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Multiaxial fatigue criteria for random stress response – theoretical and experimental comparison

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Abstract

Randomly excited structures are exposed to random stress loads that can result in fatigue failure. Prediction of such failure requires good structural dynamics and fatigue models. This study researches the multiaxial criteria which reduce the multiaxial stress state to an equivalent uniaxial stress. Several frequency-domain criteria can be found in literature. However, experimental comparison studies are scarce. This research presents and then theoretically and experimentally compares selected frequency-domain multiaxial criteria, namely: maximum normal stress, maximum shear stress and the C-S criterion. Time-to-failure results are compared across multiaxial criteria and with experimental data. Results show that selected methods give reliable estimates.

1. Introduction

Material fatigue as a consequence of a broad-band random stress load is a known phenomenon [1, 2]. However, most of established fatigue analysis procedures (*e.g.* Palmgren-Miner, S/N curve [3, 4, 5]) are usually only derived for analysis of uniaxial stress. In real structure, a complex multiaxial stress state develops. Furthermore, in the presence of more than one vibration source, the principal stresses are not proportional and the orientation of principal axes changes in time [2]. Such stress state cannot be directly analyzed for fatigue but is rather reduced to a uniaxial stress state through the use of a multiaxial criterion [1, 2, 6].

Characterization of vibration fatigue calls for special methods that can be applied in the frequency-domain, on random response states. The process of analysis consists of the following steps:

- stress response deduction [7, 8],
- equivalent uniaxial stress calculation [1, 2],
- cycle counting and damage summation [9, 10, 11].

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This study focuses on experimental comparison of multiaxial criteria in the frequency-domain, quite a few of which can be found in literature [1, 6, 12, 13, 14, 15] as well as some comparison studies [6, 14, 15]. This study experimentally compares side-by-side the following multiaxial criteria in the frequency-domain: maximum normal stress, maximum shear stress and the CS criterion. Experimental results are obtained using a Y-shaped specimen, which is excited until failure using predefined loads.

The emphasis is on the analysis of a multiaxial stress state, resulting from multiple broad-band random loads.The Y-sample is excited with two electro-dynamic shakers in perpendicular directions, each producing random vibration in the frequency range of one particular modal shape. The stresses are calculated using the FEM method and actual measured loads from experiment. Multiaxial criteria are used to get the equivalent uniaxial stress and Tovo-Benasciutti [9] method is used to calculate time-to-failure. Numerical results are then compared across different multiaxial criteria and with experiment.

Nomenclature

a	vector of coefficients for a certain spectral multiaxial criterion
a(ω)	vector of accelerations in frequency-domain
C	covariance matrix
C$_{CS}$	rotation matrix used with the C-S multiaxial criterion
$G(\omega)$	one sided power spectral density
$G_{seq}(\omega)$	equivalent uniaxial stress power spectral density
G$_{CS}(\omega)$	stresses on a critical plane as determined by the CS criterion
$G_s(\omega)$	cross-correlation matrix of stress tensor component in frequency-domain
H$_{as}(\omega)$	frequency-transfer function from acceleration to stress
l, **m**, **n**	directional cosines of principal axes
m_i	moment of power spectral density
$S_{xx,xx}(\omega)$	auto-correlation of stress tensor component
$S_{xx,yz}(\omega)$	cross-correlation of two stress tensor components
v	peak rate
ω	angular frequency
$\sigma_{xx,xx}^2$	variance of a specific stress tensor component
$\sigma_{xx,yz}^2$	covariance of specific stress tensor components
σ_{seq}^2	variance of equivalent uniaxial stress according to a certain multiaxial criterion
$\sigma_{af,-1}$	fully reversed bending fatigue limit
$\tau_{af,-1}$	fully reversed torsion fatigue limit

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2. Theoretical background

To facilitate a better understanding of vibration fatigue and multiaxial criteria, theoretical fundamentals are presented in this section. Initially, some details on the random processes [16] are given. Next, some structural dynamics [7] details are presented for their crucial role in deducing the stresses that arise in material due to vibratory loads. Finally, the procedure of calculating equivalent stresses (*i.e.* the multiaxial criteria) is presented [1].

2.1. Random processes

Most research in the field of vibration fatigue is based on normal-distributed random processes, ordinarily described by a one-sided power-spectral density G(ω) (PSD) as a function of angular frequency ω. The random process is further characterized by moments of the PSD m_i, which give information about properties of the process in time-domain, such as variance $\sigma^2 = m_0$ or peak rate ν:

$$m_i = \int_0^\infty \omega^i\, G(\omega)\, d\omega \tag{1}$$

$$\nu = \sqrt{\frac{m_4}{m_2}}$$
(2)

Based on these moments, cycle amplitude distribution of the random process can be estimated, using cycle-counting methods in frequency domain, such as Dirlik [10] or Tovo-Benasciutti [9]. Using material data in the form of an S/N curve and Palmgren-Miner hypothesis [4, 5], fatigue damage or time-to-failure can be deduced, which is part of the standard high-cycle fatigue analysis routine. This is for a simple uniaxial stress state, while complex multiaxial stress states calls for special approach with multiaxial criteria.

2.2. Structural dynamics analysis

Structure can be modeled with FEM method as a linear multi-degree-of-freedom system. Frequency analysis solution provides a modal model (*i.e.* eigenfrequencies, eigenvectors) which can be used to deduce the transfer functions from one point on the structure to the next. For the case of kinematic excitation, the structural modification approach [8] is used to obtain the appropriate transfer functions. Knowing the correct transfer function matrices $\mathbf{H_{as}}(\omega)$ (accelerations-to-stress) and $\mathbf{H_{fs}}(\omega)$ (force-to-stress) the random stress responses can then be calculated in the following way:

$$\mathbf{G}_s(\omega) = \mathbf{H_{as}}(\omega)\mathbf{a}(\omega)$$
(3)

where the stress tensor components are described in the frequency domain by a 6-by-6 cross-correlation matrix $\mathbf{G}_s$:

$$\mathbf{G}_s(\omega) = \begin{bmatrix} S_{xx,xx}(\omega) & \cdots & S_{xx,yz}(\omega) \\ \vdots & \ddots & \vdots \\ S_{yz,xx}(\omega) & \cdots & S_{yz,yz}(\omega) \end{bmatrix}$$
(4)

2.3. Multiaxial criteria

The conventional methods for fatigue analysis are based on uniaxial stress state. Multiaxial criteria are used to calculate an equivalent uniaxial stress state from a stress tensor and enable the use of cycle counting methods. These criteria are often based on failure and yield criteria used in static analyses, such as maximum normal stress criterion or von Mises. For vibration fatigue analysis, they are formulated in the frequency domain. The stress tensor in frequency domain $G_s(\omega)$, obtained using modal analysis is a 6-by-6 matrix from eq. (4), which has to be converted to a scalar function of angular frequency $G_{seq}(\omega)$. The process of reduction constitutes a multiaxial criterion in frequency domain. The equivalent stress $G_{seq}(\omega)$ on the critical plane is determined from the components of the original stress tensor, by means of directional cosines l_j, m_j and n_j with $j = 1, 2, 3$ for the respective axis. The following multiaxial criteria are presented and compared side-by-side in this study: maximum normal stress, maximum shear stress and CS criterion.

2.3.1. Criteria of maximum normal stress and maximum shear stress
The maximum normal stress and maximum shear stress can be calculated in time-domain from the time-dependent stress tensor components. A method based on six-input physical system is proposed in [1] that derives the equivalent stress PSD directly from frequency-domain PSD matrices from eq. (4):

$$G_{s_{eq}}(\omega) = \mathbf{a}\,\mathbf{G}_s(\omega)\,\mathbf{a}^{\mathrm{T}}$$
(5)

where **a** is a vector of coefficients, defined for the two criteria in Table 1.

Table 1. Coefficients for derivation of equivalent stress PSD.

a	Max. normal	Max. shear
a_1	l_1^2	$(l_1^2 - l_3^2)$
a_2	m_1^2	$(m_1^2 - m_3^2)$
a_3	n_1^2	$(n_1^2 - n_3^2)$
a_4	$2\,l_1\,m_1$	$2\,(l_1\,m_1 - l_3\,m_3)$
a_5	$2\,l_1\,n_1$	$2\,(l_1\,n_1 - l_3\,n_3)$
a_6	$2\,m_1\,n_1$	$2\,(m_1\,n_1 - m_3\,n_3)$

All of these criteria are applied on the critical plane, which has to be determined beforehand. The straightforward way of doing this would be to calculate the value of the criterion on equally spaced planes and choose the plane that produces the most damaging spectrum. However, the calculation is much faster if only the variance of the criterion is compared for equally spaced planes. In the case of using the latter approach, there is only one value to deal with at each plane as opposed to a frequency-spectrum for total damage calculation. For that purpose, the covariance matrix C is calculated first:

$$\mathbf{C} = \begin{bmatrix} \sigma_{xx,xx}^2 & \cdots & \sigma_{xx,yz}^2 \\ \vdots & \ddots & \vdots \\ \sigma_{yz,xx}^2 & \cdots & \sigma_{yz,yz}^2 \end{bmatrix} \tag{6}$$

and the plane is then searched, that gives the maximum variance of the multiaxial criterion. This is done in accordance with eq. (5):

$$\sigma_{s_{eq}}^2 = \mathbf{a}\,\mathbf{C}\,\mathbf{a}^{\mathrm{T}} \tag{7}$$

2.3.2. C-S criterion

Carpinteri et al. proposed a way to calculate the mean principal stress directions based on cross-correlation matrix $G_s(\omega)$ [6]. The rotation matrix $\mathbf{C}_{CS}$ is obtained by applying 5 successive rotations with 3 of them being based on the orientation of the mean principal angles (see [6] for more details). When the position of the critical plane is determined, the equivalent uniaxial stress G_{seq} is calculated as a combination, dependent on fully reversed bending and torsion limit $\sigma_{af,-1}$ and $\tau_{af,-1}$:

$$\mathbf{G}_{CS}(\omega) = \mathbf{C}_{CS}\,\mathbf{G}_s(\omega)\,\mathbf{C}_{CS}^{\mathrm{T}} \tag{8}$$

$$G_{s_{eq}}(\omega) = G_{CS,11}(\omega) + \left(\frac{\sigma_{af,-1}}{\tau_{af,-1}}\right) G_{CS,66}(\omega) \tag{9}$$

3. Experiment

Two excitation sources – electro-dynamic shakers are used, to provide a sufficiently complex and general stress state inside the material, as shown in Fig. 1. In the vertical direction, the sample is excited using kinematical excitation with a controlled acceleration profile. In the horizontal direction, the specimen is excited using open-loop force excitation. Measured are the acceleration responses, mounted on the top of the sample with the purpose of extracting the respective modal shapes. Force load is measured at the tip of the stinger. Masses of 104.5g are mounted on the Y-sample on each branch, pushing the mode frequencies in the desired frequency range.
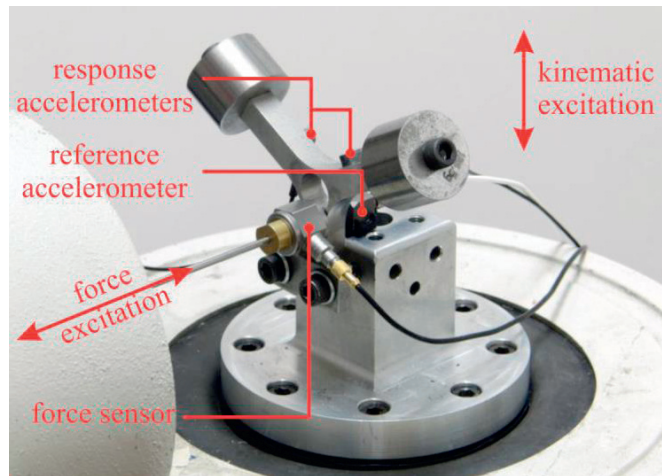
Fig. 1. Experiment setup.

At the beginning of the experiment both excitation sources are powered on with predefined loads, consisting of uniform spectra in the specified range. For kinematic excitation, a uniform profile of amplitude x is controlled in the frequency range of 380 – 480 Hz. For force excitation, the current is applied to the amplifier with a uniform profile of y in the frequency range of 290 – 390 Hz. Due to fatigue, the structural response changes; when at least one of the modes which are monitored continuously goes out of the excited frequency range, the experiment is aborted. The load amplitudes for kinematic excitation are in the range of 0.7 – 1.7 GRMS and actively controlled. The load amplitudes for force excitation are in the range of 4 – 20 NRMS and are not actively controlled. Fig. 2 shows the measured FRFs for the respective modes in the frequency-range that is excited.
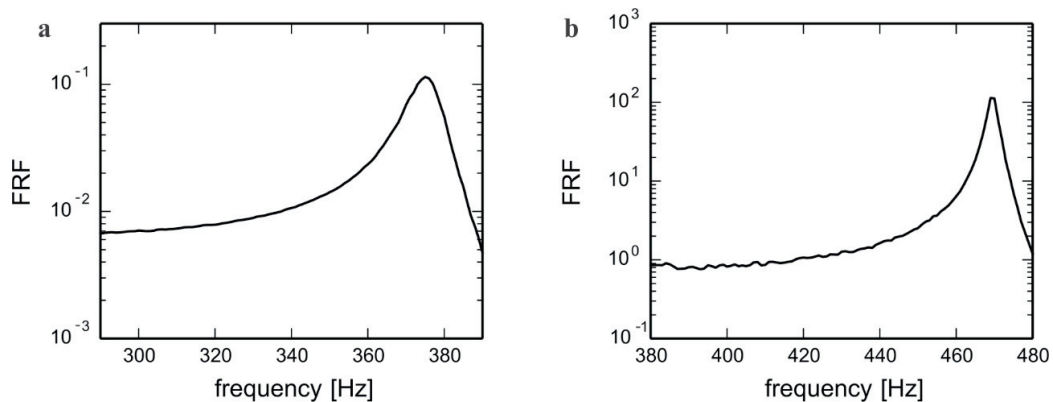

Fig. 2. (a) force excited mode; (b) kinematicaly excited mode.

4. Numerical model

The Y-shaped specimen was analyzed using FEM method to extract the eigenmodes (see Fig. 3). Acceleration and stress-response functions were deduced based on eigenmodes, according to modal superposition method. Structural modification approach [8] was used for calculating response functions for kinematic base excitation. Viscous damping was applied in this process, identified on experimental data using the circle-fit method [7].
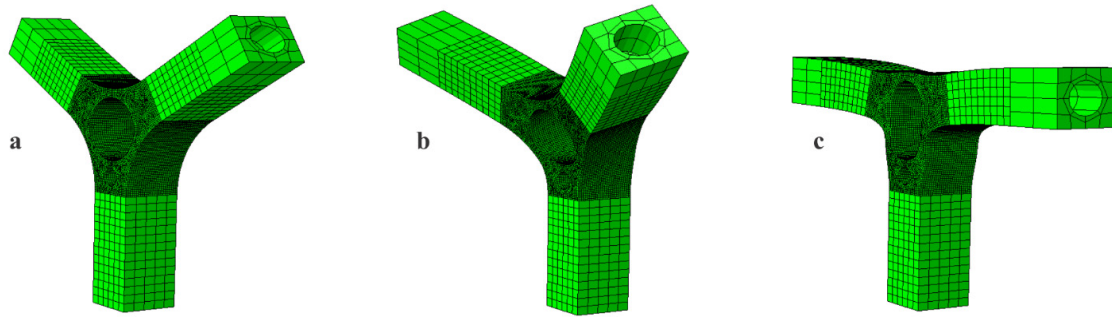
Fig. 3. (a) undeformed specimen; (b) force excited mode, calculated at 355 Hz; (c) kinematicaly excited mode, calculated at 477 Hz.

Multiaxial criteria from chapter 2.3 were applied in combination with the Tovo-Benasciutti [9] cycle counting method. Maximum variance method is applied to find the critical plane. Fully reversed bending and torsion limits, needed for the C-S criterion, were not known for the aluminum alloy tested (AlSi7Cu3). The following values were instead taken from literature [17] for alloy AlCuMg1: $\sigma_{\text{af},-1} = 161$ and $\tau_{\text{af},-1} = 97$. The time-to-failure for each specimen was obtained from experiment and was taken when any mode frequency drops for more than 5%. Material parameters were searched that minimized the deviation of calculated time-to-failure from experimental results, for each multiaxial criterion.

5. Results and comparison

The comparison of numerical and experimental results is shown in Fig. 4. Results are well within the ±300% band. It can be concluded that all specimens were loaded in the high-cycle fatigue regime, ranging from the estimated 1.5×10^6 of loading cycles for the shortest time-to-failure to the 7.9×10^6 loading cycles for the longest run.

Estimates are better for the maximum shear stress criterion and for the C-S criterion. Furthermore, both maximum shear and C-S criteria give results that are very close. It should be mentioned, that the C-S criterion performance depends on some additional material parameters, namely fully reversed bending and torsion fatigue limit. These results were not obtained for the material used but from literature for another aluminum alloy.

In general, all of the compared criteria give quite good estimate of time-to-failure for high-cycle fatigue of aluminum alloy component.
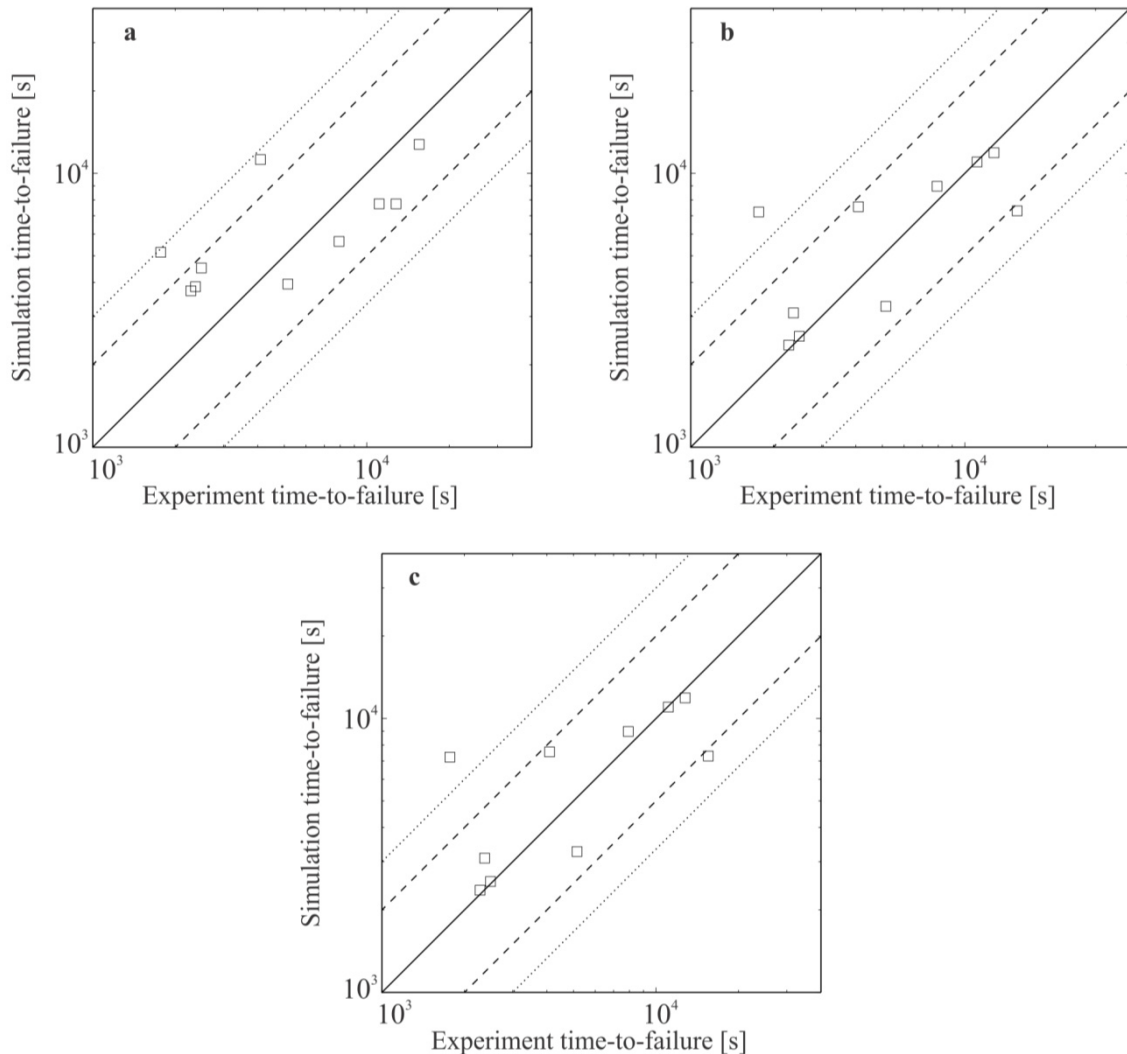
Fig. 4. (a) results for the maximum normal stress criterion; (b) results for the maximum shear stress criterion; (c) results for the C-S criterion.

6. Conclusions

Three different multiaxial criteria were experimentally compared side by side. 10 specially prepared specimen were driven to failure by simultaneous vibration originating from two sources in perpendicular directions, exciting two different modal shapes. Experiment was devised to thoroughly test multiaxial criteria in frequency domain by causing a complex multiaxial stress state inside material.

Results from experiment were compared with numerically calculated time-to-failure estimates. Analysis shows that tested criteria can be used quite reliably to produce a time-to-failure estimate. However, since these criteria can be derived using different assumptions, material parameters that give best results with one of the criteria may not be best suitable for another.

Estimate error originates from different parts of the analysis procedure. It is common to get large scatter in fatigue analyses but the source of error can also be elsewhere, *e.g.* in estimating modal frequencies and damping. With the aim of getting the best results possible, damping was measured for both excited modes at each sample and applied in numerical analysis for the respective sample. While the higher-frequency mode (kinematic excitation) was modelled well, the lower-frequency mode (force excitation) did deviate a bit from experimentally measured data. This could be a consequence of the way the force load is applied and could be one of the sources of error in the numerical model.

All in all, results show that the procedures applied in the course of this study can be used successfully for vibration fatigue analysis. Good estimates can be produced on structures that are excited with broad band loads near resonance and in multiple directions simultaneously.

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