \rangle; X^*) \xrightarrow{j^*} \mathbb{H}^*(\langle u_\alpha \rangle; X^*(H)) \to \cdots$. The proof of the localization theorem 2.1 applies to this case since $\mathbb{H}^*(\langle u_\alpha \rangle; \operatorname{Hom}(X_*/X_*(H), k))[1/t_\alpha] \cong 0$ because $X_*/X_*(H)$ is $k\langle u_\alpha \rangle$-free and bounded. A standard calculation implies (a) and (b). ∎

The following result shows that homology representations of bounded permutation complexes (permutable modules) have special types of rank varieties which arise for permutation modules.

2.8. THEOREM. *Let* X_* *be a bounded permutation kG-complex, where* $G = (\mathbb{Z}/p)^n$. *Then* $V^r_G(X_*)$ *consists of* $\mathbb{F}_p$*-rational linear subspaces of* $V^r_G(k)$ *corresponding to subgroups* $K \subseteq G$ *for which* $H_*(X_*(K)) \neq 0$.

Proof. First, let $K \subseteq G$ be a subgroup such that $H_*(X_*(K)) \neq 0$. Without loss of generality and for simplicity of notation, assume that X_* is a resolvent complex, and $H_0(X_*) = M$. By Proposition 2.7 above, we may choose $\langle u_\alpha \rangle \subset kK$ such that $X_*(\langle u_\alpha \rangle) = X_*(K)$. Then, Proposition 2.7(b) shows that $\hat{H}(\langle u_\alpha \rangle; M) \otimes_A k \cong H_*(K)) \neq 0$, hence $\hat{H}(\langle u_\alpha \rangle; M) \neq 0$. This implies that $M | k\langle u_\alpha \rangle$ is not $k\langle u_\alpha \rangle$-free. The set of such $\alpha \in V^r_K(k)$ with $X_*(\langle u_\alpha \rangle) = X_*(K)$ forms a Zariski dense open subset. Thus for all $\alpha \in V^r_K(k)$, $M | k\langle u_\alpha \rangle$ is not $k\langle u_\alpha \rangle$-free. As discussed above, the $\mathbb{F}_p$-rational linear subspace $V^r_G(\operatorname{Ind}^G_K(k)) \cong V^r_K(k)$ corresponds to K, and hence it lies in $V^r_G(M)$. Conversely, if $M | k\langle u_\alpha \rangle$ is free for such a choice of α, the localization result of 2.7(b) shows that $H_*(X_*(\langle u_\alpha \rangle)) = H_*(X_*(H)) = 0$. It remains to see that if there exists an $\alpha \in V^r_G(M)$ which does not lie in any proper $\mathbb{F}_p$-rational linear subspace of $V^r_G(k)$, then $V^r_G(M) = V^r_G(k)$ and $H_*(X_*(G)) \neq 0$. But this follows from the same argument applied above. ∎

Let us make a few useful technical remarks which will be needed for the following proof of the analogue of Carlson's conjecture (Avrunin and Scott [AS, Theorem 1] and Carlson [C2]). First, for a kG-complex Y_* and a short exact sequence $0 \to K \to G \to G/K \to 0$ of groups, there is a Lyndon–Hochschield–Serre spectral sequence with $E_2^{i,j} \cong H^i(G/K; \mathbb{H}^j(K; Y^*)) \Rightarrow \mathbb{H}^{i+j}(G; Y^*)$ when Y_* is bounded below. There is an analogue of this spectral sequence for $G = (\mathbb{Z}/p)^n$ and shifted subgroups $K \subset kG$ and $K' \subset kG$ with the property $kK \otimes kK' \cong kG$, $H^i(K'; \mathbb{H}^j(K; Y^*)) \Rightarrow \mathbb{H}^{i+j}(G; Y^*)$. This is discussed for kG-modules in Carlson [C2]. One may modify Carlson's argument and apply it to the double complex $\mathrm{Hom}_{K\times K'}(W_* \otimes W'_*, Y^*)$ (where W_* and W'_* are the free resolutions of k over kK and kK', respectively) to obtain the above spectral sequence. However, the usual spectral sequence for modules can be used in the following arguments provided that we replace Y_* by a resolvent kG-complex of Y_*.

2.9. PROPOSITION. *Suppose Y_* is a bounded permutation kG-complex for $G = (\mathbb{Z}/p)^n$, and let $\langle u_\alpha \rangle$ be a shifted cyclic subgroup of kG, and $t_\alpha \in H^i(\langle u_\alpha \rangle; k)$ a polynomial generator of $H_{\langle u_\alpha \rangle}$. Then $\mathbb{H}^*(G; Y^*)[1/t_\alpha] \cong \hat{H}^*(\langle u_\alpha \rangle; k) \otimes \mathbb{H}^*(K'; Y^*(\langle u_\alpha \rangle))$, where $kG \cong k\langle u_\alpha \rangle \otimes kK'$. In particular, $\mathbb{H}^*(G; Y^*)[1/t_\alpha] \neq 0$ if and only if $H_*(Y_*(\langle u_\alpha \rangle)) \neq 0$.*

Proof. Since localization is an exact functor, we can localize the above-mentioned spectral sequence: $H^*(K'; \mathbb{H}^*(\langle u_\alpha \rangle; Y^*))[1/t_\alpha] \Rightarrow \mathbb{H}^*(G; Y^*)[1/t_\alpha]$. But $H^*(K'; \mathbb{H}^*(\langle u_\alpha \rangle; Y^*))[1/t_\alpha] \cong H^*(K'; \mathbb{H}^*(\langle u_\alpha \rangle; Y^*)[1/t_\alpha]) \cong H^*(K'; \mathbb{H}^*(\langle u_\alpha \rangle; Y^*(\langle u_\alpha \rangle))[1/t_\alpha]) \cong H^*(K'; \hat{H}^*(\langle u_\alpha \rangle; k) \otimes H^*(Y^*(\langle u_\alpha \rangle)))$ by the localization theorem 2.1 and since $\langle u_\alpha \rangle$ acts trivially on $Y^*(\langle u_\alpha \rangle)$. To verify the formula for the E_∞-term, consider performing the localization on the E_1-level, $E_1^{**}[1/t_\alpha] \cong \mathrm{Hom}_{K'}(W'_*; \mathbb{H}^*(\langle u_\alpha \rangle; Y^*))[1/t_\alpha] \cong \mathrm{Hom}_{K'}(W'_*; \mathbb{H}^*(\langle u_\alpha \rangle; Y^*(\langle u_\alpha \rangle))[1/t_\alpha])$, and since K' acts trivially on $H^*(\langle u_\alpha \rangle; k)$ and $\langle u_\alpha \rangle$ acts trivially on $Y^*(\langle u_\alpha \rangle)$, $E_1^{**}[1/t_\alpha] \cong \mathrm{Hom}_{K'}(W'_*; H^*(Y^*(\langle u_\alpha \rangle))) \otimes \hat{H}^*(\langle u_\alpha \rangle; k)$, which clearly converges to $\mathbb{H}^*(K'; Y^*(\langle u_\alpha \rangle)) \otimes \hat{H}^*(\langle u_\alpha \rangle; k)$ and the first assertion is proved. If $H_*(Y_*(\langle u_\alpha \rangle)) \neq 0$, then $\mathbb{H}^*(K'; Y^*(\langle u_\alpha \rangle)) \neq 0$ and hence $\mathbb{H}^*(G; Y^*)[1/t_\alpha] \neq 0$. This follows from considering the second spectral sequence of the double complex $\mathrm{Hom}_{K'}(W'_*; Y^*(\langle u_\alpha \rangle))$ (which is convergent since $Y_*(\langle u_\alpha \rangle)$ is bounded) and the universal coefficients formula. If $H_*(Y_*(\langle u_\alpha \rangle)) = 0$, then the LHS-spectral sequence shows that $\mathbb{H}^*(G; Y^*)[1/t_\alpha] = 0$. ∎

We use the above to prove the analogue of Carlson's conjecture (Avrunin and Scott [AS, Theorem 1]) by a different proof for bounded permutation complexes. This proof is particularly interesting from the point of view of local-to-global properties of the homology representations of permutation complexes. It also suggests an alternative proof of Carlson's conjecture for arbitrary modules, which will be presented elsewhere.

2.10. COROLLARY (Carlson's conjecture for permutation complexes). *Let $G = (\mathbb{Z}/p)^n$ and X_* a bounded permutation kG-complex. Then $V_G^r(X_*) = V_G(X_*)$.*

Proof. $V_G(X_*)$ is defined by the annihilating ideal of the $H^*(G; k)$-modules $H^*(G; H^*(\hat{X}^*))$ or equivalently $\mathbb{H}^*(G; X^*)$, where $\hat{X}^*$ is a resolvent of X^*, if $p = 2$, otherwise the annihilating ideal as H_G-modules.

As in Theorem 2.8 above, assume $H_i(X_*)=0$ for $i>0$ and $H_0(X_*)=M$. If $K\subset G$ is any subgroup then the inclusion induces split surjections $H^*(G;k)\to H^*(K;k)$ and $H_G\to H_K$. The same is true for a shifted subgroup $K\subset kG$. The corresponding map on spectra yields an embedding $\rho_K^G\colon V_K(k)\to V_G(k)$ whose image may be identified with $V_G(\mathrm{Ind}_K^G(k))\cong V_K(k)$. Now let $\alpha\in k^n$ be chosen such that the line $V_G^r(\mathrm{Ind}_{\langle u_\alpha\rangle}^G(k))\cong V_{\langle u_\alpha\rangle}^r(k)$ does not lie in $V_G^r(X_*)$. According to the proof of Theorem 2.8 above this condition is equivalent to $H_*(X_*(\langle u_\alpha\rangle))=0$. By Proposition 2.9 above, the latter condition implies that $\mathbb{H}^*(G;X^*)[1/t_\alpha]=0$ and consequently $V_G\mathrm{Ind}_{\langle u_\alpha\rangle}^G(k))\cap \mathrm{Support}(\mathbb{H}^*(G;X^*))=0$. That is, $\rho_{\langle u_\alpha\rangle}^G(V_{\langle u_\alpha\rangle}(k))$ does not lie in $V_G(X_*)$. Conversely, if the line $V_G^r(\mathrm{Ind}_{\langle u_\alpha\rangle}^G(k))$ lies in $V_G^r(X_*)$, then $H_*(X_*(\langle u_\alpha\rangle))\neq 0$, and by Proposition 2.9, $\mathbb{H}^*(G;X^*)[1/t_\alpha]\neq 0$.

Translated into a statement about supports, this is equivalent to $V_G(X_*)\cap V_G(\mathrm{Ind}_{\langle u_\alpha\rangle}^G(k))\neq 0$. Since both varieties are homogeneous, the proof is completed. ∎

3. Homology Representations

Every RG-module M has a free RG-resolution $C_*\colon \cdots\to C_1\to C_0\to M\to 0$. That is, $H_i(C_*)=0$ for $i>0$, and $H_0(C_*)=M$. Unless M is cohomologically trivial in the sense of Tate (see Brown [B3], Cartan and Eilenberg [CE], or Rim [R]), C_* is infinite dimensional. If we choose C_i to be permutation modules, we may arrange to have a finite-dimensional chain complex C_*. This point of view has been studied by Arnold [Ar2], who has developed, for instance, analogues of the familiar homological algebraic constructions using permutation modules. For instance, Arnold proves that in this context, for cyclic groups G every $\mathbb{Z}G$-module M has a "resolution" by a complex of permutation modules of length 2. However, if we require "the resolutions" to be permutation complexes, then we get non-trivial restrictions on the type of RG-modules which arise in this way. More generally we formulate the following.

3.1. Problem. Suppose X_* is a bounded permutation complex such that for some integer d, $H_i(X_*)=0$ for $i\neq d$ and $H_d(X_*)=M$. We call X_* a "permutable resolution" of M. (1) Which RG-modules M have permutable resolutions? (2) If M is a finitely generated RG-module, when can we find a finite permutable resolution for M?

This is an algebraic analogue of the well-known Steenrod problem (see Lashof [L], Swan [Sw2], Arnold [Ar1], Smith [Sm1], [Sm2], [Sm3], [Sm4], Carlsson [Cg], Vogel [V], and Assadi [A2] for a partial survey).

As we shall see below, the class of RG-modules which arises in (1) is quite restricted. Therefore, the existence of a permutable resolution may be considered as an extra structure imposed on an RG-module which is a natural generalization of being a permutation module.

3.2. DEFINITION. An RG-module which has a permutable resolution is called a *permutable module.*

As for part (2) of the above problem, the obstruction theory of R. Swan [Sw2] generalizes to the context of permutable resolutions. Therefore, the results of Swan [Sw2] are valid in this context and show that even among permutable modules, the existence of finite permutable resolutions imposes number-theoretic conditions on finitely generated $\mathbb{Z}G$-modules.

Using the localization theorem 2.1, we may extend many results of topological transformation groups to the context of permutation complexes. For example:

3.3. THEOREM. *Let X_* be RG-chain homotopic to a bounded permutation complex, and assume that for each maximal p-elementary abelian group $E \subseteq G$ and each $p \mid |G|$ for which $p^{-1} \notin R$, the hypercohomology spectral sequence $H^*(E; H^*(X^*)) \Rightarrow \mathbb{H}^*(E; X^*)$ degenerates. Then the RG-module $M = \bigoplus_i H_i(X_*)$ is RG-projective if and only if for each subgroup $C \subseteq G$ such that $|C| = p$ and $p^{-1} \notin R$, $M \mid RC$ is RC-projective.*

Proof. The proof of Theorem 1.1 for G-spaces in Assadi [A3] is based on the localization theorem and arguments involving constructions which are valid in $\mathscr{P}(RG)$ as well; see Section 1. We leave the minor modification to the reader. ▮

Let us mention some applications to group theory. Let G be a finite group, and let π be a poset of proper subgroups of G. Let S_n be the set of chains of subgroups $P_0 < P_1 < \cdots < P_n$ of length $n+1$. Conjugation by elements of G makes S_n a G-set. The ith face map $\partial_i\colon S_n \to S_{n-1}$ is defined by dropping the ith subgroup in the chain, and $\partial\colon S_n \to R[S_{n-1}]$ is given by $\partial = \sum_{i=0}^{n} (-1)^i \partial_i$. The resulting RG-chain complex C_* is a permutation complex for suitable choices of π. In fact, C_* is the simplicial chain complex of the simplicial complex $\Delta(\pi)$ associated to the poset π by the standard construction. See Brown [B1], [B2], Quillen [Q2], Solomon [Sol], Tits [Tt], and Webb [W2] for further discussion and applications. We use Quillen's notation [Q2]: $\mathscr{A}_p(G)$ = the poset of non-trivial p-elementary abelian subgroups of G, $\mathscr{S}_p(G)$ = the poset of p-subgroups of G. If G is the finite group of $\mathbb{F}_q$-rational points of a semi-simple algebraic group $(q = p^s)$ of rank l over $\mathbb{F}_q$, then we denote the Solomon–Tits building associated to G by T; see Solomon [Sol] and Tits [Tt]. The complex of permutation modules $C_*(\mathscr{A}_p(G))$ is in fact a permutation complex, and according to Quillen [Q2, Theorem 3.1], $C_*(\mathscr{A}_p(G))$ and $C_*(T)$ are chain homotopy equivalent. Moreover, $C_*(T)$ is based and $H_i(C_*(T)) \neq 0$ only for $i = 0$ and $i = l-1$, where l is the rank. The localization theorem 2.1 and the projectivity criterion, Theorem 3.3, together imply the following well-known results. (I am grateful to Steve Smith for pointing out a correction in the statement of 3.4 below).

3.4. THEOREM. (a) *$H_{l-1}(T)$ is RG-projective, where R is a field of characteristic p or the p-adic integers.*

(b) *$\sum_i (-1)^i H_i(C_*(\mathscr{A}_p(G)); R)$ is a virtual RG-projective for an arbitrary finite group G and R as in* (a).

(c) *Let G be of p-rank 2, and $\tilde{C}_*$ be the reduced RG-chain complex associated to $\mathscr{A}_p(G)$ or $\mathscr{S}_p(G)$. Then $H_*(\tilde{C}_*)$ is RG-projective.*

Part (c) is obtained by Webb [W1] in a different context, and as pointed out in [Q2], and [W1], $H_1(C_*)$ is isomorphic to the Steinberg module if G is assumed to be a finite Chevalley group of p-rank 2.

Next, the projectivity criterion 3.3 above may be used as in Assadi [A2], [A3] to provide non-permutable modules. Note that since $\mathbb{H}^*(G; X^*)$ does not necessarily admit auxiliary structures, such as an action of the Steenrod algebra, the counterexamples to the Steenrod problem (e.g., as in [Cg]) which use such structures do not apply to Problem 3.1 above.

3.5. THEOREM. *Suppose $G \supseteq \mathbb{Z}_p \times \mathbb{Z}_p$ or Q_8 (= the quaternion group of order 8). Then there are finitely generated non-permutable $\mathbb{Z}G$-lattices.*

Proof. The examples constructed in Assadi [A2], [A3] use the projectivity criterion [A2, Theorem 1.1]. We may apply the analogous criterion, Theorem 3.3 above, to the examples of [A2], [A3]. ∎

It is worth noting that the analogue of Theorem 3.1 of [A2] also holds for homology representations of bounded permutation complexes:

3.6. THEOREM. *Let $G \supset \mathbb{Z}_p \times \mathbb{Z}_p$ or Q_8. Then*:

(a) *There are non-trivial $\mathbb{Z}G$-lattices M_1 and M_2 such that $M_1 \oplus M_2$ does not occur as the homology representation of any bounded RG-permutation complex.*

(b) *There are $\mathbb{Z}G$-lattices M_1 and M_2 such that neither M_1 nor M_2 occurs as the homology representation of a bounded permutation complex, but $M_1 \oplus M_2 \cong H_*(X_*)$ for a bounded permutation complex X_*.*

Proof. The strategy of the proof is similar to Assadi [A2] with minor modifications. The details are omitted. ∎

4. Duality

There is a "Hermitian analogue" of Problem 3.1 above which we will discuss briefly. Another property of permutation modules is their "self-duality": If M is a permutation RG-module, then $\mathrm{Hom}_R(M, R) \cong M$ as RG-modules. This property is not shared by most modules, and again, it can be thought of as an extra structure imposed on M. In particular, one may ask for the description of permutable modules which are in addition self-dual. A special case which arises in geometric topology and topological transformation groups is the homology representations of highly connected self-dual permutation complexes. Let C_* be a positive RG-complex, and $C^* = \mathrm{Hom}_R(C_*, R)$. If we use the usual convention $C_{-i} \equiv C^i$, then the duality condition is formulated as follows:

4.1. *Condition* (SD). Let C_* be a connected (augmented) RG-complex. C_* is called self-dual of formal dimension d, if there is a chain homotopy equivalence of degree d, $h: C^* \to C_*$. (We may equivalently say that C_* satisfies duality of formal dimension d.)

Let X_* be a self-dual bounded permutation complex of formal dimension $2n$ such that $H_i(X_*) = 0$ for $0 < i < n$ (and by duality for $n < i < 2n$), and $H_n(X_*) = M$ finitely generated. Then we have an RG-isomorphism $H^n(X^*) \xrightarrow{\cong} H_n(X_*)$, which shows that $M \cong \mathrm{Hom}_R(M, R)$, using the universal coefficients formula. We call X_* a *self-dual permutable structure* (*SDP-structure* for short). It is not unreasonable to conjecture that a module M with an SDP-structure is permutable. We will provide some evidence for this later. Based on this, we call an RG-module M self-dual permutable if there is an SDP-structure for M.

4.2. Problem. Determine self-dual permutable RG-modules.

4.3. Proposition. *Let $p \mid |G|$ be an odd prime. Suppose that C_* is a bounded connected RG-permutation complex such that $H_0(C_*) = H_{2n}(C_*) = R$, $H_i(C_*) = 0$ for $i > 2n$, and for $0 < i < 2n$, $H_i(C_*)$ is RG-projective. Then for each $H \in \mathscr{A}_p(G)$, $H_*(C_*(H)) \cong R \oplus R$.*

Proof. It suffices to assume that $G \approx (\mathbb{Z}_p)^r$ and $R = k$. Choose $\alpha = (\alpha_1, ..., \alpha_r) \in k^r$ such that the shifted subgroup $\langle u_\alpha \rangle$ satisfies $k\langle u_\alpha \rangle \cap kH = k[1]$ for all proper isotropy subgroups $H \neq G$ in C_*.

Consider the hypercohomology spectral sequence $H^*(\langle u_\alpha \rangle; H^*(C^*)) \Rightarrow \mathbb{H}^*(\langle u_\alpha \rangle; C^*)$ in which the only possible non-trivial differential is $d_{2n+1}: E_{2n+1}^{i,2n} \to E_{2n+1}^{i+2n+1,0}$. We note that $E_{2n+1}^{i,2n} = H^i(\langle t_\alpha \rangle; k) = H^{i+2n+1}(\langle u_\alpha \rangle; k) = E_{2n+1}^{i+2n+1,0} = k$ and d_{2n+1} is $H^*(\langle u_\alpha \rangle; k)$-linear. Since p is odd, the cohomology period of $H^*(\langle u_\alpha \rangle; k)$ is even (considering the action of the Bockstein on cohomology). Therefore $d_{2n+1} \equiv 0$ and the spectral sequence collapses. Now, the localization theorem 2.1 implies that $S^{-1}\mathbb{H}^*(\langle u_\alpha\rangle; C^*\langle u_\alpha\rangle) \cong S^{-1}H^*(\langle u_\alpha\rangle; k) \oplus H^*(\langle u_\alpha\rangle; k)) \cong \hat{H}^*(\langle u_\alpha\rangle; k) \otimes (k \oplus k)$. Since $S^{-1}\mathbb{H}^*(\langle u_\alpha\rangle; C^*\langle u_\alpha\rangle) \cong S^{-1}(H^*(\langle u_\alpha\rangle; k) \otimes H^*(C^*\langle u_\alpha\rangle)) \cong \hat{H}^*(\langle u_\alpha\rangle; k) \otimes H^*(C^*\langle u_\alpha\rangle)$. Therefore $H^*(C^*\langle u_\alpha\rangle) \cong S^{-1}\mathbb{H}^*(\langle u_\alpha\rangle; C^*\langle u_\alpha\rangle) \otimes_{\hat{H}^*(\langle u_\alpha\rangle; k)} k \cong k \oplus k$. By our choice of α, $C_*(\langle u_\alpha\rangle) \cong C_*(G)$, since for all $H \neq G$, $C_*(H)|_{k\langle u_\alpha\rangle}$ is $k\langle u_\alpha\rangle$-free. Therefore, $H^*(C^*(G)) \cong k \oplus k$ as claimed. ▮

4.4. PROPOSITION. *Let C_* be a connected bounded RG-permutation complex such that $H_i(C_*) = 0$ for $i \notin \{0, n, 2n\}$ and $H_0(C_*) = H_{2n}(C_*) = R$. For each $E \in \mathscr{A}_p(G)$ such that $C_*(E) = 0$, one has $\operatorname{rk}_A(H^*(E; H^n(C^*)) = 2$, where $A = H^*(E; R)$.*

Proof. As in the above, we may assume that $R = k$, $G = (\mathbb{Z}_p)^r$ and prove the statement for $E = G$. Again choose $\alpha \in k^r$ as in 4.3 above such that $k\langle u_\alpha\rangle \cap kH = k[1]$ for all isotropy subgroups H of C_*. We remark that the set of such α forms a Zariski open (hence dense) subset of the affine k-space k^r. Since $C_*(G) = 0$, $C_*(\langle u_\alpha\rangle) = 0$ also and $C_* \,|k\langle u_\alpha\rangle$ is $k\langle u_\alpha\rangle$-free. It follows that $H_n(C_*)|_{k\langle u_\alpha\rangle} \cong M \oplus M \oplus F$, where F is $k\langle u_\alpha\rangle$-free and $M = k$ if $n =$ odd and $M = I =$ augmentation ideal for $n =$ even. See Assadi [A4]. Thus, $\hat{H}^*(\langle u_\alpha\rangle; H^n(C)) \cong \hat{H}^*(\langle u_\alpha\rangle; k \oplus k)$. Since the set of all α for which this holds forms an open dense subset of k^r, we conclude that $H^*(G; H^n(C^*))[1/t_\alpha] \cong H^*(G; k \oplus k)[1/t_\alpha]$, and from this the claim follows. ▮

4.5. THEOREM. *Let p be an odd prime, and $E \in \mathscr{A}_p(G)$. Let M be a self-dual permutable kG-module with an SDP-structure C_*. Suppose the rank of $H^*(E; M)$ over $H^*(E; k)$ is one. Then $\dim_k H_*(C_*(E)) = 3$.*

Proof. As in the above, we may assume that $E = \mathbb{Z}_p^r = G$, and let $H^*(G; k)_{\mathrm{red}} = A$ and $K =$ quotient field of A. Recall that in the hypercohomology spectral sequence $H^*(G; H^*(C^*)) \Rightarrow \mathbb{H}^*(G; C^*)$ all E_n^{**}-terms are modules over $H^*(G; k)$ for $n \geqslant 2$, and the differentials are $H^*(G; k)$-linear. The first differential to consider is $d_{n+1}\colon E_{n+1}^{i,n+j} \to E_{n+1}^{i+n+1,j}$ for $j = 0, n$ and all i. If $C_*(G) = 0$, then rank $H^*(G; M) = 2$ by Proposition 4.4. Therefore, we may assume that $C_*(G) \neq 0$, and choose $0 \leqslant l \leqslant 2n$ to be the smallest integer such that $C_l(G) \neq 0$. As in Proposition 4.3 choose $\alpha \in k^r$ such that $k\langle u_\alpha\rangle \cap k[H] = k[1]$ for all $H \neq G$. We will need the following lemmas in order to study the above spectral sequence:

4.6. LEMMA. *In the hypercohomology spectral sequence $H^*(\langle u_\alpha\rangle; H^*(C^*)) \Rightarrow \mathbb{H}^*(\langle u_\alpha\rangle; C^*)$ the differential $d_{n+1}\colon E_{n+1}^{i,n} \to E_{n+1}^{i+n+1,0}$ vanishes for all i.*

Proof of Lemma 4.6. If $l=0$, then we have a split augmentation $C_0(G) \leftrightarrows k$ which gives a split augmentation $C_0 \leftrightarrows k$. Thus, the induced homomorphism $H^*(\langle u_\alpha \rangle; k) \to \mathbb{H}^*(\langle u_\alpha \rangle; C^*)$ is split injective. Now suppose that $l>0$. We define kG-chain complexes D_* such that $D_i = C_i$ for $0 \leqslant i \leqslant l-1$ and $D_i = 0$ for $i \geqslant l$, and $\hat{C}_*$ from the exact sequence of kG-complexes: $0 \to D_* \to C_* \xrightarrow{q} \hat{C}_* \to 0$. By the choice of $l>0$, D_* is $k\langle u_\alpha \rangle$-free, and since it is bounded, $\mathbb{H}^i(\langle u_\alpha \rangle; D^*) = 0$ for $i \gg 0$. Therefore, for all large values of i, $q^*: \mathbb{H}^i(\langle u_\alpha \rangle; \hat{C}^*) \to \mathbb{H}^i(\langle u_\alpha \rangle; C^*)$ is an isomorphism. Since $\hat{C}_*$ has a split augmentation (shifted to degree l) $\sigma: \hat{C}_l = C_l \leftrightarrows k$, the differential $\hat{d}_{n-l+1}: E^{i,n}_{n-l+1}(\hat{C}^*) \to E^{i+l+1,n-l}_{n-l+1}(\hat{C}_*)$ vanishes for all large values of i, as in the previous case. The periodicity of the cohomology of $\langle u_\alpha \rangle$ implies that $\hat{d}_{n-l+1} = 0$ for all values of i. Therefore, $\sigma^*: H^*(\langle u_\alpha \rangle; k) \to \mathbb{H}^*(\langle u_\alpha \rangle; \hat{C}^*)$ is injective. Since q^* is an $H^*(\langle u_\alpha \rangle; k)$-linear isomorphism for $i \gg 0$, $H^i(\langle u_\alpha \rangle; k) \to \mathbb{H}^i(\langle u_\alpha \rangle; C^*)$ is injective. This in turn implies that the above differential $d_{n+1} = 0$ for all i. ∎

Let $h: C^* \to C_*$ be a chain homotopy equivalence given by the self-duality of C_*, and let $h_*: H^i(C^*) \to H_{2n-i}(C^*)$ be the induce kG-isomorphism. Choose a generator $\Omega \in H^{2n}(C^*) \cong k$, and define the non-degenerate pairing $\eta: H^i(C^*) \otimes H^{2n-i}(C^*) \to k \cong H^{2n}(C^*)$ via $\eta(f \otimes g) = g(h_*(f))\,\Omega$. Here we have used the universal coefficients formula $H^{2n-i}(C^*) \xrightarrow{\cong} \operatorname{Hom}_k(H_{2n-i}(C_*), k)$. Since h_* is a kG-isomorphism, η becomes a kG-homomorphism with respect to the diagonal action on the left side. Besides, we have the following commutative diagram in which τ is the trace of an endomorphism:

$$(\square) \qquad \begin{array}{ccc} M \otimes M & \xrightarrow{\eta} & k \\ \downarrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\tau} \\ M \otimes \operatorname{Hom}(M, k) & \xrightarrow{\cong} & \operatorname{End}(M) \end{array}$$

4.7. LEMMA. *Keep the above notation and assume that $\hat{H}^i(\langle u_\alpha \rangle; M) \cong k$ for all i. Then it follows that:*

(a) *η is split surjective;*

(b) *$\eta_*: \hat{H}^*(\langle u_\alpha \rangle; M \otimes M) \to \hat{H}^*(\langle u_\alpha \rangle; k)$ is an isomorphism;*

(c) *M is stably $k\langle u_\alpha \rangle$-isomorphic either to k or to the augmentation ideal of $k\langle u_\alpha \rangle$.*

Proof of Lemma 4.7. Any indecomposable $k[\mathbb{Z}_p]$-module N is determined by the Jordan canonical form of the generator of $\mathbb{Z}_p$ acting on the k-vector space N. This shows that if $N \neq 0$ and $N \neq k\mathbb{Z}_p$, then

$1 \leqslant \dim_k(N) \leqslant p-1$, and a standard cohomology calculation and induction on $\dim_k N$ shows that $\hat{H}^i(\mathbb{Z}_p; N) \cong k$ for all $i \in \mathbb{Z}$ in this case. The assumption of Lemma 4.6 shows that $M \cong M_0 \oplus F$, where F is $k\langle u_\alpha \rangle$-free and M_0 is indecomposable such that $1 \leqslant \dim M_0 \leqslant p-1$. Hence $\dim M \not\equiv 0 \bmod p$. Define a splitting $\xi: k \to \mathrm{End}(M)$ by $\xi(1) = (1/\dim M)(\mathrm{id})$, where $\mathrm{id} \in \mathrm{End}(M)$ is the identity. The above commutative square ($\square$) yields (a). To prove (b), observe that $M \otimes M \cong M_0 \otimes M_0 \oplus F \otimes M_0 \oplus M_0 \otimes F \oplus F \oplus F \cong M' \oplus F'$, where M' is indecomposable and M' is $k\langle u_\alpha \rangle$-free. The splitting of part (a), and the Krull–Schmidt–Azumaya theorem applied to the isomorphism $k \oplus \mathrm{Ker}(\eta) \cong M' \oplus F'$ implies that $M \otimes M \cong k \oplus (k\langle u_\alpha \rangle)^s$ and $\mathrm{Ker}(\eta) \cong F'$ is $k\langle u_\alpha \rangle$-free. Thus, η_* is an isomorphism and (b) follows. An easy calculation shows that for M_0 to satisfy $M_0 \otimes M_0 \cong k \oplus (k\langle u_\alpha \rangle)^t$, the only possibilities are $\dim M_0 = 1$ or $p-1$, hence (c) follows. ▮

4.8. Lemma. *Keep the hypotheses of Lemma 4.7 and the above notation, and consider the internal cup-product in group cohomology,* $\beta: \hat{H}^r(\langle u_\alpha \rangle; M) \otimes \hat{H}^s(\langle u_\alpha \rangle; M) \to \hat{H}^{r+s}(\langle u_\alpha \rangle; M \otimes M)$.

(a) *If M is $k\langle u_\alpha \rangle$-stably isomorphic to k, then β is an isomorphism for all $r \equiv 0 \bmod 2$ and all $s \in \mathbb{Z}$.*

(b) *If M is $k\langle u_\alpha \rangle$-stably isomorphic to the augmentation ideal of $k\langle u_\alpha \rangle$, then β is an isomorphism for all $r \equiv s \equiv 1 \bmod 2$.*

Proof. The proof of (a) is immediate from periodicity of the cohomology of $\langle u_\alpha \rangle = \mathbb{Z}_p$. To see (b), we proceed as follows. Consider the exact sequence $0 \to M \to F_1 \to k \oplus F_2 \to 0$ in which F_1 and F_2 are suitable $k\langle u_\alpha \rangle$-free modules, and tensor it with M to obtain the exact sequence $0 \to M \otimes M \to F_1' \to M \oplus F_2' \to 0$, where F_1' and F_2' are also free. Let $\delta: \hat{H}^*(\langle u_\alpha \rangle; k) \to \hat{H}^{*+1}(\langle u_\alpha \rangle; M)$ and $\delta': \hat{H}^*(\langle u_\alpha \rangle; M) \to \hat{H}^{*+1}(\langle u_\alpha \rangle; M \otimes M)$ be the connecting homomorphisms in the long exact sequences of group cohomology applied to the above short exact sequences. δ and δ' are $\hat{H}^*(\langle u_\alpha \rangle; k)$-module isomorphisms and compatible with cup-products (see Brown [B3] or Cartan and Eilenberg [CE]). Therefore, we obtain the commutative diagram

$$\begin{tikzcd} H^{2i}(\langle u_\alpha \rangle; k) \otimes H^{2j-1}(\langle u_\alpha \rangle; M) \arrow[r, "\mu"] \arrow[d, "\delta \otimes \mathrm{id}"', "\cong"] & H^{2i+2j-1}(\langle u_\alpha \rangle; M) \arrow[d, "\delta'"', "\cong"] \\ H^{2i+1}(\langle u_\alpha \rangle; M) \otimes H^{2j-1}(\langle u_\alpha \rangle; M) \arrow[r, "\beta"] & H^{2i+2j}(\langle u_\alpha \rangle; M \otimes M) \end{tikzcd}$$

In the above, μ and β are given by cup-products. Since μ is an isomorphism, so is β, and (b) is proved. ▮

4.9. LEMMA. *If $\hat{H}^i(\langle u_\alpha \rangle; M) \cong k$ for all $i \in \mathbb{Z}$, then the hypercohomology spectral sequence $H^*(\langle u_\alpha \rangle; H^*(C^*)) \Rightarrow \mathbb{H}^*(\langle u_\alpha \rangle; C^*)$ collapses.*

Proof. From Lemma 4.6, it follows that we need to consider only $d_{n+1}: E_{n+1}^{i,2n} \to E_{n+1}^{i+n+1,n}$. First, note that there is a pairing in the above spectral sequence $\gamma: E_2^{i,a} \otimes E_2^{j,b} \to E_2^{i+j,a+b}$ as follows. Let $\eta_*: H^*(\langle u_\alpha \rangle; H^i(C^*) \otimes H^j(C^*)) \to H^*(\langle u_\alpha \rangle; H^{i+j}(C^*))$ be the induced homomorphism from the pairing η given above by the self-duality. Note that in this case, we need to consider $i = j = n$, and if $i = 0$ or $j = 0$, η_* is the identity. Next, we have the group cohomology cup-product β as in Lemma 4.8 above. γ is the composition $\eta_* \circ \beta$ on the E_2-level. We remark that β is constructed using a diagonal approximation in a resolution for $\langle u_\alpha \rangle$; hence, β satisfies a suitable form of the Leibnitz formula with respect to the differentials in the hypercohomology spectral sequences whose E_2-terms are $H^*(\langle u_\alpha \rangle; H^*(C^*))$ and $H^*(\langle u_\alpha \rangle; H^*(C^* \otimes C^*)) \supseteq H^*(\langle u_\alpha \rangle; H^*(C^*) \otimes H^*(C^*))$. Moreover, η_* commutes with the differentials since it is induced by coefficient homomorphisms.

Let $t \in H^2(\langle u_\alpha \rangle; k) \cong k$ and $\Omega \in \hat{H}^0(\langle u_\alpha \rangle; H^{2n}(C^*)) \cong k$ be generators. From Lemma 4.7(c) we are led to consider the two cases of Lemma 4.8. First, suppose M is stably isomorphic to k, and write $\Omega = \eta_* \beta(x \oplus y)$, where $x, y \in \hat{H}^0(\langle u_\alpha \rangle; M)$ and we have used Lemmas 4.7(b) and 4.8(a). Then $d_{n+1}(\Omega) = d_{n+1}(\eta_* \beta(x \otimes y)) = \eta_* d_{n+1}(\beta(x \otimes y)) = \eta_*(d_{n+1}(x) \otimes y \pm x \otimes d_{n+1}(y)) = 0$ since $d_{n+1}(x) = 0 = d_{n+1}(y)$ by Lemma 4.6. In the case M is stably isomorphic to the augmentation ideal of $k\langle u_\alpha \rangle$, we have $t\Omega = \eta_* B(u \otimes v)$, where $v, u \in H^1(\langle u_\alpha \rangle; M)$. Then $d_{n+1}(t\Omega) = \eta_* d_{n+1}(\beta(u \otimes v)) = \eta_*(d_{n+1}(u) \otimes v \pm u \otimes d_{n+1}(v)) = 0$ again by the same lemmas. Since the E_r-terms are modules over $H^*(\langle u_\alpha \rangle; k)$ and the differentials are $H^*(\langle u_\alpha \rangle; k)$-linear, the periodicity of cohomology of $\langle u_\alpha \rangle$ implies that $d_{n+1} \equiv 0$. For dimension reasons and using the $H^*(\langle u_\alpha \rangle; k)$-module structure, it follows that $d_{2n+1} \equiv 0$ also, and the spectral sequence collapses as claimed. ∎

4.10. LEMMA. *With the hypotheses and the notation of Lemma* 4.9 *above, we have $H_*(C_*(\langle u_\alpha \rangle)) \cong k^3$.*

Proof. This follows from Lemma 4.9 and the localization theorem 2.1 applied to the $k\langle u_\alpha \rangle$-permutation complex C_* as in Proposition 4.3 above. ∎

4.11. LEMMA. *Let p be an odd prime, and let X_* be a connected $k[\mathbb{Z}_p]$-permutation complex such that $H_i(X_*) = 0$ for $i \notin \{0, n, 2n\}$ and $H_0(X_*) = H_{2n}(X_*) = k$. If $H_*(X_*(\mathbb{Z}_p)) = k$, then $H^n(X_*)$ satisfies $\hat{H}^i(\mathbb{Z}_p; H^n(X^*)) = k$ for all $i \in \mathbb{Z}$.*

Proof. As in Lemma 4.6, the differential $d^{*,n}_{n+1}: E^{i,n}_{n+1} \to E^{i+n+1,0}_{n+1}$ vanishes. Denote by $t \in H^2(\mathbb{Z}_p; k) = k$ the generator, and localize the spectral sequence by inverting t, so that $E^{i,n}_{n+1}[1/t] \cong \hat{H}^i(\mathbb{Z}_p; H^n(X^*))$ and $E^{i,0}_{n+1}[1/t] \cong \hat{H}^i(\mathbb{Z}_p; k) \cong E^{i,2n}_{n+1}[1/t]$. By the localization theorem (see 2.1), $\mathbb{H}^*(\mathbb{Z}_p; X^*)[1/t] \cong \hat{H}^*(\mathbb{Z}_p; k)$, so that the differential $d^{*,2n}_{n+1}[1/t]$: $\hat{H}^i(\mathbb{Z}_p; H^{2n}(X^*)) \to \hat{H}^{i+n+1}(\mathbb{Z}_p; H^n(X^*))$ is an isomorphism. ∎

We complete the proof of Theorem 4.5 as follows. Suppose $\operatorname{rank}(H^*(G; M)) = 1$. In the hypercohomology spectral sequence $H^*(G; H^*(C^*)) \Rightarrow \mathbb{H}^*(G; C^*)$, the differential $d_{n+1}: E^{i,n}_{n+1} \to E^{i+n+1,0}_{n+1}$ induces k-homomorphisms $d^{*,n}_{n+1} \otimes \mathrm{id}: E^{*,n}_{n+1} \otimes_A k \to E^{*+n+1,0}_{n+1} \otimes_A k$ and $d^{*,2n}_{n+1} \otimes \mathrm{id}$: $E^{*,2n}_{n+1} \otimes_A k \to E^{*+n-1}_{n+1} \otimes_A k$. Besides, $E^{*,n}_{n+1} \otimes_A k \cong k \cong E^{*,0}_{n+1} \otimes_A k \cong E^{*,2n}_{n+1} \otimes_A k$. The proof of Lemma 4.6 applied to the hypercohomology spectral sequence of G shows that $d^{*,n}_{n+1} \otimes \mathrm{id} = 0$. (One needs to remark only that by Lemma 2.2, $\mathbb{H}^*(G; D^*) \otimes_A k = 0$ in that proof.) If $d^{*,n}_{n+1} \otimes_A k \neq 0$, then it must be an isomorphism. This implies that $\mathbb{H}^*(G; C^*) \otimes_A k \cong k$. From the localization theorem 2.1 it follows that $H^*(C^*(G)) = k$. For a choice of $\alpha \in k^r$ as in Lemma 4.6, $C_*(G) = C_*(\langle u_\alpha \rangle)$ so that $H_*(C_* \langle u_\alpha \rangle) = k$. From Lemma 4.2 above, it follows that $\hat{H}^i(\langle u_\alpha \rangle; M) = k$ for all $i \in \mathbb{Z}$. But this contradicts Lemma 4.10. This contradiction shows that $d^{*,2n}_{n+1} \otimes_A k = 0$. Since $d^{*,2n}_{2n+1} \otimes_A k = 0$ again by the proof of Lemma 4.6, and $d^{*,n}_{2n+1} = 0$ for dimension reasons, the spectral sequence collapses. Hence $\mathbb{H}^*(G; C^*) \otimes_A k \cong k^3$ and the localization theorem shows that $\dim_k H_*(C_*(G)) = 3$ as desired. ∎

4.12. Example. Let p be odd and $G = \mathbb{Z}_p$, and consider the linear representation of G on $\mathbb{C}^3$ with three non-trivial distinct weights. The induced action on the complex projective space $\mathbb{C}P^2$ has precisely three fixed points, and $H_2(\mathbb{C}P^2) = \mathbb{Z}$. If we choose m free orbits of points in $\mathbb{C}P^2$ and blow-up these points, we get another algebraic action on an algebraic surface X, and topologically $X = \mathbb{C}P^2 \# (m\overline{\mathbb{C}P^2})$ (connected sum) and $H_2(X) \cong \mathbb{Z} \oplus (\mathbb{Z}G)^m$. Similar examples can be constructed using projective actions of $G = \mathbb{Z}_p \times \mathbb{Z}_p$ on $\mathbb{C}P^2$ and by blowing up an orbit G/H of points, one obtains an algebraic surface Y with $H_2(Y) \cong \mathbb{Z} \oplus \mathbb{Z}[G/H]$. More complicated examples can be constructed by a variation on these examples. As remarked in Section 1, $C_*(X)$ and $C_*(Y)$ for suitable G-simplicial structures on X and Y provide examples of SDP-structures in which $H^*(G; M)$ has rank one over $H^*(G; k)$. The geometric consequence of Theorem 4.5 is that for a Poincaré duality complex with an effective $(\mathbb{Z}_p)^r$-action, the fixed point set of any subgroup $H \subseteq \mathbb{Z}_p^r$ is never homologically acyclic. Theorem 4.5 may be considered the algebraic analogue of the theorems of Conner and Floyd [CF1], [CF2], Atiyah and Bott [AB], and W. Browder [Bw].

4.13. COROLLARY. *Let p be an odd prime, $G=(\mathbb{Z}_p)^r$, and C_* be an SDP-structure over kG of formal dimension $2n$ and $H_n(C_*)=M$. Then the following hold*:

(1) *If* $H_0(C_*(G))\neq 0$, *then* $\dim H_*(C_*(G))\geqslant 2$.

(2) $\dim H_*(C_*(G))=2$ *if and only if* $H^*(G;M)$ *is a torsion* $H^*(G;k)$*-module*.

Proof. (1) By choosing $\alpha\in k^r$ as in Theorem 4.5 above, it follows that $\dim H_*(C_*(\langle u_\alpha\rangle))\neq 1$. Since $C_*(\langle u_\alpha\rangle)=C_*(G)$, $\dim H_*(C_*(G))\geqslant 2$. Part (2) follows from Proposition 4.3 and the following argument. $H^*(G;M)$ is a torsion $H^*(G;k)$-module if and only if the Krull dimension of the support of $H^*(G;M)$ in $\operatorname{Spec} H^{\mathrm{ev}}(G;k)$ is less than $\dim \operatorname{Spec} H^{\mathrm{ev}}(G;k)=\operatorname{rank}(G)=r$. Here, $H^{\mathrm{ev}}(G;k)=\bigoplus_{i\geqslant 0} H^{2i}(G;k)$ is a commutative k-algebra whose reduced k-algebra is isomorphic to the polynomial ring $k[t_1,...,t_n]$. From the positive answer to the Carlson conjecture (Avrunin and Scott [AS], Carlson [C1], [C2]) it follows that there is an $\alpha\in k^r$ such that $M|k\langle u_\alpha\rangle$ is $k\langle u_\alpha\rangle$-free. In fact, the set of such vectors α forms a Zariski open dense subset of k^r, namely, the complement of the proper closed subset $(\operatorname{Supp} H^*(G;M))\cap \operatorname{Max}\operatorname{Spec}(H^{\mathrm{ev}}(G;k))$. Thus, it is possible to arrange for such an α to satisfy $C_*(\langle u_\alpha\rangle)=C_*(G)$ as well. Now Proposition 4.4 shows that $H_*(C_*\langle u_\alpha\rangle)=k\oplus k$, hence $\dim H_*(C_*(G))=2$. The converse proceeds along the same lines: For any $\alpha\in k^r$ in the complement of the $\mathbb{F}_p$-rational linear subspaces corresponding to proper subgroups of G, $C_*(\langle u_\alpha\rangle)=C_*(G)$. The proof of Proposition 4.4 shows that if $\dim H_*(C_*(\langle u_\alpha\rangle))=2$, then $\hat{H}^*(\langle u_\alpha\rangle;M)=0$, so that M is $k\langle u_\alpha\rangle$-free. Therefore, the Carlson rank variety $V_G^r(M)$ (see Carlson [C1]) is a proper subset of k^r. Again, by the Avrunin–Scott theorem [AS, Theorem 1], the cohomological support variety $V_G(M)$ is a proper subset of $\operatorname{Max}\operatorname{Spec}(H^{\mathrm{ev}}(G;k))$. Hence $H^*(G;M)$ is a torsion $H^*(G;k)$-module.

5. UNITS IN THE GREEN RING

Recall that the Green ring of RG is the Grothendieck ring associated to the set of isomorphism classes of indecomposable RG-lattices. The direct sum and tensor product (over R) of RG-modules induce the ring operations. The stable Green ring is the quotient of the Green ring by the ideal generated by RG-projective modules. We use the notation $\mathbb{R}(RG)$ and $\tilde{\mathbb{R}}(RG)$ for the Green ring and its stable version. A unit in $\tilde{\mathbb{R}}(RG)$ is seen to be represented by an RG-lattice M for which there exists another RG-lattice M' such that $M\otimes M'\cong R\oplus P$, where P is RG-projective. An important class of RG-lattices is the endo-trivial modules introduced by J. Alperin and E. Dade (see Dade [D] and Alperin [Alp]) and they are characterized by $\mathrm{End}_R(M) \cong R \oplus P$ with $P=$ projective RG-module. The canonical RG-isomorphism $\mathrm{Hom}_R(M, R) \otimes M \cong \mathrm{End}_R(M)$ shows that endo-trivial modules represent units of $\tilde{\mathbb{R}}(RG)$. In the following, we determine the units of $\tilde{\mathbb{R}}(RG)$ which are permutable RG-modules arising in Steenrod's problem. It is appropriate to call an RG-module M *spherical* if there is a finite-dimensional G-space X such that non-equivariantly, X is homotopy equivalent to a bouquet of d-dimensional spheres and the homology representation $H_d(X; R)$ is RG-isomorphic to M. This is inspired by Quillen's terminology of d-spherical posets [Q2]. For example, if M is the Steinberg module of a finite Chevalley group G, or more generally the reduced homology of the simplicial complexes associated to posets $\mathscr{A}_p(G)$, $\mathscr{S}_p(G)$ or Solomon–Tits buildings (see Quillen [Q2] and Section 3 above), then M is d-spherical, where $d+1$ is the appropriate "rank" of G. Let us call M a spherical unit of $\tilde{\mathbb{R}}(G)$, if M is spherical and a unit in $\tilde{\mathbb{R}}(G)$ and such that its inverse in $\tilde{\mathbb{R}}(G)$ is also spherical.

5.1. Example. If M is finitely generated endo-trivial and spherical, then M is a spherical unit. To see this, suppose that $H_d(X; R) \cong M$ and we have arranged for X to be a finite-dimensional simplical complex with a simplicial G-action using standard approximation arguments of algebraic topology. Then we choose for G a large-dimensional real or complex representation space V, and embed X G-equivariantly in V, using the Mostow–Palais embedding theorem (cf. Bredon [Bdn]). Let V_∞ be the one-point compactification of V, which is a sphere with G-action. Let Y be the complement of X in V_∞. Then by Alexander duality, Y is connected, $H_i(Y)=0$ for $i \neq 0$, $n-d-1$, and $H_{n-d-1}(Y) \cong H^d(X)$, so that $H_{n-d-1}(Y; R) \cong \mathrm{Hom}_R(H_d(X; R), R) \cong \mathrm{Hom}_R(M, R)$. Thus, $\mathrm{Hom}_R(M, R)$ is also spherical. By endo-triviality, $\mathrm{Hom}_R(M, R) \otimes M \cong R \oplus P$, where P is RG-projective. Thus M is a spherical unit in $\tilde{\mathbb{R}}(RG)$ as claimed.

5.2. Theorem. *Suppose M is a spherical unit in the stable Green ring $\tilde{\mathbb{R}}(RG)$, where G is an abelian p-group and R is a field of characteristic p. Then M is stably isomorphic to $\Omega^n(R)$ for some $n \in \mathbb{Z}$. If M is indecomposable, then $M \cong \Omega^n(R)$.*

5.3. *Remarks.* (1) Ω is the Heller operator [Hel]. See Curtis-Reiner [CR] for the definition and properties.

(2) A deep and difficult theorem of E. Dade [D] characterizes endotrivial RG-modules, for $G=$ abelian p-group and $R=$ field of characteristic p. In a forthcoming paper, we prove that 5.2 holds without the spherical hypothesis by a proof independent of Dade's. However, the more general results require non-elementary algebraic geometry. The spherical case, however, uses elementary arguments which may help one to develop an intuitive feeling for the general results.

(3) From Section 1 it follows easily that spherical RG-modules are RG-permutable.

Proof. Let E be the maximal p-elementary abelian subgroup of G. By suspending, if necessary, we may assume that there is a G-space X such that $H_d(X; R) = M$ and $X^G \neq \phi$. By definition, $\dim X < \infty$ and X is homotopy equivalent to a bouquet of d-dimensional spheres. By standard arguments in algebraic topology, we may assume that X is a G–CW complex, so that $C_* \overset{\text{def}}{=} C_*(X)$ is a permutation complex with permutation basis given by the cells of X. Let $\Sigma(X)$ be the singular set of the G-action on X, that is, the union of fixed points X^H for all $1 \neq H \subseteq G$. Note that in the reduced regular representation $\mathbb{C}[G]/\mathbb{C}[G]^G$, we may choose a G-invariant inner product by averaging any given inner product. Call S the unit sphere in this representation. Then S is a sphere with G-action and $S^G = \phi$. Hence the join $X \circ S$ with its natural G-action is homologically only an iterated suspension of X, so that $X \circ S$ will be still spherical. Moreover. $(X \circ S)^G = X^G \neq \phi$. This operation preserves homology up to RG-isomorphism and it has the effect of increasing the codimension of the singular set, i.e., $\dim X - \dim \Sigma(X)$ will be arbitrarily large after repeated replacements of X by $X \circ S$. There is another operation which changes $H_d(X)$ by $\Omega^r H_d(X)$, $r \geqslant 0$, up to stable RG-isomorphism. This is obtained by adding free orbits of $(d+1)$-cells to X, obtaining a G–CW complex X'. We choose a surjection $(RG)^s \to M$ and regard this as $H_{d+1}(X', X) \otimes R \xrightarrow{\partial} H_d(X)$, which is geometrically realized (using Hurewicz's theorem) by attaching cells $\bigsqcup G \times D^{d+1}$ to X to obtain X'. This operation has the effect of increasing the homological codimension, i.e., since $\Sigma(X) = \Sigma(X')$, $\dim \Sigma(X)$ remains constant and the dimension d, where $H_d(X; R) \neq 0$, grows arbitrarily large. Since $\Omega^r M \otimes \Omega^{-r} N$ is RG-stably isomorphic to $M \otimes N$, $\Omega^r M$ is still a spherical unit.

Now choose Y such that $Y^G \neq \phi$ and satisfying other hypotheses which X already satisfies, and such that $H_{d'}(Y; R) \cong M'$ is an inverse of M in $\tilde{\mathbb{R}}(RG)$. That is, $M \otimes M' \cong R \oplus P$, where P is RG-projective. Note that since R is a field of characteristic p and G is a p-group, RG is local and projective modules coincide with free RG-modules. However, we will use this remark only for convenience. Consider $Z = X \wedge Y$, the smash product with the induced action (see Section 1). The Künneth formula shows that $\bar{H}_*(Z; R) \cong H_d(X; R) \otimes H_{d'}(Y; R) \cong R \oplus P$. By the localization theorem (Theorem 2.1 above, for example, or Hsiang [Hsg]), $\bar{H}_*(Z^H; R) \cong R$ for each $1 \neq H \subseteq E$. Since $Z^H = X^H \wedge Y^H$ and $\bar{H}_*(Z^H; R) \cong \bar{H}_*(X^H; R) \otimes \bar{H}_*(Y^H; R)$, it follows that $\bar{H}_*(X^H; R) \cong R$. Let $\delta(H)$ be the integer such that $\bar{H}_d(X^H; R) = R$ and note that since $X^H \supseteq X^G \neq \phi$, $\delta(H) \geqslant 0$.

Consider the set $\mathscr{U} = \{H \subseteq E : |E/H| = p\}$, and let W be the real linear representation of E which is the direct sum of $m(H)$ irreducible non-trivial linear representations of $E/H \cong \mathbb{Z}_p$ for each $H \in \mathscr{U}$. We choose $m(H)$ (depending on $p=2$ or $p>2$) such that $\dim_{\mathbb{R}} W^H = \delta(H)+1$. Let $\dim_{\mathbb{R}} W = l+1$. By shifting dimension or join operation as described above, we may arrange for X, and l, to satisfy $l+2 \geqslant \dim \Sigma(H)+2$. While this condition on X is not necessary for the proof, it will simplify and make the following argument more elementary. Consider the l-skeleton of X, call it $X^{(l)}$ and its cellular chain complex $C_*(X^{(l)}) = D_*$. Let $F_* = C_*(X)/D_*$, which is RG-free by choice of l. D_* is a permutation complex which is based and $H_l(D_* \otimes R) \cong H_{l+1}(F_* \otimes l)$. Since $H_i(F_* \otimes R) = 0$ unless $i=d$ or $i=l+1$, and F_* is RG-free and $F_i = 0$ for $i \leqslant l$ or $i > \dim X$, it follows easily that $H_d(X; R)$ is RG-stably isomorphic to $H_{l+1}(F_* \otimes l)$. Hence, up to replacing M by $\Omega^r M$ for some $r \in \mathbb{Z}$, we have reduced the problem to showing that $H_l(D_* \otimes R)$ is RG-stably isomorphic to $\Omega^n(R)$ for some $n \in \mathbb{Z}$. (In the terminology of Assadi [A2], M and $H_l(D_*; R)$ are ω-stably isomorphic. See [A2] for related discussions.)

The linear representation W satisfies the dimension equation $\dim W - \dim W^E = \sum_{H \in \mathscr{U}} (\dim W^H - \dim W^E)$, hence the restriction of the G-action on X to the E-action satisfies the Borel formula $l - \delta(E) = \sum_{H \in \mathscr{U}} (\delta(H) - \delta(E))$. (See Borel [Bor], Bredon [Bdn], or Hsiang [Hsg] for more details.) According to Dotzel [Dot], the converse to Borel's theorem holds for such a situation and $H_l(X^{(l)}; R)$ is RE-isomorphic to $R \oplus P_0$, where P_0 is RE-projective. By the above discussion, we may write $M \oplus (RE)^t \cong \Omega^{d-l}(H_l(X^{(l)}; R)) \oplus (RE)^s \cong \Omega^{d-l}(R) \oplus (RE)^u$ as RE-modules. Consider $\Omega^{l-d}(M)$ as an RG-module. By the above, $\Omega^{l-d}(M)|_E \cong R \oplus Q$, where Q is RE-free.

Consider the induced homomorphism $\rho^*\colon \hat{H}^*(G; \Omega^{l-d}(M)) \to \hat{H}^*(E; \Omega^{l-d}(M))$, which is an F-isomorphism in the terminology of Quillen [Q1]. To see this, observe that $\Omega^{l-d}(M)$ is stably isomorphic to $H_l(X^{(l)}; R)$, and for a choice of base point $x \in X^G$, $H^*_G(X^{(l)}, x; R) \cong H^*(G; H^l(X^{(l)}, x; R)$ for $* \geqslant l+1$, and similarly for E. This is true since the spectral sequences of equivariant cohomology (or equivalently hypercohomology) have only one row. By Quillen [Q1], one knows that $H^*_G(X, x; R) \to H^*_E(X, x; R)$ is an F-isomorphism since E is the unique p-elementary abelian subgroup of G. In particular, $\rho^*\colon \hat{H}^0(G; \Omega^{l-d}(M)) \to \hat{H}^0(E; \Omega^{l-d}(M)) \cong R$ is non-zero, hence surjective. Let $M' = \Omega^{l-d}(M)$. Thus, we may choose $f \in \operatorname{Hom}_{RG}(R, M')$ such that in the diagram

$$\begin{tikzcd} \hat{H}^0(G; R) \arrow[r, "f_*"] \arrow[d, "\cong"] & \hat{H}^0(G; M') \arrow[d, "\rho^*"] \\ \hat{H}^0(E; R) \arrow[r, "f_*"] & \hat{H}^0(E; M') \end{tikzcd}$$

$f_*: \hat{H}^0(G; R) \to \hat{H}^0(G; M')$ is injective. In the exact sequence of RG-modules $0 \to R \xrightarrow{f} M' \to \mathrm{Coker}(f) \to 0$, $f_\#: \hat{H}^*(E; R) \to \hat{H}^*(E; M')$ is an isomorphism, so that $\hat{H}^*(E; \mathrm{Coker}(f)) = 0$. It follows from Rim [R] that $\mathrm{Coker}(f)|_E$ is RE-free. By Chouinard's theorem (cf. [Ch], Curtis-Reiner [CR]), $\mathrm{Coker}(f)$ is RG-projective, hence the short exact sequence above splits over RG and M' is stably isomorphic to R. Therefore, M is stably isomorphic to $\Omega^{d-l}(R)$, and if M is indecomposable, $M \cong \Omega^{d-l}(R)$. ∎

In the above proof we only used the fact that G has a unique p-elementary abelian group in an essential way. Other references to the fact that G is an abelian p-group may be avoided, and a modification of the above argument proves the following more general result:

THEOREM. *Let R be a field of characteristic p, and assume that G is a finite group with a unique conjugacy class of maximal p-elementary abelian subgroups. Suppose that M is a spherical unit in the stable Green ring $\tilde{\mathbb{R}}(RG)$. Then M is RG-stably isomorphic to $\Omega^n(R)$ for some $n \in \mathbb{Z}$.*

It is also worthwhile to point out the following corollary, whose proof follows from 5.3 and the constructions of Section 1 as used in the proof of Theorem 5.2.

5.4. COROLLARY. *The spherical units of any finite group G in $\tilde{\mathbb{R}}(RG)$ form a multiplicative subgroup of the group of all units. Therefore, if M is a spherical unit, so are* $\mathrm{Hom}_R(M, R)$ *and $\Omega^n M$ for all $n \in \mathbb{Z}$.*

The above results provide some evidence for the following:

5.5. *Conjecture.* For an arbitrary finite group G and $R = \mathbb{Z}$ or a field of characteristic p, all units of $\tilde{\mathbb{R}}(RG)$ are spherical.

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