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On the partial differential equations of electrostatic MEMS devices III: Refined touchdown behavior

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Abstract

This paper is a continuation of [N. Ghoussoub, Y. Guo, On the partial differential equations of electrostatic MEMS devices: Stationary case, SIAM J. Math. Anal. 38 (2007) 1423–1449] and [N. Ghoussoub, Y. Guo, On the partial differential equations of electrostatic MEMS devices II: Dynamic case, NoDEA Nonlinear Differential Equations Appl. (2008), in press], where we analyzed nonlinear parabolic problem $u_t = \Delta u - \frac{\lambda f(x)}{(1+u)^2}$ on a bounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions. This equation models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at -1. Here u is modeled to describe dynamic deflection of the elastic membrane. When a voltage—represented here by λ—is applied, the membrane deflects towards the ground plate and a snap-through (touchdown) must occur when it exceeds a certain critical value λ^* (pull-in voltage), creating a so-called "pull-in instability" which greatly affects the design of many devices. In an effort to achieve better MEMS design, the material properties of the membrane can be technologically fabricated with a spatially varying dielectric permittivity profile $f(x)$. In this work, some a priori estimates of touchdown behavior are established, based on which the refined touchdown profiles are obtained by adapting self-similar method and center manifold analysis. Applying various analytical and numerical techniques, some properties of touchdown set—such as compactness, location and shape—are also discussed for different classes of varying permittivity profiles.

Keywords: Pull-in voltage; MEMS; Quenching profiles; Self-similar

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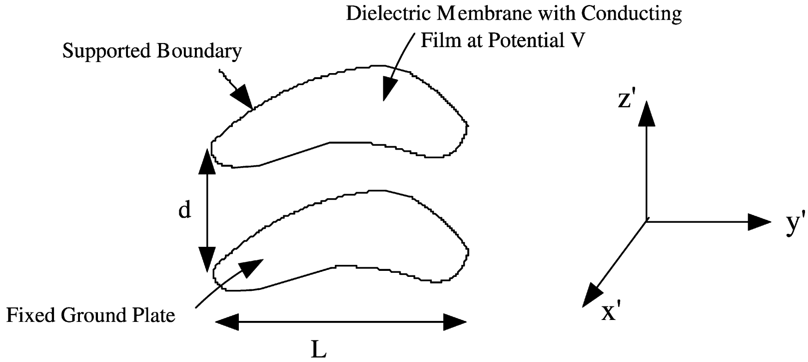

Fig. 1. The simple electrostatic MEMS device.

1. Introduction

Micro-Electromechanical Systems (MEMS) are often used to combine electronics with micro-size mechanical devices in the design of various types of microscopic machinery. MEMS devices have therefore become key components of many commercial systems, including accelerometers for airbag deployment in automobiles, ink jet printer heads, optical switches and chemical sensors and so on. The simplicity and importance of this technique have led many applied mathematicians and engineers to study mathematical models of electrostatic-elastic interactions. An overview of the physical phenomena of the mathematical models associated with the rapidly developing field of MEMS technology is given in [18].

The key component of many modern MEMS is the simple idealized electrostatic device shown in Fig. 1. The upper part of this device consists of a thin and deformable elastic membrane that is held fixed along its boundary and which lies above a rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage V is applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate, and when V is increased beyond a certain critical value V^*—known as pull-in voltage—the steady-state of the elastic membrane is lost, and proceeds to touchdown, i.e. snap through, at a finite time creating the so-called pull-in instability.

A mathematical model of the physical phenomena, leading to a partial differential equation for the dimensionless dynamic deflection of the membrane, was derived and analyzed in [5] and [15]. In the damping-dominated limit, and using a narrow-gap asymptotic analysis, the dimensionless dynamic deflection $u = u(x,t)$ of the membrane on a bounded domain Ω in $\mathbb{R}^2$, is found to satisfy the following parabolic problem

$$u_t - \Delta u = -\frac{\lambda f(x)}{u^2} \quad \text{for } x \in \Omega, \tag{1.1a}$$

$$u(x,t) = 1 \quad \text{for } x \in \partial\Omega, \tag{1.1b}$$

$$u(x,0) = 1 \quad \text{for } x \in \Omega. \tag{1.1c}$$

An outline of the derivation of (1.1) was given in Appendix A of [15]. The initial condition in (1.1c) assumes that the membrane is initially undeflected and the voltage is suddenly applied to the upper surface of the membrane at time $t = 0$. The parameter $\lambda > 0$ in (1.1a) characterizes the relative strength of the electrostatic and mechanical forces in the system, and is given in terms of the applied voltage V by $\lambda = \frac{\varepsilon_0 V^2 L^2}{2T_e d^3}$, where d is the undeflected gap size, L is the length scale of the membrane, T_e is the tension of the membrane, and ε_0 is the permittivity of free space in the gap between the membrane and the bottom plate. We shall use from now on the parameter λ and λ^* to represent the applied voltage V and pull-in voltage V^*, respectively. Referred to as the *permittivity profile*, $f(x)$ in (1.1) is defined by the ratio $f(x) = \frac{\varepsilon_0}{\varepsilon_2(x)}$, where $\varepsilon_2(x)$ is the dielectric permittivity of the thin membrane.

There are several issues that must be considered in the actual design of MEMS devices. Typically one of the primary goals is to achieve the maximum possible stable deflection before touchdown occurs, which is referred to as *pull-in distance* (cf. [15] and [17]). Another consideration is to increase the stable operating range of the device by improving the pull-in voltage λ^* subject to the constraint that the range of the applied voltage is limited by the available power supply. Such an improvement in the stable operating range is important for the design of certain MEMS devices such as microresonators. One way of achieving larger values of λ^*, while simultaneously increasing the pull-in distance, was first studied in [17] and [15], and consists of introducing a spatially varying dielectric permittivity $\varepsilon_2(x)$ of the membrane. The idea is to locate the region where the membrane deflection would normally be largest under a spatially uniform permittivity, and then make sure that a new dielectric permittivity $\varepsilon_2(x)$ is largest—and consequently the profile $f(x)$ smallest—in that region.

J.A. Pelesko studied in [17] the steady-states of (1.1), when $f(x)$ is assumed to be bounded away from zero, i.e., $0 < C \leqslant f(x) \leqslant 1$ for all $x \in \Omega$. He established in this case an upper bound for λ^*, and derived numerical results for the power-law permittivity profile, from which the larger pull-in voltage and thereby the larger pull-in distance, the existence and multiplicity of the steady-states were observed. Recently, Y. Guo, Z. Pan and M.J. Ward studied in [15] the dynamic behavior of (1.1), which is also of great practical interest. They considered a more general class of profiles $f(x)$, where the membrane is allowed to be perfectly conducting, i.e., $0 \leqslant f(x) \leqslant 1$ for all $x \in \Omega$, with $f(x) > 0$ on a subset of positive measure of Ω. By using both analytical and numerical techniques, they obtained larger pull-in voltage λ^* and larger pull-in distance for different classes of varying permittivity profiles. These results were extended and sharpened in [8] and [1], where we focused on the steady-state solutions of (1.1) in the form

$$\begin{aligned} -\Delta v &= \frac{\lambda f(x)}{(1-v)^2}, && x \in \Omega, \\ v(x) &= 0, && x \in \partial\Omega, \end{aligned} \tag{$(S)_\lambda$}$$

with $0 < v < 1$ on $\Omega \subset \mathbb{R}^N$, and $f(x)$ was assumed to satisfy

$$\begin{gathered} f \in C^\alpha(\bar{\Omega}) \quad \text{for some } \alpha \in (0,1], \quad 0 \leqslant f \leqslant 1 \quad \text{and} \\ f > 0 \quad \text{on a subset of } \Omega \text{ with positive measure.} \end{gathered} \tag{1.2}$$

Theorem A. *(See [8, Theorem 1.1].) Assume f satisfies* (1.2) *on a bounded domain Ω, then there exists a finite pull-in voltage $\lambda^* := \lambda^*(\Omega, f) > 0$ such that*:

(1) *If $0 \leqslant \lambda < \lambda^*$, there exists at least one solution for $(S)_\lambda$.*
(2) *If $\lambda > \lambda^*$, there is no solution for $(S)_\lambda$.*

The rigorous bounds of pull-in voltage λ^* were also given in Theorem 1.1 of [8]. Fine properties of steady states—such as regularity, stability, uniqueness, multiplicity, energy estimates and comparison results—were shown in [8] and [1] to depend on the dimension of the ambient space and on the permittivity profile. For any solution v of $(S)_\lambda$, we introduced in [8] the linearized operator at v defined by $L_{v,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-v)^3}$, and its corresponding eigenvalues $\{\mu_{k,\lambda}(v);\ k = 1, 2, \ldots\}$. In particular, the following properties of positive minimal solutions of $(S)_\lambda$ were established in [8]. Here a solution v_λ of $(S)_\lambda$ is said to be a minimal solution, if $v_\lambda(x) \leqslant v(x)$ in Ω whenever v is any solution of $(S)_\lambda$.

Theorem B. *(See [8, Theorem 1.2].) Assume f satisfies* (1.2) *on a bounded domain Ω, and consider $\lambda^* := \lambda^*(\Omega, f)$ as defined in Theorem* A. *Then*:

(1) *For any $0 \leqslant \lambda < \lambda^*$, there exists a unique minimal solution v_λ of $(S)_\lambda$ such that $\mu_{1,\lambda}(v_\lambda) > 0$. Moreover for each $x \in \Omega$, the function $\lambda \to v_\lambda(x)$ is strictly increasing and differentiable on* $(0, \lambda^*)$.
(2) *If $1 \leqslant N \leqslant 7$ then—by means of energy estimates—one has $\sup_{\lambda\in(0,\lambda^*)} \|v_\lambda\|_\infty < 1$ and consequently, $v^* = \lim_{\lambda\uparrow\lambda^*} v_\lambda$ exists in $C^{1,\alpha}(\bar{\Omega})$ with $0 < \alpha < 1$ and is a solution for $(S)_{\lambda^*}$ such that $\mu_{1,\lambda^*}(v^*) = 0$. In particular, v^*—often referred to as the extremal solution of problem $(S)_\lambda$—is unique.*
(3) *On the other hand, if $N \geqslant 8$, $f(x) = |x|^\alpha$ with $0 \leqslant \alpha \leqslant \alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$ and Ω is the unit ball, then the extremal solution is necessarily $v^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ and is therefore singular.*

We remark that in general, the function v^* exists in any dimension, does solve $(S)_{\lambda^*}$ in a suitable weak sense and is the unique solution in an appropriate class. The above theorem says that it is however a classical solution in dimensions $1 \leqslant N \leqslant 7$.

For the dynamic problem (1.1), we now define

Definition 1.1. (1) A steady-state $u_\lambda(x)$ of (1.1) is said to be a maximal steady-state, if $u_\lambda(x) \geqslant u(x)$ in Ω whenever $u(x)$ is any steady-state of (1.1).

(2) A solution $u(x,t)$ of (1.1) is said to touchdown, i.e. quenching, at finite (infinite) time $T = T(\lambda, \Omega, f)$ if the minimum value of u reaches 0 at the time $T < \infty$ $(T = \infty)$.

More recently, in [9] we dealt with issues of global convergence as well as finite and infinite time touchdown of (1.1), together with [10], where one of the main results was the following analysis of the relationship between the applied voltage λ and dynamic solution u of (1.1):

Theorem C. *(See [9 and 10, Theorem 1.1].) Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose λ^* is as in Theorem* A. *Then the followings hold*:

(1) *If $\lambda \leqslant \lambda^*$, then there exists a unique solution $u(x,t)$ for* (1.1) *which globally converges pointwise as $t \to +\infty$ to its unique maximal steady-state.*
(2) *If $\lambda > \lambda^*$, then a unique solution $u(x,t)$ of* (1.1) *must touchdown at a finite time.*

Theorems B and C show that the solution u of (1.1) may touchdown at infinite time in higher dimension $(N \geqslant 8)$, which exactly occurs at $\lambda = \lambda^*$; however, infinite-time touchdown cannot occur in MEMS because the dimension of its ambient space is $N = 1$ or 2. Recall that pull-in distance of MEMS devices refers to the maximum possible stable deflection before touchdown occurs. Therefore, Theorems B and C also show that pull-in distance is exactly achieved at $\lambda = \lambda^*$ in MEMS devices.

In this paper we consider the case $\lambda > \lambda^*$ and we shall give a refined description of finite-time touchdown behavior for u, including some touchdown estimates, touchdown rates, as well as some information on the properties of touchdown set—such as compactness, location and shape. This paper is organized as follows: the purpose of Section 2 is mainly to derive some a priori estimates of touchdown profiles under the assumption that touchdown set of u is a compact subset of Ω. In Section 2.1, we establish the following lower bound estimate of touchdown profiles.

Theorem 1.1. *Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. If touchdown set of u is a compact subset of Ω, then*

(1) *any point $a \in \bar{\Omega}$ satisfying $f(a) = 0$ is not a touchdown point of $u(x,t)$;*
(2) *there exists a bounded positive constant M such that*

$$M(T-t)^{\frac{1}{3}} \leqslant u(x,t) \quad \text{in } \Omega \times (0,T). \tag{1.3}$$

Whether the compactness of touchdown set holds for any $f(x)$ satisfying (1.2) is a quite challenging problem. We shall prove in Proposition 2.1 of Section 2 that the compactness of touchdown set holds for the case where the domain Ω is convex and $f(x)$ satisfies the additional condition

$$\frac{\partial f}{\partial \nu} \leqslant 0 \quad \text{on } \Omega_\delta^c := \bigl\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \leqslant \delta\bigr\} \text{ for some } \delta > 0. \tag{1.4}$$

Here ν is the outward unit norm vector to $\partial\Omega$. On the other hand, when $f(x)$ does not satisfy (1.4), the compactness of touchdown set was numerically observed, see [9,15] or Section 4 of the present paper. Therefore, it is our conjecture that under the convexity of Ω, the compactness of touchdown set holds for any $f(x)$ satisfying (1.2). In Section 2.2 we estimate the derivatives of touchdown solution u, see Lemma 2.6; and as a byproduct, an integral estimate is also given in Section 2.2, see Theorem 2.7.

Motivated by Theorem 1.1, the key point of studying touchdown profiles is a similarity variable transformation of (1.1). For the touchdown solution $u = u(x,t)$ of (1.1) at finite time T, we use the associated similarity variables

$$y = \frac{x-a}{\sqrt{T-t}}, \qquad s = -\log(T-t), \qquad u(x,t) = (T-t)^{\frac{1}{3}} w_a(y,s), \tag{1.5}$$

where a is any interior point of Ω. Then $w_a(y,s)$ is defined in $W_a := \{(y,s)\colon a + ye^{-s/2} \in \Omega,\ s > s' = -\log T\}$, and it solves

$$\rho(w_a)_s - \nabla \cdot (\rho \nabla w_a) - \frac{1}{3}\rho w_a + \frac{\lambda \rho f(a + ye^{-\frac{s}{2}})}{w_a^2} = 0,$$

where $\rho(y) = e^{-|y|^2/4}$. Here $w_a(y,s)$ is always strictly positive in W_a. The slice of W_a at a given time s^1 is denoted by $\Omega_a(s^1) := W_a \cap \{s = s^1\} = e^{s^1/2}(\Omega - a)$. Then for any interior point a of Ω, there exists $s_0 = s_0(a) > 0$ such that $B_s := \{y\colon |y| < s\} \subset \Omega_a(s)$ for $s \geqslant s_0$. We now introduce the frozen energy functional

$$E_s[w_a](s) = \frac{1}{2}\int_{B_s} \rho|\nabla w_a|^2\,dy - \frac{1}{6}\int_{B_s} \rho w_a^2\,dy - \int_{B_s} \frac{\lambda\rho f(a)}{w_a}\,dy. \tag{1.6}$$

By estimating the energy $E_s[w_a](s)$ in B_s, one can establish the following upper bound estimate.

Theorem 1.2. *Assume f satisfies* (1.2) *on a bounded domain Ω in $\mathbb{R}^N$, suppose u is a touchdown solution of* (1.1) *at finite time T and $w_a(y,s)$ is defined by* (1.5). *Assume touchdown set of u is a compact subset of Ω. If $w_a(y,s) \to \infty$ as $s \to \infty$ uniformly for $|y| \leqslant C$, where C is any positive constant, then a is not a touchdown point for u.*

Based on a priori estimates of Section 2, we shall establish refined touchdown profiles in Section 3, where self-similar method and center manifold analysis will be applied. Here is the statement of refined touchdown profiles:

Theorem 1.3. *Assume f satisfies* (1.2) *on a bounded domain Ω in $\mathbb{R}^N$, and suppose u is a touchdown solution of* (1.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, then*:

(1) *If $N = 1$ and $x = a$ is a touchdown point of u, then we have*

$$\lim_{t\to T^-} u(x,t)(T-t)^{-\frac{1}{3}} \equiv \big(3\lambda f(a)\big)^{\frac{1}{3}} \tag{1.7}$$

uniformly on $|x-a| \leqslant C\sqrt{T-t}$ for any bounded constant C. Moreover, when $t \to T^-$,

$$u \sim \big[3\lambda f(a)(T-t)\big]^{1/3}\left(1 - \frac{1}{4|\log(T-t)|} + \frac{|x-a|^2}{8(T-t)|\log(T-t)|} + \cdots\right), \quad N = 1. \tag{1.8}$$

(2) *If $\Omega = B_R(0) \subset \mathbb{R}^N$ is a bounded ball with $N \geqslant 2$, $f(r) = f(|x|)$ is radially symmetric, and suppose $r = 0$ is a touchdown point of u, then we have*

$$\lim_{t\to T^-} u(r,t)(T-t)^{-\frac{1}{3}} \equiv \big(3\lambda f(0)\big)^{\frac{1}{3}} \tag{1.9}$$

uniformly on $r \leqslant C\sqrt{T-t}$ for any bounded constant C. Moreover, when $t \to T^-$,

$$u \sim \left[3\lambda f(0)(T-t)\right]^{1/3}\left(1 - \frac{1}{2|\log(T-t)|} + \frac{r^2}{4(T-t)|\log(T-t)|} + \cdots\right), \quad N = 2. \tag{1.10}$$

Note that the uniqueness of solutions for (1.1) gives the radial symmetry of u in Theorem 1.3(2). When dimension $N \geqslant 2$, it should remark from Theorem 1.3(2) that we are only able to discuss the refined touchdown profiles for special touchdown point $x = 0$ in the radial situation, and it seems unknown for the general case.

Adapting various analytical and numerical techniques, Section 4 will be focused on the set of touchdown points. This may provide useful information on the design of MEMS devices. In Section 4.1 we discuss the radially symmetric case of (1.1) as follows:

Theorem 1.4. *Assume $f(r) = f(|x|)$ satisfies* (1.2) *and $f'(r) \leqslant 0$ in a bounded ball $B_R(0) \subset \mathbb{R}^N$ with $N \geqslant 1$, and suppose u is a touchdown solution of* (1.1) *at finite time T. Then, $r = 0$ is the unique touchdown point of u.*

Remark 1.1. Assume $f(r) = f(|x|)$ satisfies (1.2) and $f'(r) \leqslant 0$ in a bounded ball $B_R(0) \subset \mathbb{R}^N$ with $N \geqslant 1$. Together with Proposition 2.1 below, Theorems 1.1 and 1.4 show an interesting phenomenon: finite-time touchdown point is not the zero of $f(x)$, but the maximum value point of $f(x)$, see also [10].

Remark 1.2. Numerical simulations in Section 4.1 show that the assumption $f'(r) \leqslant 0$ in Theorem 1.4 is sufficient, but not necessary. This gives that Theorem 1.3(2) does hold for a larger class of profiles $f(r) = f(|x|)$.

For one-dimensional case, Theorem 1.4 already implies that touchdown points must be unique when permittivity profile $f(x)$ is uniform. In Section 4.2 we further discuss one-dimensional case of (1.1) for varying profile $f(x)$, where numerical simulations show that touchdown points may be composed of finite points or finite compact subsets of the domain. Finally, Section 5 is a conclusion, where we review the main results of this paper, and address their applications to the understanding of dynamic deflection of MEMS devices.

2. A priori estimates of touchdown behavior

Under the assumption that touchdown set of u is a compact subset of Ω, in this section we study some a priori estimates of touchdown behavior, and establish the claims in Theorems 1.1 and 1.2. In Section 2.1 we establish a lower bound estimate, from which we complete the proof of Theorem 1.1. Using the lower bound estimate, in Section 2.2 we shall prove some estimates for the derivatives of touchdown solution u, and an integral estimate will be also obtained as a byproduct. In Section 2.3 we shall study the upper bound estimate by energy methods, which gives Theorem 1.2.

We first prove the following compactness result for a large class of profiles $f(x)$ satisfying (1.2) and (1.4).

Proposition 2.1. *Assume f satisfies* (1.2) *and* (1.4) *on a bounded convex domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. Then, the set of touchdown points for u is a compact subset of Ω.*

Proof. We prove Proposition 2.1 by adapting moving plane method from Theorem 3.3 in [4], where it is used to deal with blow-up problems. Take any point $y_0 \in \partial\Omega$, and assume for simplicity that $y_0 = 0$ and that the half space $\{x_1 > 0\}$ ($x = (x_1, x')$) is tangent to Ω at y_0. Let $\Omega_\alpha^+ = \Omega \cap \{x_1 > \alpha\}$ where $\alpha < 0$ and $|\alpha|$ is small, and also define $\Omega_\alpha^- = \{(x_1, x')\colon (2\alpha - x_1, x') \in \Omega_\alpha^+\}$, the reflection of Ω_α^+ with respect to the plane $\{x_1 = \alpha\}$, where $x' = (x_2, \dots, x_N)$.

Consider the function

$$w(x,t) = u(2\alpha - x_1, x', t) - u(x_1, x', t)$$

for $x \in \Omega_\alpha^-$, then w satisfies

$$w_t - \Delta w = \frac{\lambda(u(x_1, x', t) + u(2\alpha - x_1, x', t)) f(x)}{u^2(x_1, x', t) u^2(2\alpha - x_1, x', t)} w.$$

It is clear that $w = 0$ on $\{x_1 = \alpha\}$. Since $u(x,t) = 1$ along $\partial\Omega$ and since the maximum principle gives $u_t < 0$ for $0 < t < T$, we may choose a small $t_0 > 0$ such that

$$\frac{\partial u(x, t_0)}{\partial \nu} > 0 \quad \text{along } \partial\Omega, \tag{2.1}$$

where ν is the outward unit norm vector to $\partial\Omega$. Then for sufficiently small $|\alpha|$, (2.1) implies that $w(x, t_0) \geqslant 0$ in Ω_α^- and also $w = 1 - u(x_1, x', t) > 0$ on $(\partial\Omega_\alpha^- \cap \{x_1 < \alpha\}) \times (t_0, T)$. Applying the maximal principle we now conclude that $w > 0$ in $\Omega_\alpha^- \times (t_0, T)$ and $\frac{\partial w}{\partial x_1} = -2\frac{\partial u}{\partial x_1} < 0$ on $\{x_1 = \alpha\}$. Since α is arbitrary, it follows by varying α that

$$\frac{\partial u}{\partial x_1} > 0, \quad (x,t) \in \Omega_{\alpha_0}^+ \times (t_0, T), \tag{2.2}$$

provided $|\alpha_0| = |\alpha_0(t_0)| > 0$ is sufficiently small.

Fix $0 < |\alpha_0| \leqslant \delta$, where δ is as in (1.4), we now consider the function

$$J = u_{x_1} - \varepsilon_1(x_1 - \alpha_0) \quad \text{in } \Omega_{\alpha_0}^+ \times (t_0, T),$$

where $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0) > 0$ is a constant to be determined later. The direct calculations show that

$$J_t - \Delta J = \frac{2\lambda f}{u^3} u_{x_1} - \frac{\lambda f_{x_1}}{u^2} = \frac{2\lambda f}{u^3} u_{x_1} - \frac{\lambda}{u^2} \frac{\partial f}{\partial \nu} \frac{\partial \nu}{\partial x_1} \geqslant 0 \quad \text{in } \Omega_{\alpha_0}^+ \times (t_0, T) \tag{2.3}$$

due to (1.4). Therefore, J cannot attain negative minimum in $\Omega_{\alpha_0}^+ \times (t_0, T)$. Next, $J > 0$ on $\{x_1 = \alpha_0\}$ by (2.2). Since (2.1) gives $\frac{\partial u(x,t_0)}{\partial x_1} \geqslant C > 0$ along $(\partial\Omega_{\alpha_0}^+ \cap \partial\Omega)$ for some $C > 0$, we have $J > 0$ on $\{t = t_0\}$ provided $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0) > 0$ is sufficiently small. We now claim that for small $\varepsilon_1 > 0$,

$$J > 0 \quad \text{on } \left(\partial\Omega_{\alpha_0}^+ \cap \partial\Omega\right) \times (t_0, T). \tag{2.4}$$

To prove (2.4), we compare the solution $U := 1 - u$ satisfying

$$U_t - \Delta U = \frac{\lambda f(x)}{(1-U)^2}, \quad (x,t) \in \Omega \times (t_0, T),$$
$$U(x,t_0) = 1 - u(x,t_0), \qquad U(x,t) = 0, \quad x \in \partial\Omega,$$

with the solution v of the heat equation

$$v_t = \Delta v, \quad (x,t) \in \Omega \times (t_0, T),$$

where $0 \leqslant v(x,t_0) = U(x,t_0) < 1$ and $v = 0$ on $\partial\Omega$. Then we have $U \geqslant v$ in $\Omega \times (t_0, T)$. Consequently,

$$\frac{\partial U}{\partial \nu} \leqslant \frac{\partial v}{\partial \nu} \leqslant -C_0 < 0 \quad \text{on } \left(\partial\Omega^+_{\alpha_0} \cap \partial\Omega\right) \times (t_0, T),$$

and hence $\frac{\partial u}{\partial \nu} \geqslant C_0 > 0$ on $(\partial\Omega^+_{\alpha_0} \cap \partial\Omega) \times (t_0, T)$. It then follows that $J \geqslant C_0 \frac{\partial v}{\partial x_1} - \varepsilon_1(x_1 - \alpha_0) > 0$ provided $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0)$ is small enough, which gives (2.4).

The maximum principle now yields that there exists $\varepsilon_1 = \varepsilon_1(\alpha_0, t_0) > 0$ so small that $J \geqslant 0$ in $\Omega^+_{\alpha_0} \times (t_0, T)$, i.e.,

$$u_{x_1} \geqslant \varepsilon_1(x_1 - \alpha_0), \tag{2.5}$$

if $x' = 0$ and $\alpha_0 \leqslant x_1 < 0$. Integrating (2.5) with respect to x_1 on $[\alpha_0, y_1]$, where $\alpha_0 < y_1 < 0$, yields that

$$u(y_1, 0, t) - u(\alpha_0, 0, t) \geqslant \frac{\varepsilon_1}{2}|y_1 - \alpha_0|^2.$$

It follows that

$$\varliminf_{t \to T^-} u(0,t) = \varliminf_{t \to T^-} \lim_{y_1 \to 0^-} u(y_1, 0, t) \geqslant \varepsilon_1 \alpha_0^2/2 > 0,$$

which shows that $y_0 = 0$ cannot be a touchdown point of $u(x,t)$.

The proof of (2.2) can be slightly modified to show that $\frac{\partial u}{\partial \nu} > 0$ in $\Omega^+_{\alpha_0} \times (t_0, T)$ for any direction ν close enough to the x_1-direction. Together with (1.4), this enables us to deduce that any point in $\{x' = 0, \alpha_0 < x_1 < 0\}$ cannot be a touchdown point. Since above proof shows that α_0 can be chosen independently of initial point y_0 on $\partial\Omega$, by varying y_0 along $\partial\Omega$ we deduce that there is an Ω-neighborhood Ω' of $\partial\Omega$ such that each point $x \in \Omega'$ cannot be a touchdown point. This completes the proof of Proposition 2.1. □

2.1. *Lower bound estimate*

Define for $\eta > 0$,

$$\Omega_\eta := \left\{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \eta\right\}, \qquad \Omega^c_\eta := \left\{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) \leqslant \eta\right\}. \tag{2.6}$$

Since touchdown set of u is assumed to be a compact subset of Ω, in the rest of this section we may choose a small $\eta > 0$ such that any touchdown point of u must lie in Ω_η. Our first aim of this subsection is to prove that any point $x_0 \in \bar{\Omega}_\eta$ satisfying $f(x_0) = 0$ cannot be a touchdown point of u at finite time T, which then leads to the following proposition.

Proposition 2.2. *Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose $u(x,t)$ is a touchdown solution of* (1.1) *at finite time T. If touchdown set of u is a compact subset of Ω, then any point $x_0 \in \bar{\Omega}$ satisfying $f(x_0) = 0$ cannot be a touchdown point of $u(x,t)$.*

This claim is based on the following Harnack-type estimate, which was proved in Lemma 3.2 of [9].

Lemma 2.3. *For any compact subset K of $\bar{\Omega}$ and any $m > 0$, there exists a constant $C = C(K,m) > 0$ such that $\|v\|_\infty \leqslant C < 1$ on K, whenever v satisfies*

$$\Delta v \geqslant \frac{m}{(1-v)^2}, \quad x \in \Omega, \qquad 0 \leqslant v < 1, \quad x \in \Omega. \tag{2.7}$$

Proof of Proposition 2.2. Since touchdown set of u is assumed to be a compact subset of Ω, it now suffices to discuss the point x_0 lying in the interior domain Ω_η for some small $\eta > 0$, such that there is no touchdown point on Ω_η^c.

For any $t_1 < T$, we first recall that the maximum principle gives $u_t < 0$ for all $(x,t) \in \Omega \times (0,t_1)$. Further, the boundary point lemma shows that the outward normal derivative of $v = u_t$ on $\partial\Omega$ is positive for $t > 0$. This implies that for taking small $0 < t_0 < T$, there exists a positive constant $C = C(t_0, \eta)$ such that $u_t(x,t_0) \leqslant -C < 0$ for all $x \in \bar{\Omega}_\eta$. For any $0 < t_0 < t_1 < T$, we next claim that there exists $\varepsilon = \varepsilon(t_0,t_1,\eta) > 0$ such that

$$J^\epsilon(x,t) = u_t + \frac{\varepsilon}{u^2} \leqslant 0 \quad \text{for all } (x,t) \in \Omega_\eta \times (t_0,t_1). \tag{2.8}$$

Indeed, it is now clear that there exists $C_\eta = C_\eta(t_0,t_1,\eta) > 0$ such that $u_t(x,t) \leqslant -C_\eta$ on the parabolic boundary of $\Omega_\eta \times (t_0,t_1)$. And further, we can choose $\varepsilon = \varepsilon(t_0,t_1,\eta) > 0$ so small that $J^\epsilon \leqslant 0$ on the parabolic boundary of $\Omega_\eta \times (t_0,t_1)$, due to the local boundedness of $\frac{1}{u^2}$ on $\partial\Omega_\eta \times (t_0,t_1)$. Also, direct calculations imply that

$$J_t^\epsilon - \Delta J^\epsilon = \frac{2\lambda f}{u^3} J^\epsilon - \frac{6\varepsilon|\nabla u|^2}{u^4} \leqslant \frac{2\lambda f}{u^3} J^\epsilon.$$

Now (2.8) follows again from the maximum principle.

Combining (2.8) and (1.1) we deduce that for a small neighborhood B of x_0 where $\lambda f(x) \leqslant \varepsilon/2$ is in $B \subset \bar{\Omega}_\eta$, we have for $v := 1 - u$,

$$\Delta v \geqslant \frac{\varepsilon}{2} \frac{1}{(1-v)^2}, \quad (x,t) \in B \times (t_0,t_1).$$

Proposition 2.2 is now a direct result of Lemma 2.3, since $t_1 < T$ is arbitrary. □

Essentially, the claim (2.8) is ready to give the following lower bound estimate, which completes the proof of Theorem 1.1.

Lemma 2.4. *Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, then there exists a bounded positive constant M such that*

$$M(T-t)^{\frac{1}{3}} \leqslant u(x,t) \tag{2.9}$$

for all $0<t<T$. Moreover, $u_t \to -\infty$ as u touches down.

Proof. Given any small $\eta > 0$, applying the same argument used for (2.8) yields that for any $0 < t_0 < t_1 < T$, there exists $\varepsilon = \varepsilon(t_0, t_1, \eta) > 0$ such that

$$u_t \leqslant -\frac{\varepsilon}{u^2} \quad \text{in } \Omega_\eta \times (t_0, t_1).$$

This inequality shows that $u_t \to -\infty$ as u touchdown, and there exists $M > 0$ such that

$$M_1(T-t)^{\frac{1}{3}} \leqslant u(x,t) \quad \text{in } \Omega_\eta \times (0,T) \tag{2.10}$$

due to the arbitrary of t_0 and t_1, where M_1 depends only on λ, f and η. Furthermore, one can obtain (2.9) because of the boundedness of u on Ω_η^c. □

2.2. Gradient estimates

As a preliminary of next section, it is now important to know a priori estimates for the derivatives of touchdown solution u, which are the contents of this subsection. Following the analysis in [4], our first lemma is about the derivatives of first order without the compactness assumption of touchdown set.

Lemma 2.5. *Assume f satisfies* (1.2) *on a bounded convex domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. Then for any $0 < t_0 < T$, there exists a bounded constant $C > 0$ such that*

$$\frac{1}{2}|\nabla u|^2 \leqslant \frac{C}{\underline{u}} - \frac{C}{u} \quad \textit{in } \Omega \times (0, t_0), \tag{2.11}$$

where $\underline{u} = \underline{u}(t_0) = \min_{x\in\Omega} u(x,t_0)$, and C depends only on λ, f and Ω.

Proof. Fix any $0 < t_0 < T$ and treat $\underline{u}(t_0)$ as a fixed constant. Let $w = u - \underline{u}$, then w satisfies

$$\begin{aligned} w_t - \Delta w &= -\frac{\lambda f(x)}{(w+\underline{u})^2} && \text{in } \Omega \times (0, t_0), \\ w &= 1 - \underline{u} && \text{in } \partial\Omega \times (0, t_0), \\ w(x,0) &= 1 - \underline{u} && \text{in } \Omega. \end{aligned}$$

We introduce the function

$$P = \frac{1}{2}|\nabla w|^2 + \frac{C}{w+\underline{u}} - \frac{C}{\underline{u}}, \tag{2.12}$$

where the bounded constant $C \geqslant 2\lambda \sup_{x\in\bar\Omega} f$ will be determined later. Then we have

$$\begin{aligned}
P_t - \Delta P &= \frac{C\lambda f(x)}{(w+\underline{u})^4} - \frac{\lambda \nabla f(x)\nabla w}{(w+\underline{u})^2} + \frac{2(\lambda f(x)-C)|\nabla w|^2}{(w+\underline{u})^3} - \sum_{i,j=1}^N w_{ij}^2 \\
&\leqslant \frac{\lambda C \sup_{x\in\bar\Omega} f}{(w+\underline{u})^4} + \frac{-2\lambda|\nabla w|^2 \sup_{x\in\bar\Omega} f + \lambda|\nabla w| \sup_{x\in\bar\Omega}|\nabla f|}{(w+\underline{u})^3} - \sum_{i,j=1}^N w_{ij}^2 \\
&\leqslant \frac{\lambda(C\sup_{x\in\bar\Omega} f + C_1)}{(w+\underline{u})^4} - \sum_{i,j=1}^N w_{ij}^2,
\end{aligned} \tag{2.13}$$

where $C_1 := \frac{(\sup_{x\in\bar\Omega}|\nabla f|)^2}{8\sup_{x\in\bar\Omega} f} \geqslant 0$ is bounded. Since (2.12) gives

$$\sum_{i=1}^N \left(P_i + \frac{C}{(w+\underline{u})^2} w_i\right)^2 = \sum_{i,j=1}^N (w_j w_{ij})^2 \leqslant |\nabla w|^2 \sum_{i,j=1}^N w_{ij}^2, \tag{2.14}$$

we now take

$$C := \max\left\{2\lambda \sup_{x\in\bar\Omega} f, \frac{\lambda \sup_{x\in\bar\Omega} f + \lambda\sqrt{(\sup_{x\in\bar\Omega} f)^2 + 4C_1}}{2}\right\} \geqslant 2\lambda \sup_{x\in\bar\Omega} f$$

so that $C^2 \geqslant \lambda(C\sup_{x\in\bar\Omega} f + C_1)$, where C clearly depends only on λ, f and Ω. From the choice of C, a combination of (2.13) and (2.14) gives that

$$P_t - \Delta P \leqslant \vec{b}\cdot\nabla P,$$

where $\vec{b} = -|\nabla w|^{-2}(\nabla P + \frac{2C\nabla w}{(w+\underline{u})^2})$ is a locally bounded when $\nabla w \not\equiv 0$. Therefore, P can only attain positive maximum either at the point where $\nabla w = 0$, or on the parabolic boundary of $\Omega \times (0, t_0)$. But when $\nabla w = 0$, we have $P \leqslant 0$.

On the initial boundary, $P = \frac{C}{1+\underline{u}} - \frac{C}{\underline{u}} < 0$. Let (y, s) be any point on $\partial\Omega \times (0, t_0)$, if we can prove that

$$\frac{\partial P}{\partial \nu} \leqslant 0 \quad \text{at } (y, s), \tag{2.15}$$

it then follows from the maximum principle that $P \leqslant 0$ in $\Omega \times (0, t_0)$. And therefore, the assertion (2.11) is reduced from (2.12) together with $w = u - \underline{u}$.

To prove (2.15), we recall the fact that since $w = \text{const}$ on $\partial\Omega$ (for $t = s$), we have

$$\Delta w = w_{\nu\nu} + (N-1)\kappa w_\nu \quad \text{at } (y, s),$$

where κ is the non-negative mean curvature of $\partial\Omega$ at y. It then follows that

$$\frac{\partial P}{\partial \nu} = w_\nu w_{\nu\nu} - \frac{C w_\nu}{(w+\underline{u})^2} \leqslant w_\nu \left[\Delta w - (N-1)\kappa w_\nu - \frac{\lambda f(x)}{(w+\underline{u})^2} \right]$$
$$= w_\nu \big[w_t - (N-1)\kappa w_\nu \big] = -(N-1)\kappa w_\nu^2 \leqslant 0$$

at (y, s), and we are done. □

The following lemma is dealt with the derivatives of higher order, and the idea of its proof is similar to Proposition 1 of [11].

Lemma 2.6. *Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, and $x = a$ is any point of Ω_η for some small $\eta > 0$. Then there exists a positive constant M' such that*

$$\big|\nabla^m u(x,t)\big|(T-t)^{-\frac{1}{3}+\frac{m}{2}} \leqslant M', \quad m = 1, 2, \tag{2.16}$$

holds for $|x - a| \leqslant R$.

Proof. It suffices to consider the case $a = 0$ by translation, and we may focus on $\frac{1}{2}R^2 < r^2 < R^2$ and denote $Q_r = B_r \times (T[1 - (\frac{r}{R})^2], T)$.

Our first task is to show that $|\nabla u|$ and $|\nabla^2 u|$ are uniformly bounded on compact subsets of Q_R. Indeed, since $f(x)/u^2$ is bounded on any compact subset D of Q_R, standard L^p estimates for heat equations (cf. [16]) give

$$\iint_D \big(\big|\nabla^2 u\big|^p + |u_t|^p\big)\,dx\,dt < C, \quad 1 < p < \infty.$$

Choosing p to be large enough, we then conclude from Sobolev's inequality that $f(x)/u^2$ is Hölder continuous on D. Therefore, Schauder's estimates for heat equations (cf. [16]) show that $|\nabla u|$ and $|\nabla^2 u|$ are uniformly bounded on compact subsets of D. In particular, there exists M_1 such that

$$|\nabla u| + \big|\nabla^2 u\big| \leqslant M_1 \quad \text{for } (x,t) \in B_r \times \left(T\left[1 - \left(\frac{r}{R}\right)^2\right], T\left[1 - \frac{1}{2}\left(1 - \frac{r}{R}\right)^2\right]\right), \tag{2.17}$$

where M_1 depends only on R, N and M given in (2.9).

We next prove (2.16) for $|x| < r$ and $T[1 - \frac{1}{2}(1 - \frac{r}{R})^2] \leqslant t < T$. Fix such a point (x,t), let $\mu = [\frac{2}{T}(T-t)]^{1/2}$ and consider

$$v(z,\tau) = \mu^{-\frac{2}{3}} u\big(x + \mu z, T - \mu^2(T-\tau)\big). \tag{2.18}$$

For above given point (x,t), we now define $O := \{z\colon (x + \mu z) \in \Omega\}$ and $g(z) := f(x + \mu z) \geqslant 0$ on O. One can verify that $v(z,\tau)$ is a solution of

$$v_\tau - \Delta_z v = -\frac{\lambda g(z)}{v^2}, \quad z \in O,$$
$$v(z,0) = v_0(z) > 0, \qquad v(z,\tau) = \mu^{-\frac{2}{3}}, \quad z \in \partial O, \tag{2.19}$$

where Δ_z denotes the Laplacian operator with respect to z, and $v_0(z) = \mu^{-\frac{2}{3}} u(x + \mu z, T - \mu^2 T) > 0$ satisfies $\Delta_z v_0 - \frac{\lambda g(z)}{v_0^2} \leqslant 0$ on O. The formula (2.18) implies that T is also the finite touchdown time of v, and the domain of v includes Q_{r_0} for some $r_0 = r_0(R) > 0$. Since touchdown set of u is assumed to be a compact subset of Ω, one can observe that touchdown set of v is also a compact subset of O. Therefore, the argument of Lemma 2.4 can be applied to (2.19), yielding that there exists a constant $M_2 > 0$ such that

$$v(z, \tau) \geqslant M_2 (T - \tau)^{\frac{1}{3}},$$

where M_2 depends only on R, λ, f and Ω again. The argument used for (2.17) then yields that there exists $M_1' > 0$, depending on R, N and M_2, such that

$$|\nabla_z v| + \left|\nabla_z^2 v\right| \leqslant M_1' \quad \text{for } (z, \tau) \in B_r \times \left(T\left[1 - \left(\frac{r}{r_0}\right)^2\right], T\left[1 - \frac{1}{2}\left(1 - \frac{r}{r_0}\right)^2\right]\right), \tag{2.20}$$

where we assume $\frac{1}{2} r_0^2 < r^2 < r_0^2$. Applying (2.18) and taking $(z, \tau) = (0, \frac{T}{2})$, this estimate reduces to

$$\mu^{-\frac{2}{3}+1} |\nabla u| + \mu^{-\frac{2}{3}+2} \left|\nabla^2 u\right| \leqslant M_1'.$$

Therefore, (2.16) follows since $\mu = [\frac{2}{T}(T - t)]^{\frac{1}{2}}$. $\square$

Before concluding this subsection, we now apply gradient estimates to establishing integral estimates.

Theorem 2.7. *Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, then for $\gamma > \frac{3}{2} N$ we have*

$$\varliminf_{t \to T^-} \int_\Omega f(x) u^{-\gamma}(x, t)\, dx = +\infty.$$

Proof. For any given $t_0 \in (0, T)$ close to T, Lemma 2.5 implies that

$$\frac{1}{2} |\nabla u|^2 \leqslant \frac{C}{\underline{u}^2} (u - \underline{u}) \quad \text{in } \Omega \times (0, t_0) \tag{2.21}$$

for some bounded constant $C > 0$, where $\underline{u} = u(x_0, t_0) = \min_{x \in \Omega} u(x, t_0)$. Considering any t sufficiently close to t_0, we now introduce polar coordinates (r, θ) about the point x_0. Then in any direction θ, there is a smallest value of $r_0 = r_0(\theta, t)$ such that $u(r_0, t) = 2\underline{u}$. Note that r_0 is very small as $t < t_0$ sufficiently approach to T. Furthermore, since x_0 approaches to one of touchdown points of u as $t \to T^-$, Proposition 2.2 shows that as $t < t_0$ sufficiently approach to T, we have $f(x) \geqslant C_0 > 0$ in $\{r < r_0\}$ for some $C_0 > 0$. Since (2.21) and the definition of $\underline{u}$

imply that $\frac{u_r}{\sqrt{u-\underline{u}}} \leqslant \frac{\sqrt{2C}}{\underline{u}}$, which is $2\sqrt{u-\underline{u}} \leqslant \frac{\sqrt{2C}}{\underline{u}} r$, we attain $\sqrt{\frac{2}{C}}\underline{u}^{3/2} \leqslant r_0$ by taking $r = r_0$. Therefore, for $\gamma > \frac{3}{2}N$ we have

$$\begin{aligned}\int_\Omega u^{-\gamma}\,dx &\geqslant C\int_\Omega f(x)u^{-\gamma}\,dx \geqslant CC_0 \int_{\{r\leqslant r_0\}} u^{-\gamma}\,dx \geqslant C\int_\theta dS_\theta \int_{\{r\leqslant r_0\}} u^{-\gamma} r^{N-1}\,dr \\ &\geqslant C\int_\theta dS_\theta \int_{\{r\leqslant r_0\}} (2\underline{u})^{-\gamma} r^{N-1}\,dr \\ &\geqslant C\int_\theta dS_\theta\, \underline{u}^{-\gamma} r_0^N \geqslant C\int_\theta dS_\theta\, \underline{u}^{-\gamma+\frac{3}{2}N} = +\infty\end{aligned}$$

as $t \to T^-$, which completes the proof of Theorem 2.7. □

2.3. Upper bound estimate

In this subsection, we discuss the upper bound estimate of touchdown solution u, and we shall apply energy methods to establishing Theorem 1.2 already stated in the introduction.

First, we note the following local upper bound estimate.

Proposition 2.8. *Suppose u is a touchdown solution of* (1.1) *at finite time T. Then, there exists a bounded constant $C = C(\lambda, f, \Omega) > 0$ such that*

$$\min_{x\in\Omega} u(x,t) \leqslant C(T-t)^{\frac{1}{3}} \quad \textit{for } 0 < t < T. \tag{2.22}$$

Proof. Set

$$U(t) = \min_{x\in\Omega} u(x,t), \quad 0 < t < T,$$

and let $U(t_i) = u(x_i, t_i)$ $(i = 1, 2)$ with $h = t_2 - t_1 > 0$. Then,

$$\begin{aligned} U(t_2) - U(t_1) &\leqslant u(x_1,t_2) - u(x_1,t_1) = hu_t(x_1,t_1) + o(h), \\ U(t_2) - U(t_1) &\geqslant u(x_2,t_2) - u(x_2,t_1) = hu_t(x_2,t_2) + o(h). \end{aligned}$$

It follows that $U(t)$ is Lipschitz continuous. Hence, for $t_2 > t_1$ we have

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} \geqslant u_t(x_2,t_2) + o(1).$$

On the other hand, since $\Delta u(x_2,t_2) \geqslant 0$, we obtain

$$u_t(x_2,t_2) \geqslant -\frac{\lambda f(x_2)}{u^2(x_2,t_2)} = -\frac{\lambda f(x_2)}{U^2(t_2)} \geqslant -\frac{C}{U^2(t_2)} \quad \text{for } 0 < t_2 < T.$$

Consequently, at any point of differentiability of $U(t)$, it deduces from above inequalities that

$$U^2 U_t \geqslant -C \quad \text{a.e. } t \in (0, T). \tag{2.23}$$

Integrating (2.23) from t to T we obtain (2.22). □

For the touchdown solution $u = u(x,t)$ of (1.1) at finite time T, we now introduce the associated similarity variables

$$y = \frac{x-a}{\sqrt{T-t}}, \qquad s = -\log(T-t), \qquad u(x,t) = (T-t)^{\frac{1}{3}} w_a(y,s), \tag{2.24}$$

where a is any point of Ω_η for some small $\eta > 0$. Then $w_a(y,s)$ is defined in

$$W_a := \big\{(y,s)\colon\ a + ye^{-s/2} \in \Omega,\ s > s' = -\log T\big\},$$

and it solves

$$\frac{\partial}{\partial s} w_a - \Delta w_a + \frac{1}{2} y \cdot \nabla w_a - \frac{1}{3} w_a + \frac{\lambda f(a + ye^{-s/2})}{w_a^2} = 0. \tag{2.25}$$

Here $w_a(y,s)$ is always strictly positive in W_a. Note that the form of w_a defined by (2.24) is motivated by Theorem 1.1 and Proposition 2.8. The slice of W_a at a given time s^1 will be denoted by $\Omega_a(s^1)$:

$$\Omega_a\big(s^1\big) := W_a \cap \big\{s = s^1\big\} = e^{s^1/2}(\Omega - a).$$

Then for any $a \in \Omega_\eta$, there exists $s_0 = s_0(\eta, a) > 0$ such that

$$B_s := \big\{y\colon\ |y| < s\big\} \subset \Omega_a(s) \quad \text{for } s \geqslant s_0. \tag{2.26}$$

From now on, we often suppress the subscript a, writing w for w_a, etc.

In view of (2.24), one can combine Lemmas 2.4 and 2.6 to reaching the following estimates on $w = w_a$:

Corollary 2.9. *Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, then the rescaled solution $w = w_a$ satisfies*

$$M \leqslant w \leqslant e^{\frac{s}{3}}, \qquad |\nabla w| + |\Delta w| \leqslant M' \quad \text{in } W,$$

where M is a constant as in Lemma 2.4 *and while M' is a constant as in Lemma* 2.6. *Moreover, it satisfies*

$$M \leqslant w(y_1, s) \leqslant w(y_2, s) + M'|y_2 - y_1|$$

for any $(y_i, s) \in W$, $i = 1, 2$.

We now rewrite (2.25) in divergence form:

$$\rho w_s - \nabla\cdot(\rho\nabla w) - \frac{1}{3}\rho w + \frac{\lambda\rho f(a+ye^{-\frac{s}{2}})}{w^2} = 0, \tag{2.27}$$

where $\rho(y) = e^{-|y|^2/4}$. We also introduce the frozen energy functional

$$E_s[w](s) = \frac{1}{2}\int\limits_{B_s}\rho|\nabla w|^2\,dy - \frac{1}{6}\int\limits_{B_s}\rho w^2\,dy - \int\limits_{B_s}\frac{\lambda\rho f(a)}{w}\,dy, \tag{2.28}$$

which is defined in the compact set B_s of $\Omega_a(s)$ for $s \geqslant s_0$.

Lemma 2.10. *Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, then the rescaled solution $w = w_a$ satisfies*

$$\frac{1}{2}\int\limits_{B_s}\rho|w_s|^2\,dy \leqslant -\frac{d}{ds}E_s[w](s) + g_\eta(s) \quad \text{for } s \geqslant s_0, \tag{2.29}$$

where $g_\eta(s)$ is positive and satisfies $\int_{s_0}^{\infty} g_\eta(s)\,ds < \infty$.

Proof. Multiply (2.27) by w_s and use integration by parts to get

$$\begin{aligned}
&\int\limits_{B_s}\rho|w_s|^2\,dy \\
&\quad= \int\limits_{B_s} w_s\nabla(\rho\nabla w)\,dy + \frac{1}{3}\int\limits_{B_s}\rho w w_s\,dy - \int\limits_{B_s}\frac{\lambda\rho w_s f(a+ye^{-\frac{s}{2}})}{w^2}\,dy \\
&\quad= -\frac{1}{2}\int\limits_{B_s}\frac{d}{ds}|\nabla w|^2\rho\,dy + \int\limits_{B_s}\frac{d}{ds}\left(\frac{1}{6}w^2 + \frac{\lambda f(a)}{w}\right)\rho\,dy \\
&\qquad+ \int\limits_{\partial B_s}\rho w_s\frac{\partial w}{\partial\nu}\,dS + \int\limits_{B_s}\frac{\lambda\rho w_s[f(a) - f(a+ye^{-\frac{s}{2}})]}{w^2}\,dy \\
&\quad= -\frac{d}{ds}E_s[w](s) + \int\limits_{\partial B_s}\rho w_s\frac{\partial w}{\partial\nu}\,dS + \frac{1}{2s}\int\limits_{\partial B_s}\rho|\nabla w|^2(y\cdot\nu)\,dS \\
&\qquad- \frac{1}{s}\int\limits_{\partial B_s}\rho\left(\frac{1}{6}w^2 + \frac{\lambda f(a)}{w}\right)(y\cdot\nu)\,dS + \int\limits_{B_s}\frac{\lambda\rho w_s[f(a) - f(a+ye^{-\frac{s}{2}})]}{w^2}\,dy \\
&\quad\leqslant -\frac{d}{ds}E_s[w](s) + \int\limits_{\partial B_s}\rho w_s\frac{\partial w}{\partial\nu}\,dS + \frac{1}{2s}\int\limits_{\partial B_s}\rho|\nabla w|^2(y\cdot\nu)\,dS
\end{aligned}$$

$$+\int\limits_{B_s} \frac{\lambda\rho w_s[f(a)-f(a+ye^{-\frac{s}{2}})]}{w^2}\,dy$$

$$:= -\frac{d}{ds}E_s[w](s) + I_1 + I_2 + I_3, \tag{2.30}$$

where ν is the exterior unit norm vector to $\partial\Omega$ and dS is the surface area element. The following formula is applied in the third equality of (2.30): if $g(y,s): W \mapsto R$ is a smooth function, then

$$\frac{d}{ds}\int\limits_{B_s} g(y,s)\,dy = \int\limits_{B_s} g_s(y,s)\,dy + \frac{1}{s}\int\limits_{\partial B_s} g(y,s)(y\cdot\nu)\,dS.$$

For $s \geqslant s_0$, we next estimate integration terms I_1, I_2 and I_3 as follows:

Considering $|y| \leqslant S$ in B_s, Corollary 2.9 gives

$$|w_s| = \left|\Delta w - \frac{1}{2}y\cdot\nabla w + \frac{1}{3}w - \frac{\lambda f(a+ye^{-\frac{s}{2}})}{w^2}\right| \leqslant C\big(1+|y|\big) + \frac{1}{3}w \leqslant C_1 s + \frac{1}{3}e^{\frac{s}{3}},$$

which implies

$$I_1 \leqslant Cs^{N-1}e^{-\frac{s^2}{4}}\left(C_1 s + \frac{1}{3}e^{\frac{s}{3}}\right) \leqslant C_2 s^N e^{-\frac{s^2}{4}+\frac{s}{3}}. \tag{2.31}$$

It is easy to observe that

$$I_2 \leqslant C_3 s^{N-1}e^{-\frac{s^2}{4}}. \tag{2.32}$$

As for I_3, since w has a lower bound and since $f(x) \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1]$, we apply Young's inequality to deduce

$$I_3 \leqslant Ce^{-\frac{\alpha}{2}s}\int\limits_{B_s} \rho|y|^\alpha w_s\,dy \leqslant Ce^{-\frac{\alpha}{2}s}\left[\varepsilon\int\limits_{B_s} \rho w_s^2\,dy + C(\varepsilon)\int\limits_{B_s} \rho|y|^{2\alpha}\,dy\right],$$

where the constant $\varepsilon > 0$ is arbitrary. Because $e^{-\frac{\alpha}{2}s} < \infty$, one can take sufficiently small ε such that

$$I_3 \leqslant \frac{1}{2}\int\limits_{B_s} \rho w_s^2\,dy + C_4 e^{-\frac{\alpha}{2}s}. \tag{2.33}$$

Combining (2.30)–(2.33) then yields

$$\frac{1}{2}\int\limits_{B_s} \rho|w_s|^2\,dy \leqslant -\frac{d}{ds}E_s[w](s) + \bar{C}_1 s^N e^{-\frac{s^2}{4}+\frac{s}{3}} + \bar{C}_2 e^{-\frac{\alpha}{2}s}$$

$$:= -\frac{d}{ds}E_s[w](s) + g_\eta(s),$$

where $g_\eta(s)$ is positive and satisfies $\int_{s_0}^\infty g_\eta(s)\,ds < \infty$, and we are done. □

Remark 2.1. Supposing the convexity of Ω, one can establish an energy estimate in the whole domain $\Omega_a(s)$:

$$\int\limits_{\Omega_a(s)} \rho|w_s|^2\,dy \leqslant -\frac{d}{ds}E_{\Omega_a(s)}[w](s) + K_\eta(s) \quad \text{for } s \geqslant s_0, \tag{2.34}$$

where $K_\eta(s)$ is positive and satisfies $\int_{s_0}^{\infty} K_\eta(s)\,ds < \infty$, and $E_{\Omega_a(s)}[w](s)$ is defined by

$$E_{\Omega_a(s)}[w](s) = \frac{1}{2}\int\limits_{\Omega_a(s)} \rho|\nabla w|^2\,dy - \frac{1}{6}\int\limits_{\Omega_a(s)} \rho w^2\,dy - \int\limits_{\Omega_a(s)} \frac{\lambda\rho f(a)}{w}\,dy. \tag{2.35}$$

However, by estimating the energy functional $E_s[w](s)$ in B_s, instead of $\Omega_a(s)$, it is sufficient to obtain the desirable upper bound estimate of w, see Theorem 2.12 below.

The following lemma is also necessary for establishing the desirable upper bound estimate.

Lemma 2.11. *Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, and a is any point of Ω_η for some $\eta > 0$. Then there exists a constant $\varepsilon > 0$, depending only on λ, f and Ω, such that if*

$$u(x,t)(T-t)^{-\frac{1}{3}} \geqslant \varepsilon \tag{2.36}$$

for all $(x,t) \in Q_\delta := \{(x,t)\colon |x-a| < \delta,\ T-\delta < t < T\}$, then a is not a touchdown point for u. Here $\delta > 0$ is an arbitrary constant.

Proof. Setting $v(x,t) = \frac{1}{u(x,t)}$, then $v(x,t)$ blows up at finite time T, and v satisfies

$$v_t - \Delta v = -\frac{2|\nabla v|^2}{v} + \lambda f(x)v^4 \leqslant K\big(1+v^4\big) \quad \text{in } Q_\delta, \tag{2.37}$$

where $K := \lambda \sup_{x\in\bar{\Omega}} f(x) > 0$. We now apply Theorem 2.1 of [13] to (2.37), which gives that there exists a constant $\frac{1}{\varepsilon} > 0$, depending only on λ, f and Ω, such that if

$$v(x,t) \leqslant \frac{1}{\varepsilon}(T-t)^{-\frac{1}{3}} \quad \text{in } Q_\delta,$$

then a is not a blow-up point for v, and hence (2.36) follows. □

Theorem 2.12. *Assume f satisfies* (1.2) *on a bounded domain Ω, and suppose u is a touchdown solution of* (1.1) *at finite time T. Assume touchdown set of u is a compact subset of Ω, and a is any point of Ω_η for some $\eta > 0$. If $w_a(y,s) \to \infty$ as $s \to \infty$ uniformly for $|y| \leqslant C$, where C is any positive constant, then a is not a touchdown point for u.*

Proof. We first claim that if $w_a(y,s)\to\infty$ as $s\to\infty$ uniformly for $|y|\leqslant C$, then

$$E_s[w_a](s)\to-\infty \quad \text{as } s\to\infty. \tag{2.38}$$

Indeed, it is obvious from Corollary 2.9 that the first term and the third term in $E_s[w_a](s)$ are uniformly bounded. As for the second term, we can write

$$\int_{B_s}\rho w^2\,dy=\int_{B_C}\rho w^2\,dy+\int_{B_s\setminus B_C}\rho w^2\,dy\geqslant\int_{B_C}\rho w^2\,dy.$$

Since $w_a\to\infty$ as $s\to\infty$ uniformly on B_C, we have $\int_{B_C}\rho w^2\,dy\to\infty$ as $s\to\infty$, which gives $-\frac{1}{6}\int_{B_C}\rho w^2\,dy\to-\infty$ as $s\to\infty$, and hence (2.38) follows.

Let K be a large positive constant to be determined later. Then (2.38) implies that there exists $\bar{s}$ such that $E_{\bar{s}}[w_a](\bar{s})\leqslant -4K$. Using the same argument as in [12], it is easy to show that for any fixed s, $E_s[w_a](s)$ varies smoothly with $a\in\Omega$. Therefore, there exists $r_0>0$ such that

$$E_{\bar{s}}[w_b](\bar{s})\leqslant -3K \quad \text{for } |b-a|<r_0.$$

Since touchdown set of u is assumed to be a compact subset of Ω, we have $\operatorname{dist}(a,\partial\Omega)>\eta$ for some $\eta>0$. Therefore, it now follows from Lemma 2.10 that

$$E_s[w_b](s)\leqslant -2K \quad \text{for } |b-a|<r_0,\ s\geqslant\bar{s},$$

provided $K\geqslant M_1:=\int_{s_0}^{\infty}g_\eta(s)\,ds$, where $g_\eta(s)$ is as in Lemma 2.10. Since the first term and the third term in $E_s[w_b](s)$ are uniformly bounded, we have

$$\int_{B_s}\rho w_b^2\,dy\geqslant 6K \quad \text{for } |b-a|<r_0,\ s\geqslant\bar{s}. \tag{2.39}$$

Recalling from Corollary 2.9,

$$w_b^2(y,s)\leqslant 2\big(w_b^2(0,s)+M'^2|y|^2\big),$$

we obtain from (2.39) that

$$3K\leqslant w_b^2(0,s)\int_{B_s}\rho\,dy+M'^2\int_{B_s}\rho|y|^2\,dy\leqslant C_1w_b^2(0,s)+C_2.$$

We now choose $K\geqslant\max\{M_1,\frac{2}{3}C_2\}$ so large that

$$w_b(0,s)\geqslant\sqrt{\frac{3K}{2C_1}}:=\varepsilon. \tag{2.40}$$

Setting $\bar{t}:=T-e^{-\bar{s}}$, it reduces from (2.40) that

$$u(b,t)(T-t)^{-\frac{1}{3}}\geqslant\varepsilon \quad \text{for } |b-a|<r_0,\ \bar{t}<t<T.$$

Applying Lemma 2.11 with a small r_0, we finally conclude that a is not a touchdown point for u, and the theorem is proved. □

3. Refined touchdown profiles

In this section we first establish touchdown rates by applying self-similar method [6,11]. Then the refined touchdown profiles for $N = 1$ and $N = 2$ will be separately derived by using center manifold analysis of a PDE [3], which will be discussed for $N = 1$ in Section 3.1 and for $N \geqslant 2$ in Section 3.2, respectively. It should be pointed out that for $N = 1$ we may establish the refined touchdown profiles for any touchdown point, see Theorem 1.3(1); while for $N \geqslant 2$, we are only able to deal with the refined touchdown profiles in the radial situation for the special touchdown point $r = 0$, see Theorem 1.3(2). Throughout this section and unless mentioned otherwise, touchdown set for u is assumed to be a compact subset of Ω, and a is always assumed to be any touchdown point of u. Therefore, all a priori estimates of last section can be adapted here.

Our starting point of studying touchdown profiles is a similarity variable transformation of (1.1). For the touchdown solution $u = u(x,t)$ of (1.1) at finite time T, as before we use the associated similarity variables

$$y = \frac{x-a}{\sqrt{T-t}}, \qquad s = -\log(T-t), \qquad u(x,t) = (T-t)^{\frac{1}{3}} w(y,s), \tag{3.1}$$

where a is any touchdown point of u. Then $w(y,s)$ is defined in $W = \{(y,s)\colon |y| < Re^{s/2}, s > s' = -\log T\}$, where $R = \max\{|x-a|\colon x \in \Omega\}$, and it solves

$$w_s - \frac{1}{\rho}\nabla(\rho\nabla w) - \frac{1}{3}w + \frac{\lambda f(a + ye^{-\frac{s}{2}})}{w^2} = 0 \tag{3.2}$$

with $\rho(y) = e^{-|y|^2/4}$, where $f(a) > 0$ since a is assumed to be a touchdown point. Therefore, studying touchdown behavior of u is equivalent to studying large time behavior of w.

Lemma 3.1. *Suppose w is a solution of* (3.2). *Then, $w(y,s) \to w_\infty(y)$ as $s \to \infty$ uniformly on $|y| \leqslant C$, where $C > 0$ is any bounded constant, and $w_\infty(y)$ is a bounded positive solution of*

$$\Delta w - \frac{1}{2}y \cdot \nabla w + \frac{1}{3}w - \frac{\lambda f(a)}{w^2} = 0 \quad \text{in } \mathbb{R}^N, \tag{3.3}$$

where $f(a) > 0$.

Proof. We adapt the arguments from the proofs of Propositions 6 and 7 in [11]: let $\{s_j\}$ be a sequence such that $s_j \to \infty$ and $s_{j+1} - s_j \to \infty$ as $j \to \infty$. We define $w_j(y,s) = w(y, s + s_j)$. According to Theorem 1.1, Corollary 2.9 and Arzela–Ascoli theorem, there is a subsequence of $\{w_j\}$, still denoted by w_j, such that

$$w_j(y,s) \to w_\infty(y,s)$$

uniformly on compact subsets of W, and

$$\nabla w_j(y,m) \to \nabla w_\infty(y,m)$$

for almost all y and for each integer m. We obtain from Corollary 2.9 that either $w_\infty \equiv \infty$ or $w_\infty < \infty$ in $\mathbb{R}^{N+1}$. Since a is a touchdown point for u, the case $w_\infty \equiv \infty$ is ruled out by

Theorem 2.12, and hence $w_\infty < \infty$ in $\mathbb{R}^{N+1}$. Therefore, we conclude again from Corollary 2.9 that

$$w \leqslant C_1\big(1+|y|\big) \tag{3.4}$$

for some constant $C_1 > 0$.

Define the associated energy of w at time s,

$$E_R[w](s) = \frac{1}{2}\int\limits_{B_R} \rho|\nabla w|^2\,dy - \frac{1}{6}\int\limits_{B_R} \rho w^2\,dy - \int\limits_{B_R} \frac{\lambda\rho f(a)}{w}\,dy. \tag{3.5}$$

Taking $R(s) = s$, the same calculations as in (2.30) give

$$-\frac{d}{ds}E_s[w](s) = \int\limits_{B_s} \rho(y)|w_s|^2\,dy - K(s) \tag{3.6}$$

with

$$\begin{aligned} K(s) &= \int\limits_{\partial B_s} \rho w_s \frac{\partial w}{\partial \nu}\,dS + \frac{1}{2s}\int\limits_{\partial B_s} \rho|\nabla w|^2 (y\cdot\nu)\,dS \\ &\quad - \frac{1}{s}\int\limits_{\partial B_s} \rho\left(\frac{1}{6}w^2 + \frac{\lambda f(a)}{w}\right)(y\cdot\nu)\,dS \\ &\quad + \lambda\int\limits_{B_s} \frac{\rho w_s[f(a) - f(a+ye^{-s/2})]}{w^2}\,dy. \end{aligned}$$

We note that the expression $K(s)$ can be estimated as $s \gg 1$. Essentially, since $f(x) \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1]$, using (3.4) and applying the same estimates as in Lemma 2.10 one can deduce that

$$K(s) - \frac{1}{2}\int\limits_{B_s} \rho w_s^2\,dy \leqslant G(s) := C_1 s^N e^{-\frac{s^2}{4}} + C_2 e^{-\frac{\alpha}{2}s} \quad \text{for } s \gg 1. \tag{3.7}$$

Together with (3.7), integrating (3.6) in time yields an energy inequality

$$\frac{1}{2}\int\limits_a^b \int\limits_{B_s} \rho|w_s|^2\,dy\,ds \leqslant E_a[w](a) - E_b[w](b) + \int\limits_a^b G(s)\,ds, \tag{3.8}$$

whenever $a < b$.

We now use (3.8) to prove that w_∞ is independent of s. We set $a = s_j + m$ and $b = s_{j+1} + m$ in (3.8) to obtain

$$\frac{1}{2}\int_{m}^{m+s_{j+1}-s_j}\int_{B_{s_j+s}}\rho|w_{js}|^2\,dy\,ds$$

$$\leqslant E_{s_j+m}[w_j](m)-E_{s_{j+1}+m}[w_{j+1}](m)+\int_{s_j+m}^{s_{j+1}+m}G(s)\,ds \tag{3.9}$$

for any integer m, where we use $w_j(y,s)=w(y,s+s_j)$. Since $\nabla w_j(y,m)$ is bounded and independent of j, and since we have assumed that $\nabla w_j(y,m)\to\nabla w_\infty(y,m)$ a.e. as $j\to\infty$, the dominated convergence theorem shows that

$$\int\rho(y)\big|\nabla w_j(y,m)\big|^2\,dy\to\int\rho(y)\big|\nabla w_\infty(y,m)\big|^2\,dy\quad\text{as } j\to\infty.$$

Arguing similarly for the other terms we can deduce that

$$\lim_{j\to\infty}E_{s_j+m}[w_j](m)=\lim_{j\to\infty}E_{s_{j+1}+m}[w_{j+1}](m):=E[w_\infty]. \tag{3.10}$$

On the other hand, because $m+s_j\to\infty$ as $j\to\infty$, (3.7) assures that the term involving G in (3.9) tends to zero as $j\to\infty$. Therefore, the right side of (3.9) tends to zero as $j\to\infty$. It now follows from $s_{j+1}-s_j\to\infty$ that

$$\lim_{j\to\infty}\int_{m}^{M}\int_{B_{s_j+s}}\rho|w_{js}|^2\,dy\,ds=0 \tag{3.11}$$

for each pair of integers $m<M$. Further, since (3.4) implies $|w_{js}(y,s)|\leqslant C(1+|y|)$ with C independently of j, one can deduce that w_{js} converges weakly to $w_{\infty s}$. Because ρ decreases exponentially as $|y|\to\infty$, the integral of (3.11) is lower semi-continuous, and hence

$$\int_{m}^{M}\int_{\mathbb{R}^N}\rho|w_{\infty s}|^2\,dy\,ds=0,$$

where m and M are arbitrary, which shows that w_∞ is independent of the choice of s.

We now notice from (3.5) that (3.10) defines $E[w_\infty]$ by

$$E[v]=\frac{1}{2}\int_{\mathbb{R}^N}\rho|\nabla_y v|^2\,dy-\frac{1}{6}\int_{\mathbb{R}^N}\rho|v|^2\,dy-\int_{\mathbb{R}^N}\frac{\lambda\rho f(a)}{v}\,dy.$$

We claim that $E[w_\infty]$ is independent of the choice of the sequence $\{s_j\}$. If this is not the case, then there is another $\{\bar{s}_j\}$ such that $E[w_\infty]\neq E[\bar{w}_\infty]$, where $\bar{w}_\infty=\lim_{j\to\infty}\bar{w}_j$ with $\bar{w}_j(y,s)=$

$w(y, s+\bar{s}_j)$. Relabeling and passing to a sequence if necessary, we may suppose that $E[w_\infty] < E[\bar{w}_\infty]$ with $s_j < \bar{s}_j$. Now the energy inequality (3.8), with $a = s_j$ and $b = \bar{s}_j$, gives that

$$\frac{1}{2}\int_{s_j}^{\bar{s}_j}\int_{B_s} \rho|w_s|^2\,dy\,ds \leqslant E_{s_j}[w_j](0) - E_{\bar{s}_j}[\bar{w}_j](0) + \int_{s_j}^{\bar{s}_j} G(s)\,ds. \tag{3.12}$$

Since $E_{s_j}[w_j](0) - E_{\bar{s}_j}[\bar{w}_j](0) \to E[w_\infty] - E_{\bar{w}_\infty}[\bar{w}_\infty] < 0$ and $\int_{s_j}^{\bar{s}_j} G(s)\,ds \to 0$ as $j \to \infty$, the right side of (3.12) is negative for sufficiently large j. This leads to a contradiction, because the left side of (3.12) is non-negative. Hence $E[w_\infty] = E[\bar{w}_\infty]$, which implies that $E[w_\infty]$ is independent of the choice of the sequence $\{s_j\}$.

Therefore, we conclude that $w(y,s) \to w_\infty(y)$ as $s \to \infty$ uniformly on $|y| \leqslant C$, where C is any bounded constant, and $w_\infty(y)$ is a bounded positive solution of (3.3). □

3.1. Refined touchdown profiles for $N = 1$

In this subsection, we establish refined touchdown profiles for the deflection $u = u(x,t)$ in one-dimensional case. We begin with the discussions on the solution $w_\infty(y)$ of (3.3). For one-dimensional case, Fila and Hulshof proved in Theorem 2.1 of [2] that every non-constant solution $w(y)$ of

$$w_{yy} - \frac{1}{2}yw_y + \frac{1}{3}w - \frac{1}{w^2} = 0 \quad \text{in } (-\infty, \infty)$$

must be strictly increasing for all $|y|$ sufficiently large, and $w(y)$ tends to ∞ as $|y| \to \infty$. So it reduces from Lemma 3.1 that it must have $w_\infty(y) \equiv \text{const}$. Therefore, by scaling we conclude that

$$\lim_{s\to\infty} w(y,s) \equiv \big(3\lambda f(a)\big)^{\frac{1}{3}}$$

uniformly on $|y| \leqslant C$ for any bounded constant C. This gives the following touchdown rate.

Lemma 3.2. *Assume f satisfies* (1.2) *on a bounded domain $\Omega \subset \mathbb{R}^1$, and suppose u is a unique touchdown solution of* (1.1) *at finite time T. Assume touchdown set for u is a compact subset of Ω. If $x = a$ is a touchdown point of u, then we have*

$$\lim_{t\to T^-} u(x,t)(T-t)^{-\frac{1}{3}} \equiv \big(3\lambda f(a)\big)^{\frac{1}{3}}$$

uniformly on $|x-a| \leqslant C\sqrt{T-t}$ for any bounded constant C.

We next determine the refined touchdown profiles for one-dimensional case. Our method is based on the center manifold analysis of a PDE that results from a similarity group transformation of (1.1). Such an approach was used in [15] for the uniform permittivity profile $f(x) \equiv 1$. A closely related approach was used in [3] to determine the refined blow-up profile for a semilinear heat equation. We now briefly outline this method and the results that can be extended to the varying permittivity profile $f(x)$.

Continuing from (3.2) with touchdown point $x = a$, for $s \gg 1$ and $|y|$ bounded we have $w \sim w_\infty + v$, where $v \ll 1$ and $w_\infty \equiv (3\lambda f(a))^{1/3} > 0$. Keeping the quadratic terms in v, we obtain for $N = 1$ that

$$\begin{aligned} v_s - v_{yy} + \frac{y}{2} v_y - v &= \frac{w_\infty}{3}\left[1 - \frac{f(a + ye^{-s/2})}{f(a)}\right] + \frac{2[f(a+ye^{-s/2}) - f(a)]}{3f(a)} v \\ &\quad - \frac{3\lambda f(a+ye^{-s/2})}{w_\infty^4} v^2 + O\big(v^3\big) \\ &\approx -\big(3\lambda f(a)\big)^{-\frac{1}{3}} v^2 + O\big(v^3 + e^{-\frac{\alpha}{2}s}\big) \end{aligned} \tag{3.13}$$

for $s \gg 1$ and bounded $|y|$, due to the assumption (1.2) that $f(x) \in C^\alpha(\bar{\Omega})$ for some $0 < \alpha \leqslant 1$. As shown in [3] (see also [15]), the linearized operator in (3.13) has a one-dimensional nullspace when $N = 1$. By projecting the nonlinear term in (3.13) against the nullspace of the linearized operator, the following far-field behavior of v for $s \to +\infty$ and $|y|$ bounded is obtained (see (1.7) of [3]):

$$v \sim -\frac{(3\lambda f(a))^{\frac{1}{3}}}{4s}\left(1 - \frac{|y|^2}{2}\right), \quad N = 1. \tag{3.14}$$

The refined touchdown profile is then obtained from $w \sim w_\infty + v$, (3.1) and (3.14), which is for $t \to T^-$,

$$u \sim \big[3\lambda f(a)(T-t)\big]^{\frac{1}{3}}\left(1 - \frac{1}{4|\log(T-t)|} + \frac{|x-a|^2}{8(T-t)|\log(T-t)|} + \cdots\right), \quad N = 1. \tag{3.15}$$

Combining Lemma 3.2 and (3.15) completes the proof of Theorem 1.3(1).

We finally remark that applying formal asymptotic methods, when $N = 1$ the refined touchdown profile of (1.1) was also established in (4.11) of [15]. By making a binomial approximation, it is easy to compare that (3.15) agrees asymptotically with (4.11) of [15].

3.2. *Refined touchdown profiles for $N \geqslant 2$*

For obtaining refined touchdown profiles in higher dimension, in this subsection we assume that $f(r) = f(|x|)$ is radially symmetric and $\Omega = B_R(0)$ is a bounded ball in $\mathbb{R}^N$ with $N \geqslant 2$. Then the uniqueness of solutions for (1.1) implies that the solution u of (1.1) must be radially symmetric. We study the refined touchdown profile for the special touchdown point $r = 0$ of u at finite time T. In this situation, the fact that the solution u of (1.1) is radially symmetric implies the radial symmetry of $w(y,s)$ in y, and hence the radial symmetry of $w_\infty(y)$ (cf. [14]). Note that $w_\infty(y)$ is a radially symmetric solution of

$$w_{yy} + \left(\frac{N-1}{y} - \frac{y}{2}\right) w_y + \frac{1}{3} w - \frac{\lambda f(0)}{w^2} = 0 \quad \text{for } y > 0, \tag{3.16}$$

where $w_y(0) = 0$ and $f(0) > 0$. For this case, applying Theorem 1.6 of [7] yields that every non-constant radial solution $w(y)$ of (3.16) must be strictly increasing for all y sufficiently large, and $w(y)$ tends to ∞ as $y \to \infty$. It now reduces again from Lemma 3.1 that

$$\lim_{s\to\infty} w(y,s) \equiv \big(3\lambda f(0)\big)^{\frac{1}{3}}$$

uniformly on $|y| \leqslant C$ for any bounded constant C. This gives the following touchdown rate.

Lemma 3.3. *Assume $f(r) = f(|x|)$ satisfies* (1.2) *on a bounded ball $B_R(0) \subset \mathbb{R}^N$ with $N \geqslant 2$, and suppose u is a unique touchdown solution of* (1.1) *at finite time T. Assume touchdown set for u is a compact subset of Ω. If $r = 0$ is a touchdown point of u, then we have*

$$\lim_{t\to T^-} u(r,t)(T-t)^{-\frac{1}{3}} \equiv \big(3\lambda f(0)\big)^{\frac{1}{3}}$$

uniformly for $r \leqslant C\sqrt{T-t}$ for any bounded constant C.

For completing Theorem 1.3(2), the rest is to derive the refined touchdown profile (1.10). Similar to one-dimensional case, indeed we can establish the refined touchdown profiles for varying permittivity profile $f(|x|)$ defined in higher dimension $N \geqslant 2$. Specially, applying a result from [3], the refined touchdown profile for $N = 2$ is given by

$$u \sim \big[3\lambda f(0)(T-t)\big]^{1/3}\left(1 - \frac{1}{2|\log(T-t)|} + \frac{|x-a|^2}{4(T-t)|\log(T-t)|} + \cdots\right), \quad N = 2.$$

Remark 3.1. Applying analytical and numerical techniques, next section we shall show that Theorem 1.3(2) does hold for a larger class of profiles $f(r) = f(|x|)$.

Before concluding this section, it is interesting to compare the solution of (1.1) with that of the ordinary differential equation obtained by omitting Δu. For that we focus on one-dimensional case, and we compare the solutions of

$$u_t - u_{xx} = -\frac{\lambda f(x)}{u^2} \quad \text{in } (-a,a), \tag{3.17a}$$

$$u(\pm a,t) = 1, \qquad u(x,0) = 1, \tag{3.17b}$$

and

$$v_t = -\frac{\lambda f(x)}{v^2} \quad \text{in } (-a,a), \tag{3.18a}$$

$$v(\pm a,t) = 1, \qquad v(x,0) = 1, \tag{3.18b}$$

where f is assumed to satisfy (1.2) and (1.4). The ordinary differential equation (3.18) is explicitly solvable, and the solution touches down at finite time

$$v(x,t) = \big(1 - 3\lambda f(x)t\big)^{\frac{1}{3}}, \tag{3.19}$$

which shows that touchdown point of v is the maximum value point of $f(x)$. In the partial differential equation (3.17), there is a contest between the dissipating effect of the Laplacian u_{xx} and the singularizing effect of the nonlinearity $f(x)/u^2$; when u touches down at $x = x_0$ in finite time T, then the nonlinear term dominates. (Essentially, for some special cases, touchdown point x_0 of u is also the maximum value point of $f(x)$, see Theorem 1.4 for details.)

However, we claim that a smoothing effect of the Laplacian can be still observed in the different character of touchdown. Indeed, letting $f(y_0) = \max\{f(x)\colon x \in (-a, a)\}$, then $f'(y_0) = 0$ and $f''(y_0) \leqslant 0$. And (3.19) gives the finite touchdown time T_0 for v satisfying $T_0 = 1/[3\lambda f(y_0)]$. Furthermore, we can get from (3.19), together with the Taylor series of $f(x)$,

$$\lim_{t\to T_0^-} (T_0 - t)^{-\frac{1}{3}} v\big(y_0 + (T_0 - t)^{\frac{1}{2}} y, t\big) = \big(3\lambda f(y_0)\big)^{\frac{1}{3}} \left[1 - \frac{f''(y_0)}{2f^2(y_0)}|y|^2\right]^{\frac{1}{3}} \geqslant \big(3\lambda f(x_0)\big)^{\frac{1}{3}}. \tag{3.20}$$

And our Theorem 1.3(1) says that for such u we have

$$\lim_{t\to T^-} (T - t)^{-\frac{1}{3}} u\big(x_0 + (T - t)^{\frac{1}{2}} y, t\big) = \big(3\lambda f(x_0)\big)^{\frac{1}{3}}. \tag{3.21}$$

Comparing (3.20) with (3.21), we see that the touchdown of the partial differential equation (3.17) is “flatter” than that of the ordinary differential equation (3.18).

4. Set of touchdown points

This section is focused on the set of touchdown points for (1.1), which may provide useful information on the design of MEMS devices. In Section 4.1, we consider the radially symmetric case where $f(r) = f(|x|)$ with $r = |x|$ is a radial function and Ω is a ball $B_R = \{|x| \leqslant R\} \subset \mathbb{R}^N$ with $N \geqslant 1$. In Section 4.2, numerically we compute some simulations for one-dimensional case, from which we discuss the compose of touchdown points for some explicit permittivity profiles $f(x)$.

4.1. Radially symmetric case

In this subsection, $f(r) = f(|x|)$ is assumed to be a radial function and Ω is assumed to be a ball $B_R = \{|x| \leqslant R\} \subset \mathbb{R}^N$ with any $N \geqslant 1$. For this radially symmetric case, the uniqueness of solutions for (1.1) implies that the solution $u = u(x,t)$ of (1.1) must be radially symmetric. We begin with the following lemma for proving Theorem 1.4:

Lemma 4.1. *Suppose $f(r)$ satisfies* (1.2) *and $f'(r) \leqslant 0$ in B_R, and let $u = u(r,t)$ be a touchdown solution of* (1.1) *at finite time T. Then $u_r > 0$ in $\{0 < r < R\} \times (t_0, T)$ for some $0 < t_0 < T$.*

Proof. Setting $w = r^{N-1}u_r$, then (1.1) gives

$$u_t - \frac{1}{r^{N-1}} w_r = -\frac{\lambda f(r)}{u^2}, \quad 0 < t < T. \tag{4.1}$$

Differentiating (4.1) with respect to r, we obtain

$$w_t - w_{rr} + \frac{N-1}{r} w_r - \frac{2\lambda f}{u^3} w = -\frac{\lambda f' r^{N-1}}{u^2} \geqslant 0, \quad 0 < t < T, \tag{4.2}$$

since $f'(r) \leqslant 0$ in B_R. Therefore, w cannot attain negative minimum in $\{0 < r < R\} \times (0, T)$. Since $w(0,t) = w(r,0) = 0$ and $u_t < 0$ for all $t \in (0,T)$, we have $w = r^{N-1} u_r > 0$ on $\partial B_R \times (0,T)$. So the maximum principle shows that $w \geqslant 0$ in $\{0 < r < R\} \times (0,T)$. This gives

$$w_t - w_{rr} + \frac{N-1}{r} w_r \geqslant 0 \quad \text{in } \{0 < r < R\} \times (t_1, T),$$

where $t_1 > 0$ is chosen so that $w(r, t_1) \not\equiv 0$ in $\{0 < r < R\}$.

Now compare w with the solution z of

$$z_t - z_{rr} + \frac{N-1}{r} z_r = 0 \quad \text{in } \{0 < r < R\} \times (t_1, T)$$

subject to $z(r, t_1) = w(r, t_1)$ for $0 \leqslant r \leqslant R$, $z(R,t) = w(R,t) > 0$ and $z(0,t) = 0$ for $t_1 \leqslant t < T$. The comparison principle yields $w \geqslant z$ in $\{0 < r < R\} \times (t_1, T)$. On the other hand, for any $t_0 > t_1$ we have $z > 0$ in $\{0 < r < R\} \times (t_0, T)$. Consequently we conclude that $w > 0$, i.e. $u_r > 0$ in $\{0 < r < R\} \times (t_0, T)$. □

Proof of Theorem 1.4. For $w = r^{N-1} u_r$, we set $J(r,t) = w - \varepsilon \int_0^{r^\theta} f(s)\,ds$, where $\theta \geqslant N$ and $\varepsilon = \varepsilon(\theta) > 0$ are constants to be determined. We calculate from (4.1) and (4.2) that

$$\begin{aligned} J_t - J_{rr} + \frac{N-1}{r} J_r &= b_1 J + \frac{2\lambda\varepsilon f \int_0^{r^\theta} f(s)\,ds}{u^3} - \frac{\lambda f' r^{N-1}}{u^2} + \theta\varepsilon r^{\theta-1} f' \\ &\geqslant b_1 J - r^{N-1}\big(\lambda - \theta\varepsilon r^{\theta-N}\big) f' \geqslant b_1 J, \end{aligned}$$

provided ε is sufficiently small, where b_1 is a locally bounded function. Here we have applied the assumption $f'(r) \leqslant 0$ and the relations $u_r = w/r^{N-1}$ and $w = J + \varepsilon \int_0^{r^\theta} f(s)\,ds$. Note that $J(0,t) = 0$, and hence it follows that J cannot obtain negative minimum in $B_R \times (0,T)$.

We next observe that J cannot obtain negative minimum on $\{r = R\}$ provided ε is sufficiently small, which comes from the fact

$$\begin{aligned} J_r(R,t) = w_r - \theta\varepsilon R^{\theta-1} f(R) &= \frac{\lambda R^{N-1} f(R)}{u^2} - \theta\varepsilon R^{\theta-1} f(R) \\ &\geqslant R^{N-1} f(R)\big[\lambda - \theta\varepsilon R^{\theta-N}\big] \geqslant 0 \end{aligned}$$

for sufficiently small $\varepsilon > 0$, where (4.1) is applied. We now choose some $0 < t_0 < T$ such that $w(r, t_0) > 0$ for $0 < r \leqslant R$ in view of Lemma 4.1. This gives $u_r(r, t_0) > 0$ for $0 < r \leqslant R$. Since $u_r(0, t_0) = 0$, there exists some $\alpha > 0$ such that

$$u_{rr}(0, t_0) = \lim_{r \to 0} \frac{u_r(r, t_0)}{r^\alpha} = \lim_{r \to 0} \frac{w(r, t_0)}{r^{N+\alpha-1}} > 0.$$

We now choose $\theta = \max\{N, N+\alpha-1\}$, from which one can further deduce that $J(r,t_0) \geqslant 0$ for $0 \leqslant r < R$ provided $\varepsilon = \varepsilon(t_0,\theta) > 0$ is sufficiently small.

It now concludes from the maximum principle that $J \geqslant 0$ in $B_R \times (t_0, T)$ provided $\varepsilon = \varepsilon(t_0) > 0$ is sufficiently small. This leads to

$$u(r,t) \geqslant u(r,t) - u(0,t) \geqslant \varepsilon \int_0^r \frac{\int_0^{s^\theta} f(\mu)\,d\mu}{s^{N-1}}\,ds. \tag{4.3}$$

Given small $C_0 > 0$, then the assumption of $f(r)$ implies that there exists $0 < r_0 = r_0(C_0) \leqslant R$ such that $f(r) \geqslant C_0$ on $[0, r_0]$. Denote $r_m = \min\{r_0, r\}$, and then (4.3) gives

$$\begin{aligned} u(r,t) &\geqslant \varepsilon \int_0^{r_m} \frac{\int_0^{s^\theta} f(\mu)\,d\mu}{s^{N-1}}\,ds \geqslant \varepsilon \int_0^{r_m} \frac{C_0 s^\theta}{s^{N-1}}\,ds \\ &= \frac{1}{\theta - N + 2}\varepsilon C_0 r_m^{\theta-N+2}, \quad \text{where } \theta - N + 2 \geqslant 2, \end{aligned}$$

which implies that $r = 0$ must be the unique touchdown point of u. □

Before ending this subsection, we now present a few numerical simulations on Theorem 1.4. Here we apply the implicit Crank–Nicholson scheme (see §3.2 of [15] for details). In the following simulations $1 \sim 3$, we always take $\lambda = 8$ and the number of meshpoints $N = 1000$, and consider (1.1) in the following symmetric slab or unit disk domains:

$$\Omega\colon [-1/2, 1/2] \quad \text{(slab)}, \qquad \Omega\colon x^2 + y^2 \leqslant 1 \quad \text{(unit disk)}. \tag{4.4}$$

Simulation 1. $f(|x|) = 1 - |x|^2$ is chosen as a permittivity profile. In Fig. 2(a), u versus x is plotted at different times for (1.1) in the symmetric slab domain. For this touchdown behavior, touchdown time is $T = 0.044727$ and the unique touchdown point is $x = 0$. In Fig. 2(b), u versus $r = |x|$ is plotted at different times for (1.1) in the unit disk domain. For this touchdown behavior, touchdown time is $T = 0.0455037$ and the unique touchdown point is $r = 0$.

Simulation 2. $f(|x|) = e^{-|x|^2}$ is chosen as a permittivity profile. In Fig. 3(a), u versus x is plotted at different times for (1.1) in the symmetric slab domain. For this touchdown behavior, touchdown time is $T = 0.044675$ and the unique touchdown point is $x = 0$. In Fig. 3(b), u versus $r = |x|$ is plotted at different times for (1.1) in the unit disk domain. For this touchdown behavior, touchdown time is $T = 0.0450226$ and the unique touchdown point is $r = 0$ too.

Simulation 3. $f(|x|) = e^{|x|^2-1}$ is chosen as a permittivity profile. In Fig. 4(a), u versus x is plotted at different times for (1.1) in the symmetric slab domain. For this touchdown behavior, touchdown time is $T = 0.147223$ and touchdown point is still uniquely at $x = 0$. In Fig. 4(b), u versus $r = |x|$ is plotted at different times for (1.1) in the unit disk domain. For this touchdown behavior, touchdown time is $T = 0.09065363$, but touchdown points are at $r_0 = 0.51952$, which compose into the surface of $B_{r_0}(0)$. This simulation shows that the assumption $f'(r) \leqslant 0$ in Theorem 1.4 is just sufficient, not necessary.

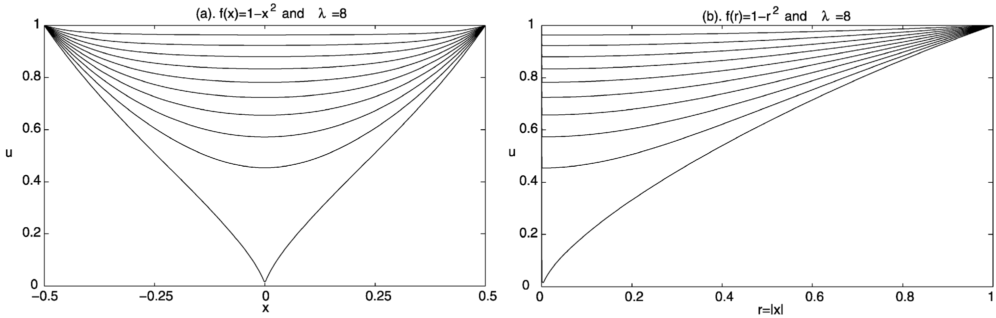

Fig. 2. Left figure: plots of u versus x at different times with $f(x) = 1 - x^2$ in the slab domain, where the unique touchdown point is $x = 0$. Right figure: plots of u versus $r = |x|$ at different times with $f(r) = 1 - r^2$ in the unit disk domain, where the unique touchdown point is $r = 0$ too.

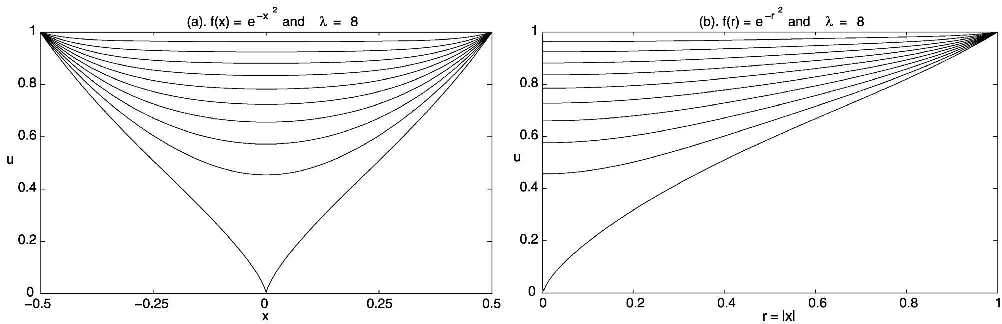

Fig. 3. Left figure: plots of u versus x at different times with $f(x) = e^{-x^2}$ in the slab domain, where the unique touchdown point is $x = 0$. Right figure: plots of u versus $r = |x|$ at different times with $f(r) = e^{-r^2}$ in the unit disk domain, where the unique touchdown point is $r = 0$ too.

4.2. One-dimensional case

For one-dimensional case $\Omega = [-a, a]$, Theorem 1.4 already gives that touchdown points must be unique if the permittivity profile $f(x)$ is uniform. In the following, we choose some explicit varying permittivity profiles $f(x)$ to perform two numerical simulations. Here we apply the implicit Crank–Nicholson scheme again.

Simulation 4. Monotone function $f(x)$. We take $\lambda = 8$ and the number of meshpoints $N = 1000$, and we consider (1.1) in the slab domain Ω defined in (4.4). In Fig. 5(a), the monotonically decreasing profile $f(x) = 1/2 - x/2$ is chosen, and u versus x is plotted for (1.1) at different times. For this touchdown behavior, the touchdown time is $T = 0.09491808$ and the unique touchdown point is $x = -0.10761$. In Fig. 5(b), the monotonically increasing profile $f(x) = x + 1/2$ is chosen, and u versus x is plotted for (1.1) at different times. For this touchdown behavior, the touchdown time is $T = 0.0838265$ and the unique touchdown point is $x = 0.17467$.

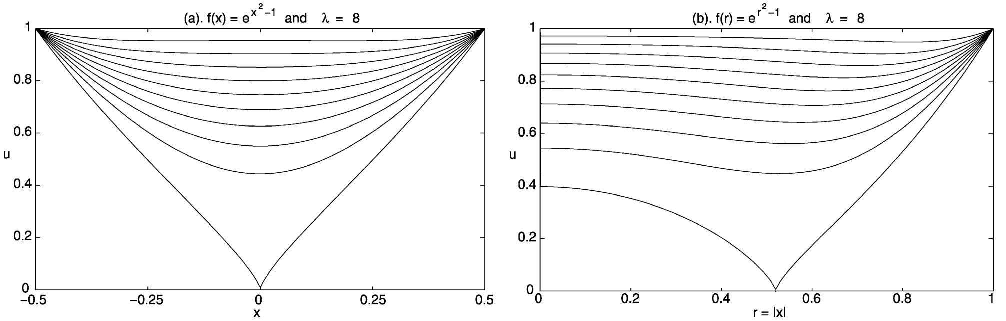

Fig. 4. Left figure: plots of u versus x at different times with $f(x) = e^{x^2-1}$ in the slab domain, where the unique touchdown point is still at $x = 0$. Right figure: plots of u versus $r = |x|$ at different times with $f(r) = e^{r^2-1}$ in the unit disk domain, where the touchdown points satisfy $r = 0.51952$.

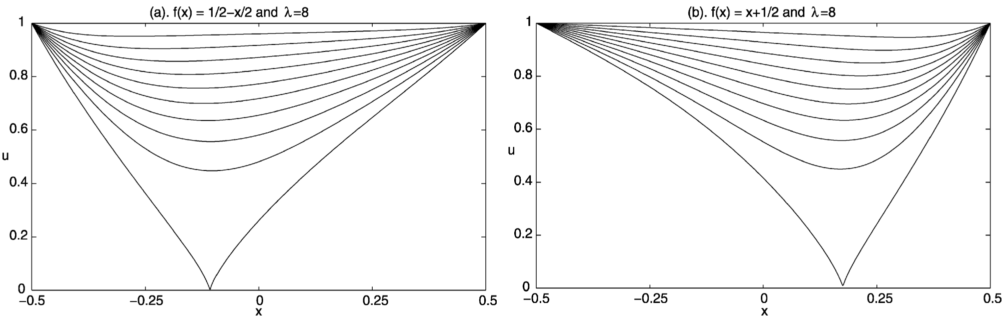

Fig. 5. Left figure: plots of u versus x at different times with $f(x) = 1/2 - x/2$ in the slab domain, where the unique touchdown point is $x = -0.10761$. Right figure: plots of u versus $r = |x|$ at different times with $f(x) = x + 1/2$ in the slab domain, where the unique touchdown point is $x = 0.17467$.

Simulation 5. "M"-form function $f(x)$. In this simulation, we consider (1.1) in the slab domain Ω defined in (4.4). Here we take $\lambda = 8$ and the number of the meshpoints $N = 2000$, and the varying dielectric permittivity profiles satisfy

$$f[\alpha](x) = \begin{cases} 1 - 16(x + 1/4)^2, & \text{if } x < -1/4, \\ \alpha + (1-\alpha)|\sin(2\pi x)|, & \text{if } |x| \leqslant 1/4, \\ 1 - 16(x - 1/4)^2, & \text{if } x > 1/4, \end{cases} \tag{4.5}$$

with $\alpha \in [0, 1]$, which has "M"-form. In Fig. 6, u versus x is plotted at different times for (1.1) for different α, i.e. for different permittivity profiles $f[\alpha](x)$. In Fig. 6(a): when $\alpha = 0.5$, the touchdown time is $T = 0.05627054$ and two touchdown points are at $x = \pm 0.12631$. In Fig. 6(b): when $\alpha = 1$, the touchdown time is $T = 0.0443323$ and the unique touchdown point is at $x = 0$. In Fig. 6(c): when $\alpha = 0.785$, the touchdown time is $T = 0.04925421$ and touchdown points are observed to compose into a closed interval $[-0.0021255, 0.0021255]$. In Fig. 6(d): local amplified plots of (c) at touchdown time $t = T$. This simulation shows for dimension $N = 1$ that the set of touchdown points may be composed of finite points or finite compact subsets of the domain, if the permittivity profile is ununiform.

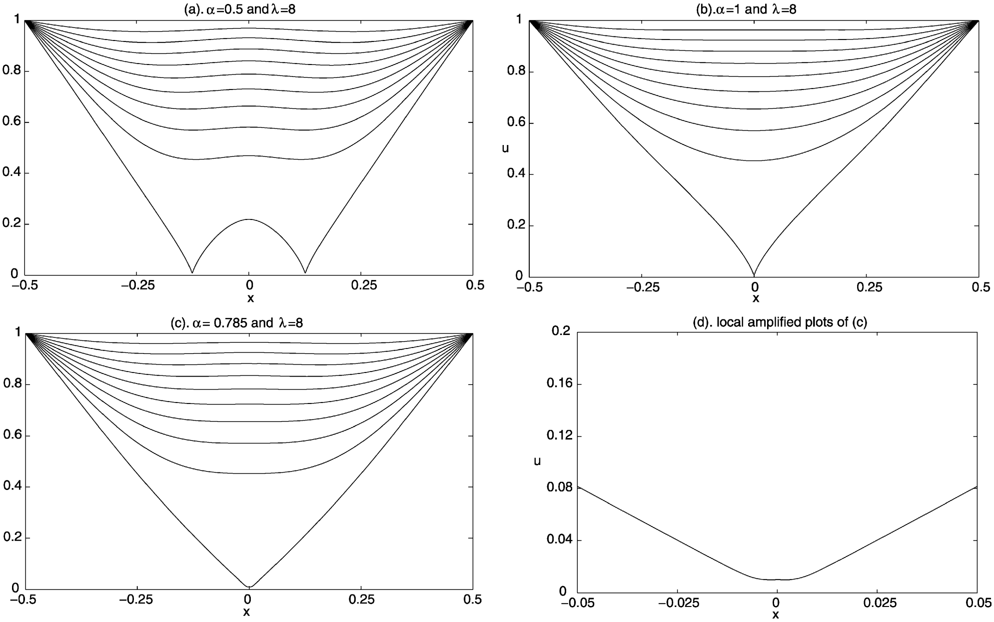

Fig. 6. Plots of u versus x at different times in the slab domain, for different permittivity profiles $f[\alpha](x)$ given by (4.5). Top left (a): when $\alpha = 0.5$, two touchdown points are at $x = \pm 0.12631$. Top right (b): when $\alpha = 1$, the unique touchdown point is at $x = 0$. Bottom left (c): when $\alpha = 0.785$, touchdown points are observed to consist of a closed interval $[-0.0021255, 0.0021255]$. Bottom right (d): local amplified plots of (c).

5. Conclusion

We have analyzed finite-time touchdown (i.e. quenching) behavior of the electrostatic deflection of an elastic membrane, in terms of a spatially variable dielectric permittivity profile $f(x)$. Suppose the domain of the membrane is convex, we have derived in Proposition 2.1 the compactness of touchdown set under the assumption (1.4), which implies the impossibility of touchdown near the boundary of the membrane in MEMS devices. An interesting open problem is to address whether the assumption (1.4) of Proposition 2.1 can be removed for the compactness of touchdown set.

Under the compactness of touchdown set, some a priori estimates of finite-time touchdown behavior have been discussed in Section 2. In particular, it was proved in Proposition 2.2 that any finite-time touchdown point cannot be the zero point of the profile $f(x)$, which was firstly observed in [15]. This shows that touchdown cannot occur at the location where the dielectric permittivity $\epsilon_2(x)$ of the membrane is largest, which gives useful information on the actual design of MEMS devices. Interestingly, we recently proved in [10] that touchdown must occur near the maximum point of profile $f(x)$ for sufficiently large voltage λ. Based on a priori estimates of Section 2, refined touchdown profiles have been obtained in Section 3, which allow us to gain information on how snap-through of MEMS devices occurs.

Touchdown points were analyzed and simulated in Section 4, which is also great practical interest in the actual construction of the dielectric membrane for MEMS devices. When $f(|x|)$ is nonincreasing in $|x|$, the analytic and numerical results of Section 4.1 show that touchdown only occurs at the center of the membrane, provided that the domain Ω of the membrane is radially symmetric. However, when $f(|x|)$ is nondecreasing in $|x|$, we observed in Fig. 4 that all touchdown points of u compose into the surface of $B_{r_0}(0)$ for some $0 \leqslant r_0 < R$, provided that the domain Ω of the membrane is also radially symmetric. Other different shapes of touchdown set were also observed in Section 4.2 for different classes of profiles $f(x)$. It may be interesting to analyze these simulations observed in Section 4.

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