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On T-duality transformations for the three-sphere

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Abstract

We study collective T-duality transformations along one, two and three directions of isometry for the three-sphere with H-flux. Our aim is to obtain new non-geometric backgrounds along lines similar to the example of the three-torus. However, the resulting backgrounds turn out to be geometric in nature. To perform the duality transformations, we develop a novel procedure for non-abelian T-duality, which follows a route different compared to the known literature, and which highlights the underlying structure from an alternative point of view.

1. Introduction

String theory is a theory of extended objects, which distinguishes it from ordinary quantum field theories of point particles. In particular, string theory contains closed strings, for which two types of excitations can be found in the spectrum: left-moving and right-moving modes. When a closed string is probing a background in which these two sectors behave in the same way, roughly speaking, both sectors "see" the same geometry. Hence, one can give a geometric interpretation of the background (at least in the large volume regime). However, in general the left- and right-moving sectors do not need to be the same, but can detect the background differently.

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In this case, no geometric description is available and the corresponding background is called non-geometric.

Usually, string theory is studied in the geometric regime for which a large variety of background spaces is known, however, in the non-geometric setting it is more difficult to obtain explicit examples. One of the strategies to construct backgrounds for the non-geometric case is to apply T-duality transformations to a known geometric space with non-vanishing NS–NS field strength H. The prime example for this approach [1] is the flat three-torus with $H \neq 0$, leading to

$$H_{xyz} \quad \overset{T_z}{\longleftrightarrow} \quad f_{xy}{}^{z} \quad \overset{T_y}{\longleftrightarrow} \quad Q_x{}^{yz} \quad \overset{T_x}{\longleftrightarrow} \quad R^{xyz}, \tag{1.1}$$

where this chain of T-duality transformations can be explained as follows.

- The starting point is a flat three-torus with non-trivial H-flux, on which one performs a first T-duality transformation. This results in a twisted torus with vanishing field strength, where the topology is characterized by a so-called geometric flux f [2,3].
- A second T-duality transformation leads to a background with a locally-geometric description, which is however globally non-geometric [4]. The latter means that when considering a covering of the torus by open neighborhoods, the transition functions on the overlap of the charts are not solely given by diffeomorphisms, and hence such a manifold cannot be described by Riemannian geometry. However, if in addition to diffeomorphisms one includes T-duality transformations as transition maps [5], this space can be globally defined. This construction is called a T-fold [6], and carries a so-called Q-flux [1]. The Q-flux is related to non-commutative features of this background, and non-commutativity in this context has been studied for instance in [7–16], and has been reviewed recently in [17].
- It has also been argued that formally a third T-duality transformation can be performed [1], but the resulting R-flux background is not even locally geometric and exhibits a non-associative structure. These spaces have been studied from a mathematical point of view in [18,19], later in [20], and have been reconsidered in a series of papers [21–23,11,24,25,13,15,26,27,16] more recently. A review from a mathematical perspective can be found in [28].

Another class of backgrounds showing non-geometric features are asymmetric orbifolds. In the context of non-geometry these have been studied for instance in [5,4,29–31,22,12,32], but they will not be the focus of this work.

There are a number of different approaches to investigate non-geometric backgrounds. In addition to the above-mentioned line of research, we note that non-geometric flux configurations have been studied from a doubled-geometry point of view in [6,33,34]. More recently, non-geometric backgrounds have been investigated via field redefinitions for the ten-dimensional supergravity action in [35–42], and have been analyzed from a world-sheet point of view for instance in [30,43–45]. Also, there exists an extensive literature for non-geometry in the context of double-field theory, for which we would like to refer the reader to the reviews [46,47].

The main purpose of the present paper is to study the chain of T-duality transformations shown in (1.1) not for the three-torus, but for the three-sphere with H-flux. One of the appealing features of the latter is that, in contrast to the torus, the string equations of motion can be solved when the flux is appropriately adjusted. The main question we want to answer is the following:

> When applying two T-duality transformations to the three-sphere with H-flux, does one obtain a non-geometric Q-flux background?

In order to address this point, a proper understanding of T-duality transformations is required. More concretely, since the isometry group of the three-sphere is non-abelian, we would like to be able to perform non-abelian T-duality transformations. These have been studied extensively in the past and some of the corresponding references are [48–55]; more recently non-abelian T-duality has been discussed for instance in [56–59]. However, in this paper we are going to approach non-abelian T-duality from a slightly different point of view, which highlights some of the structure important for our purposes. Let us furthermore mention that some of the examples we will be discussing are related to results known in the literature; nevertheless, our investigation here is in view of the chain of T-duality transformations shown in Eq. (1.1).

This paper is organized as follows: in Sections 2 and 3 we develop a novel formalism for studying collective, and more generally non-abelian, T-duality transformations. Our approach is based on [60], which for instance does not require a gauge-fixing procedure and which is not based on Wess–Zumino–Witten models. Furthermore, we are able to make explicit a particular constraint, shown in Eq. (2.10), which explains some of the structure found in the context of non-geometric backgrounds.

In Section 4 we apply collective T-duality transformations to the well-known example of the three-torus, thereby illustrating and checking our formalism. In Section 5 we study the chain of T-dualities (1.1) for the example of the three-sphere with H-flux; we find that after two T-duality transformations not a non-geometric but a geometric background is obtained.

In Section 6 we summarize and discuss our findings, and in Appendix A we collect results on collective (and non-abelian) T-duality transformations for the twisted three-torus with H-flux.

2. Preliminaries: non-linear sigma-model

We begin our discussion by reviewing the sigma-model action for the NS-NS sector of the closed string, which encodes the dynamics of a target-space metric G, an anti-symmetric Kalb–Ramond field B, and a dilaton ϕ. In the second part of this section, we study gaugings of this action, thereby generalizing some results of [61,62,50,63]

The action

The sigma model is usually defined on a compact two-dimensional manifold without boundaries, corresponding to the world-sheet of a closed string. However, in order to incorporate non-trivial field strengths $H = dB \neq 0$ for the Kalb–Ramond field B, it turns out to be convenient to work with a Wess–Zumino term, which is defined on a compact three-dimensional Euclidean world-sheet Σ with two-dimensional boundary $\partial\Sigma$. In this case, the sigma-model action takes the form

$$
\begin{aligned}
\mathcal{S} = &-\frac{1}{4\pi\alpha'}\int_{\partial\Sigma}\Big[G_{ij}\,dX^i\wedge\star dX^j + \alpha' R\phi\star 1\Big] \\
&-\frac{i}{2\pi\alpha'}\int_{\Sigma}\tfrac{1}{3!}H_{ijk}dX^i\wedge dX^j\wedge dX^k,
\end{aligned}
\tag{2.1}
$$

where the Hodge-star operator $\star$ is defined on $\partial\Sigma$, and the differential is understood as $dX^i(\sigma^\alpha) = \partial_\alpha X^i d\sigma^\alpha$ with $\{\sigma^\alpha\}$ coordinates on $\partial\Sigma$ and on Σ. The indices take values $i, j \in \{1, \dots, d\}$ with d the dimension of the target space, and R denotes the curvature scalar corresponding to the world-sheet metric $h_{\alpha\beta}$ on $\partial\Sigma$.

Note that the choice of three-manifold Σ for a given boundary $\partial\Sigma$ is not unique. However, if the field strength H is quantized, the path integral only depends on the data of the two-dimensional theory [64]. In the above conventions, the quantization condition reads

$$\frac{1}{2\pi\alpha'}\int_{\Sigma} H \in 2\pi\mathbb{Z}. \tag{2.2}$$

Symmetries of the world-sheet action

The classical world-sheet action (2.1) is invariant under the standard world-sheet diffeomorphisms, but it can also have pure target-space symmetries of the form

$$\delta_\epsilon X^i = \epsilon^\alpha k_\alpha^i(X) \tag{2.3}$$

for ϵ^α constant, provided that three requirements are satisfied. First, k_α with $\alpha = 1, \ldots, N$ are Killing vectors of the metric $G = G_{ij} dX^i \wedge \star dX^j$. Second, there exist one-forms v_α such that $\iota_{k_\alpha} H = dv_\alpha$ [61,62], and third, the Lie derivative of the dilaton ϕ in the direction of k_α vanishes. In terms of equations, these three conditions can be summarized as

$$\mathcal{L}_{k_\alpha} G = 0, \qquad \iota_{k_\alpha} H = dv_\alpha, \qquad \mathcal{L}_{k_\alpha}\phi = 0, \tag{2.4}$$

where the Lie derivative is given by $\mathcal{L}_k = d\circ\iota_k + \iota_k\circ d$. We also note that the isometry algebra generated by the Killing vectors is in general non-abelian with structure constants $f_{\alpha\beta}{}^\gamma$,

$$\big[k_\alpha, k_\beta\big]_{\mathrm{L}} = f_{\alpha\beta}{}^\gamma k_\gamma. \tag{2.5}$$

Gauging a symmetry

Let us now promote the global symmetries (2.3) to local ones, with ϵ^α depending on the world-sheet coordinates $\{\sigma^\alpha\}$. To do so, we introduce world-sheet gauge fields A^α and replace $dX^i \to dX^i + k_\alpha^i A^\alpha$ for the term involving the metric. For the Wess–Zumino term dX^i is kept unchanged, but additional scalar fields χ_α have to be introduced. The resulting gauged action reads

$$\begin{aligned}\widehat{\mathcal{S}} = &-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} \tfrac{1}{2} G_{ij}(dX^i + k_\alpha^i A^\alpha)\wedge\star(dX^j + k_\beta^j A^\beta) \\ &- \frac{i}{2\pi\alpha'}\int_{\Sigma} \tfrac{1}{3!} H_{ijk} dX^i\wedge dX^j\wedge dX^k \\ &- \frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\Big[(v_\alpha + d\chi_\alpha)\wedge A^\alpha + \tfrac{1}{2}\big(\iota_{k_{[\alpha}} v_{\underline{\beta}]} + f_{\alpha\beta}{}^\gamma\chi_\gamma\big)A^\alpha\wedge A^\beta\Big],\end{aligned} \tag{2.6}$$

where we omitted the dilaton term, which does not get modified. Now, given this action, there are two slightly different ways to implement the local symmetry transformations:

1. In the first approach, developed in detail in the two papers [61,62], the scalar fields χ_α do not play a role; in fact, they are not mentioned at all. In the present context, the local symmetry transformations then read as follows

$$\hat{\delta}_\epsilon X^i = \epsilon^\alpha k_\alpha^i, \qquad \hat{\delta}_\epsilon A^\alpha = -d\epsilon^\alpha - \epsilon^\beta A^\gamma f_{\beta\gamma}{}^\alpha, \qquad \hat{\delta}_\epsilon\chi = 0, \tag{2.7}$$

which have to be supplemented by the constraints[1]

$$\mathcal{L}_{k_{[\underline{\alpha}}} v_{\underline{\beta}]} = f_{\alpha\beta}{}^{\gamma} v_{\gamma}, \qquad \iota_{k_{(\bar{\alpha}}} v_{\bar{\beta})} = 0. \tag{2.8}$$

2. In the second approach, the scalar fields χ_α participate in the local symmetry transformations and cannot be left out. For the abelian case, this realization first appeared in [50] (see also [63]), but here we present the generalization to the non-abelian case. To our knowledge, this has not appeared in the literature before.[2] The local variations of the action (2.6) in the second approach read

$$\begin{aligned} \hat{\delta}_\epsilon X^i &= \epsilon^\alpha k_\alpha^i, \qquad & \hat{\delta}_\epsilon A^\alpha &= -d\epsilon^\alpha - \epsilon^\beta A^\gamma f_{\beta\gamma}{}^\alpha, \\ & & \hat{\delta}_\epsilon \chi_\alpha &= -\iota_{k_{(\bar{\alpha}}} v_{\bar{\beta})} \epsilon^\beta - f_{\alpha\beta}{}^\gamma \epsilon^\beta \chi_\gamma. \end{aligned} \tag{2.9}$$

   However, in this case the constraints are weaker as compared to (2.8). In particular, they read

$$\mathcal{L}_{k_{[\underline{\alpha}}} v_{\underline{\beta}]} = f_{\alpha\beta}{}^{\gamma} v_{\gamma}, \qquad \iota_{k_{[\underline{\alpha}}} f_{\underline{\beta\gamma}]}{}^{\delta} v_\delta = \frac{1}{3} \iota_{k_\alpha} \iota_{k_\beta} \iota_{k_\gamma} H. \tag{2.10}$$

Since the local variations (2.9) are in general less restrictive as compared to (2.7), in the following we focus on the second approach of implementing the symmetry transformations.

Global properties on the world-sheet

Let us now have a closer examination of the symmetry transformations (2.9), although we note that the same line of arguments applies to (2.7). When varying the action (2.6), besides trivial cancellations one is left with

$$\hat{\delta}_\epsilon \widehat{\mathcal{S}} = -\frac{i}{2\pi\alpha'} \int_{\partial\Sigma} d\epsilon^\alpha \wedge (v_\alpha + d\chi_\alpha) - \frac{i}{2\pi\alpha'} \int_{\Sigma} d\epsilon^\alpha \wedge dv_\alpha. \tag{2.11}$$

In order to show that this variation is vanishing, we assume that $d\epsilon^\alpha \wedge (v_\alpha + d\chi_\alpha)$ is globally defined on the world-sheet $\partial\Sigma$. We can then apply Stoke's theorem for the first term in (2.11), canceling the second term, and leading to $\hat{\delta}_\epsilon \widehat{\mathcal{S}} = 0$. This assumption follows from a more general requirement, which will be needed later on. In particular,

> We demand that the last line in the gauged action (2.6) is globally defined on the world-sheet $\partial\Sigma$, such that Stoke's theorem can be applied.

This condition imposes some constraints on the fields appearing in the gauged world-sheet action (2.6), however, a derivation of their global properties from first principles appears to be difficult. In the case of a single abelian isometry this can be done (see e.g. [65,49,50]), but for the general situation we were not able to perform a corresponding analysis. We thus leave the global properties of the world-sheet fields unspecified at this point.

Generalized geometry

Let us also give an interpretation of the constraints (2.10) in terms of generalized geometry. For the latter, the formal sum of a vector and a one-form is considered to be an element of

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1 Our convention is that the symmetrization and anti-symmetrization contains a factor of $1/n!$.

2 We thank F. Rennecke for collaboration on this part.

the generalized tangent space which, with M the target-space manifold, (locally) takes the form $TM \oplus T^*M$.[3] The algebraic structure of interest for us is the so-called H-twisted Courant bracket defined as follows

$$\big[k_\alpha + v_\alpha, k_\beta + v_\beta\big]^H_{\mathrm{C}} \big[k_\alpha, k_\beta\big]_{\mathrm{L}} + \mathcal{L}_{k_\alpha} v_\beta - \mathcal{L}_{k_\beta} v_\alpha - \tfrac{1}{2} d\big(\iota_{k_\alpha} v_\beta - \iota_{k_\beta} v_\alpha\big) - \iota_{k_\alpha}\iota_{k_\beta} H. \quad (2.12)$$

Using then the relations in (2.4) and (2.5), and defining the generalized vectors $K_\alpha = k_\alpha + v_\alpha$, the constraints (2.10) can be written as

$$\big[K_\alpha, K_\beta\big]^H_{\mathrm{C}} = f_{\alpha\beta}{}^\gamma K_\gamma, \qquad \mathrm{Nij}^H_{\mathrm{C}}\big(K_\alpha, K_\beta, K_\gamma\big) = 0. \quad (2.13)$$

The Nijenhuis tensor for the H-twisted Courant bracket is expressed in terms of the inner product $\langle K_\alpha, K_\beta\rangle = \frac{1}{2}(\iota_{k_\alpha} v_\beta + \iota_{k_\beta} v_\alpha)$ and reads [67]

$$\mathrm{Nij}^H_{\mathrm{C}}\big(K_\alpha, K_\beta, K_\gamma\big) = \big\langle \big[K_{[\underline{\alpha}}, K_{\underline{\beta}}\big]^H_{\mathrm{C}}, K_{\underline{\gamma}]}\big\rangle. \quad (2.14)$$

To summarize, the constraints (2.10) for gauging the non-linear sigma model (2.1) by isometries of the target-space manifold are 1) that the H-twisted Courant algebra of generalized vectors $K_\alpha = k_\alpha + v_\alpha$ closes, and 2) that the corresponding Nijenhuis tensor vanishes.

Global symmetries of the gauged action

We finally discuss global symmetries of the gauged action. Suppose that only a subgroup $H \subset G_{\mathrm{iso}}$ of the full isometry group G_{iso} has been gauged in (2.6). We denote the Killing vectors corresponding to the gauged isometry group H by $\{k_{\tilde\alpha}\}$, and we denote the remaining Killing vectors by $\{Z_\alpha\}$. For this setting we find that the gauged action (2.6) is invariant under global symmetries parametrized by Z_α if

$$\big[k_{\tilde\alpha}, Z_\beta\big]_{\mathrm{L}} = 0 \quad \text{and} \quad \mathcal{L}_{Z_\alpha} v_{\tilde\beta} = 0. \quad (2.15)$$

Thus, the gauging procedure can break some of the remaining global symmetries in the gauged action.

3. Collective T-duality

In this section, we study collective T-duality transformations in detail. These have been discussed mainly in the context of non-abelian T-duality, for which some of the main references are [48–55]. However, collective T-dualities also include the case of multiple abelian duality transformations, which have been investigated for instance in [63].

As compared to the older references, we approach non-abelian T-duality from a slightly different point of view, which for instance makes a particular constraint apparent, and which does not depend on a gauge-fixing procedure. In particular, when following Buscher's procedure [69–71] of gauging a sigma model and integrating out either the gauge fields or the Lagrange multiplies, it is known how to obtain the dual theory. However, to our knowledge, in the non-abelian case it is not known how to recover the original model without fixing a particular gauge. Here, we present a mechanism of how the original model can indeed be recovered, at least at the classical level, and we discuss the construction of the dual model in the formalism of [60].

[3] For an introduction to generalized geometry, we would like to refer the reader to the original papers [66] and [67], and for instance to [68] for a discussion in the physics literature.

3.1. Recovering the original model

Given the gauged action (2.6), one can ask how the original model can be recovered. Usually, this is achieved by using the equations of motion for the scalar fields χ_α, and for an abelian isometry algebra this has been discussed in [65,49,50] (see also [72–74] for previous as well as for related work on T-duality transformations), but for the non-abelian case we are not aware of results in the literature (without fixing a gauge).

Equations of motion for χ_α

We start by determining the equations of motion for the scalar fields χ_α. For the variation of the action (2.6) with respect to χ_α we obtain

$$\delta_\chi \widehat{\mathcal{S}} = +\frac{i}{2\pi\alpha'} \int_{\partial\Sigma} \delta\chi_\alpha \Big(dA^\alpha - \tfrac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma \Big), \tag{3.1}$$

from which we can read off the equations of motion as

$$0 = dA^\alpha - \tfrac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma . \tag{3.2}$$

Rewriting the action

We now want to recover the original theory (2.1) from the gauged version (2.6) by employing the equations of motion for χ_α. To this end, let us define

$$DX^i = dX^i + k^i_\alpha A^\alpha , \tag{3.3}$$

and use Stoke's theorem together with the equation of motion (3.2) and the constraints (2.10). After some manipulations we find

$$\begin{aligned} \widehat{\mathcal{S}} = &-\frac{1}{4\pi\alpha'} \int_{\partial\Sigma} \Big[G_{ij} DX^i \wedge \star DX^j + \alpha' R\phi \star 1 \Big] \\ &- \frac{i}{2\pi\alpha'} \int_{\Sigma} \tfrac{1}{3!} H_{ijk} DX^i \wedge DX^j \wedge DX^k . \end{aligned} \tag{3.4}$$

The structure of this action suggests that in order to obtain the original model, we should perform a field redefinition and identify DX^i with the differentials of new coordinates Y^i, that is $DX^i \to dY^i$. However, in general the one-forms DX^i are not closed, that is

$$d(DX^i) = \big(\partial_m k^i_\alpha\big) DX^m \wedge A^\alpha , \tag{3.5}$$

and therefore such a naive field redefinition would be inconsistent. An exception is the case of constant Killing-vector components $\partial_m k^i_\alpha = 0$, corresponding to an abelian isometry algebra, where the simple replacement $DX^i \to dY^i$ is indeed possible [65,49,50]. For the general case with non-constant Killing vectors, a more involved procedure has to be followed. Schematically, it consists of the following steps:

1. Perform a change of basis of the cotangent space, such that the exterior derivative d acting on $\{DX^a\}$ in the new basis forms a closed algebra with some structure constants $C_{bc}{}^a$

$$d(DX^a) = -\frac{1}{2} C_{bc}{}^a DX^b \wedge DX^c . \tag{3.6}$$

2. Identify the one-forms $\{DX^a\}$ with vielbeins $E^a = E^a{}_i dY^i$, expressed in terms of new local coordinates $\{Y^i\}$. Note that the vielbeins $\{E^a\}$ satisfy the algebra (3.6).
3. Perform an inverse change of basis and express the vielbeins $\{E^a\}$ in terms of the new differentials $\{dY^i\}$. The action (3.4) then takes the same form as the original model (2.1).

Note that these steps are simply the generalization from the abelian to the non-abelian case. In the following paragraphs, the technical details of this procedure will be explained; the reader not interested in those can safely skip to page 266.

Change of basis

Before we begin our discussion, let us impose one technical requirement: we demand that the target-space manifold M under consideration has been split as

$$M = M_0 \times M_1, \tag{3.7}$$

where the Killing vectors $\{k_\alpha\}$ appearing in the gauged action (2.6) form a basis of the tangent space of M_0, but are not contained in TM_1. Note that the separation (3.7) corresponds to choosing so-called *adapted coordinates*. Physically, it means that we perform a T-duality transformation only on M_0 and leave M_1 unchanged. In the remainder of this section, we only focus on M_0.

In order to perform the field redefinition for a non-abelian isometry algebra, let us introduce a new basis for the tangent and co-tangent space by considering invertible matrices $e^a{}_i = e^a{}_i(X)$ with $a, i = 1, \ldots, d_0$, and d_0 the dimension of M_0. These matrices do not need to diagonalize the metric, but in the following we nevertheless refer to them as a vielbein basis. We then define

$$e^a = e^a{}_i dX^i, \qquad e_a = e_a{}^i \partial_i, \tag{3.8}$$

where $e_a{}^i \equiv (e^{-1})_a{}^i$. The structure constants for the dual basis of vector fields $\{e_a\}$ will be denoted by $C_{ab}{}^c$, and they appear in the commutator

$$[e_a, e_b]_{\mathrm{L}} = C_{ab}{}^c e_c. \tag{3.9}$$

Let us note that by requiring a torsion-free connection, we see that the one-forms $\{e^a\}$ satisfy the following algebra with respect to the exterior derivative

$$de^a = -\frac{1}{2} C_{bc}{}^a e^b \wedge e^c. \tag{3.10}$$

We also mention that in regard to this basis the standard notation will be employed, that is indices are changed from $\{i, j, k, \ldots\}$ to $\{a, b, c, \ldots\}$ by appropriately contracting with $e^a{}_i$ or $e^i{}_a$. Now, the main requirement for the vector fields $\{e_a\}$ defined in (3.8) is that they should commute with the Killing vector fields $\{k_\alpha\}$, that is

$$\big[k_\alpha, e_a\big]_{\mathrm{L}} = 0. \tag{3.11}$$

It is not clear whether a basis of vielbeins satisfying this condition can always be found, however, in Section 5 and in Appendix A we give two explicit examples where this condition is indeed satisfied.

Coordinate dependence of the metric, H-flux and dilaton

For the new basis introduced in the previous paragraph we can determine the exterior derivative of the one-forms $DX^a = e^a{}_i DX^i$. Employing the equation of motion shown in (3.2), the algebra (2.5), and the condition (3.11), we find that the one-forms $\{DX^a\}$ form a closed algebra under d

$$d(DX^a) = -\frac{1}{2}C_{bc}{}^a DX^b \wedge DX^c. \tag{3.12}$$

Furthermore, using the condition (3.11) together with (2.4) and $dH = 0$, we observe that the components of the metric and H-flux in the vielbein basis satisfy

$$k_\alpha^m \partial_m G_{ab} = 0, \qquad k_\alpha^m \partial_m H_{abc} = 0. \tag{3.13}$$

Since the Killing vectors $\{k_\alpha\}$ span TM_0, Eqs. (3.13) imply that these components are constant on M_0. Including then the condition $k_\alpha^m \partial_m \phi$ following from (2.4), in formulas we have that on M_0

$$G_{ab} = \text{const.}, \qquad H_{abc} = \text{const.}, \qquad \phi = \text{const.} \tag{3.14}$$

Recovering the original model

We are now in the position to show how the original action (2.1) can be recovered from the gauged action (2.6). To do so, we first define the one-forms

$$E^a = DX^a = e^a + k_\alpha^a A^\alpha, \tag{3.15}$$

which by definition satisfy the algebra shown in (3.12), that is

$$dE^a = -\frac{1}{2}C_{bc}{}^a E^b \wedge E^c. \tag{3.16}$$

We observe that this is the same algebra as in (3.10) which is obeyed by the original vielbein one-forms $\{e^a\}$. It is therefore clear that a local basis $\{dY^i\}$ of the cotangent space T^*M_0 exists, for which we can write

$$E^a = E^a{}_i dY^i, \tag{3.17}$$

with $\{E^a{}_i\}$ invertible matrices. Now, since the dilaton and the components of the metric and H-flux are constant in the vielbein basis, cf. (3.14), we can rewrite for instance the metric term in the action (3.4) in the following way

$$G_{ij}DX^i \wedge \star DX^j = G_{ab}DX^a \wedge \star DX^b = G_{ab}E^a \wedge \star E^b = G_{ij}dY^i \wedge \star dY^j, \tag{3.18}$$

where in the last step we performed the inverse change of basis. An analysis similar to that of the metric can be performed for the H-field and dilaton term, so that after the above field redefinition we recover from (3.4) the original action

$$\begin{aligned}\mathcal{S} = &-\frac{1}{4\pi\alpha'}\int_{\partial\Sigma}\Big[G_{ij}dY^i \wedge \star dY^j + \alpha' R\phi \star 1\Big] \\ &-\frac{i}{2\pi\alpha'}\int_{\Sigma}\tfrac{1}{3!}H_{ijk}dY^i \wedge dY^j \wedge dY^k.\end{aligned} \tag{3.19}$$

This action may take a different form in the local coordinates $\{Y^i\}$ as compared to the action in the coordinates $\{X^i\}$. However, since both of these actions can be expressed in a vielbein basis with the same structure constants, shown in (3.10) and (3.16), both choices are related by a change of basis.

3.2. Obtaining the dual model

Let us now turn to the dual model. As usual, it is obtained by using the equations of motion for the gauge fields A^α in the gauged action (2.6). This part of the duality is rather well-understood; here, we extend the formalism of [60] from the abelian to the non-abelian case.

Equations of motion for A^α

We begin by deriving the equations of motion for the gauge fields A^α from the gauged action (2.6). Setting to zero the variation with respect to the gauge fields and solving for A^α, we find

$$A^\alpha = -\Big(\big[\mathcal{G} - \mathcal{D}\mathcal{G}^{-1}\mathcal{D}\big]^{-1}\Big)^{\alpha\beta}\Big(\mathbb{1} + i\star\mathcal{D}\mathcal{G}^{-1}\Big)_\beta{}^\gamma\big(k + i\star\xi\big)_\gamma, \tag{3.20}$$

where we remind the reader that $\alpha, \beta, \gamma = 1, \ldots, N$ label the isometries which have been gauged. In the expression shown in (3.20), we have employed the notation

$$\begin{aligned} \mathcal{G}_{\alpha\beta} &= k^i_\alpha G_{ij} k^j_\beta, & \xi_\alpha &= d\chi_\alpha + v_\alpha, \\ \mathcal{D}_{\alpha\beta} &= \iota_{k_{[\underline{\alpha}}} v_{\underline{\beta}]} + f_{\alpha\beta}{}^\gamma \chi_\gamma, & k_\alpha &= k^i_\alpha G_{ij} dX^j, \end{aligned} \tag{3.21}$$

and have assumed the matrix $\mathcal{G}_{\alpha\beta}$ to be invertible. In the case of a single Killing vector this corresponds to the usual requirement that $|k|^2 \neq 0$, and in formulas it reads

$$\det \mathcal{G} \neq 0. \tag{3.22}$$

Finally, for later purposes, let us define the symmetric and invertible matrix

$$\mathcal{M} = \mathcal{G} - \mathcal{D}\mathcal{G}^{-1}\mathcal{D}. \tag{3.23}$$

Enlarged target-space

In order to obtain the dual model, we follow the procedure which has been described in detail in [60]. To do so, we first use the solution (3.20) to equations of motion for the gauge fields in the gauged action (2.6). We then obtain

$$\check{\mathcal{S}} = -\frac{1}{4\pi\alpha'}\int_{\partial\Sigma}\Big(\check{G} + \alpha' R\phi\star 1\Big) - \frac{i}{2\pi\alpha'}\int_{\Sigma}\check{H}, \tag{3.24}$$

where the tensor fields $\check{G}$ and $\check{H}$ are given by

$$\begin{aligned} \check{G} &= G + \begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} -\mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} \\ +\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} & +\mathcal{M}^{-1} \end{pmatrix} \wedge\star \begin{pmatrix} k \\ \xi \end{pmatrix}, \\ \check{H} &= H + \tfrac{1}{2} d\left[\begin{pmatrix} k \\ \xi \end{pmatrix}^T \begin{pmatrix} +\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} & +\mathcal{M}^{-1} \\ -\mathcal{M}^{-1} & -\mathcal{M}^{-1}\mathcal{D}\mathcal{G}^{-1} \end{pmatrix} \wedge \begin{pmatrix} k \\ \xi \end{pmatrix}\right]. \end{aligned} \tag{3.25}$$

Here and in the following, matrix multiplication for the indices $\alpha, \beta, \ldots$ is understood. We observe that these two tensor fields can be interpreted as being defined on an enlarged $(d_0 + N)$-dimensional target space, which is locally described by the coordinates $\{X^i, \chi_\alpha\}$ with $i = 1, \ldots, d_0$ and $\alpha = 1, \ldots, N$. For the enlarged cotangent space, a convenient basis of one-forms is given by $\{dX^i, \xi_\alpha\}$.

As observed in [60] for the abelian case, the component matrix $\check{G}_{IJ}$ of the enlarged metric tensor has null-eigenvectors. Indeed, consider the following vector in the basis dual to $\{dX^i, \xi_\beta\}$

$$\check{n}_\alpha = \begin{pmatrix} k^i_\alpha \\ \mathcal{D}_{\alpha\beta} \end{pmatrix}, \tag{3.26}$$

for which we find after a somewhat lengthy computation that

$$\iota_{\check{n}_\alpha} \check{G} = 0, \qquad \iota_{\check{n}_\alpha} \check{H} = 0. \tag{3.27}$$

Note that the first of these conditions implies that the component matrix $\check{G}_{IJ}$ has N eigenvectors with vanishing eigenvalue. We also mention that the vectors (3.26) are Killing vectors for the enlarged metric $\check{G}$ and enlarged field strength $\check{H}$. In particular, including the result for the dilaton, we find

$$\check{\mathcal{L}}_{\check{n}_\alpha} \check{G} = 0, \qquad \check{\mathcal{L}}_{\check{n}_\alpha} \check{H} = 0, \qquad \check{\mathcal{L}}_{\check{n}_\alpha} \phi = 0. \tag{3.28}$$

Obtaining the dual model

In order to obtain the dual model from the enlarged target space, we proceed as in the abelian case. We do not repeat the general discussion of [60] for the non-abelian case here, but only want to outline the main idea.

- First, we note that since the metric $\check{G}$ has N eigenvectors with vanishing eigenvalue, we can perform a change of basis such that

$$\check{G}_{IJ} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \check{G}_{\alpha\beta} \end{array} \right), \tag{3.29}$$

  with I, J collectively labeling $\{dX^i, \xi_\alpha\}$. As can be verified, the same change of basis results in vanishing components of the field strength $\check{H}$ along one or more dX^i directions, that is

$$\check{H}_{iJK} = 0. \tag{3.30}$$

  This means, after the change of basis, in the action (3.24) no one-forms dX^i with $i = 1, \ldots, d_0$ are appearing.
- Second, the components $\check{G}_{\alpha\beta}$ and $H_{\alpha\beta\gamma}$ as well as the dilaton ϕ may still depend on the coordinates X^i. However, due to the isometries (3.28) of the enlarged target space, we may go to a convenient but fixed point in the X^i-space. Hence, also the components do not depend on X^i and we have arrived at the dual model.

Note that here we have only outlined the main idea of how the dependence on $\{X^i\}$ and $\{dX^i\}$ in the action (3.24) vanishes. However, in the next two sections we discuss explicit examples for this procedure.

Remark on isometries of the dual background

It is well known that non-abelian T-duality transformations can in general not be inverted. We do not want to address this question in detail in this paper, but only consider the case when part of the isometry group has been gauged in the action.

Let us therefore recall our discussion from page 262 about the remaining global symmetries after the gauging procedure. There, we saw that only those Killing vectors which satisfy (2.15) survive as global isometries in the gauged theory, in addition to the gauged Killing vectors. Hence, in general the isometry group for the dual background is reduced.

4. Examples I: three-torus

We now want to illustrate the formalism introduced in the last section with the example of the three-torus with H-flux. After performing one T-duality transformation, one arrives at the so-called twisted torus with vanishing field strength, for which the topology is characterized by a geometric flux f [2,3]. Two successive T-dualities result in a locally-geometric but globally non-geometric background which carries a Q-flux [4,1], and which is also called a T-fold [6]. Finally, three successive T-dualities have been argued to give a locally non-geometric background carrying so-called R-flux [18,1,20].

In this section, we re-derive these results not using successive but *collective* T-duality transformations. In Section 5, we then turn to the example of the three-sphere, and in Appendix A the results for the twisted three-torus with H-flux have been summarized.

Setup

Let us start by introducing some notation. We consider a flat three-torus with non-trivial field strength H. The components of the metric tensor in the standard basis of one-forms $\{dX^1, dX^2, dX^3\}$ are chosen to be of the form

$$G_{ij} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \tag{4.1}$$

and the topology is characterized by the identifications $X^i \simeq X^i + \ell_s$ for $i = 1, 2, 3$. The components of the field strength $H = dB$ of the Kalb–Ramond field are taken to be constant, which, keeping in mind the quantization condition (2.2), leads to

$$H = h\, dX^1 \wedge dX^2 \wedge dX^3, \qquad h \in \ell_s^{-1}\mathbb{Z}. \tag{4.2}$$

The Killing vectors for this configuration in the basis $\{\partial_1, \partial_2, \partial_3\}$, dual to the above one-forms, can be chosen as

$$k_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad k_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad k_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4.3}$$

which satisfy an abelian algebra, that is

$$[k_\alpha, k_\beta]_{\mathrm{L}} = 0. \tag{4.4}$$

The one-forms v_α corresponding to (4.3) are defined through Eq. (2.4), and up to exact terms they can be written as

$$\begin{aligned} v_1 &= h\alpha_1 X^2 dX^3 - h\alpha_2 X^3 dX^2, \qquad && \alpha_1 + \alpha_2 = 1, \\ v_2 &= h\beta_1 X^3 dX^1 - h\beta_2 X^1 dX^3, && \beta_1 + \beta_2 = 1, \\ v_3 &= h\gamma_1 X^1 dX^2 - h\gamma_2 X^2 dX^1, && \gamma_1 + \gamma_2 = 1. \end{aligned} \tag{4.5}$$

Note that here α_m, β_m and γ_m are constants which parametrize a gauge freedom. In general these one-forms are not globally defined on the torus, however, due to the equivalence $v_\alpha \simeq v_\alpha + d\Lambda$

for a function Λ, we can define the v_α on local charts and cover the torus consistently (see for instance [63] for more details).

Constraints on gauging the sigma model

As we discussed in Section 2, in the presence of a non-vanishing field strength H there are restrictions on which isometries of the sigma model can be gauged, cf. Eq. (2.10). In the present situation, these imply

$$\iota_{k_\alpha}\iota_{k_\beta}\iota_{k_\gamma}H = 0, \tag{4.6}$$

so that for the example of the three-torus we can distinguish the following cases:

- For vanishing H-flux, one, two, or three isometries can be gauged. These situations are well known in the literature, and so in Section 4.3 we discuss briefly only the case of gauging all three isometries.
- For non-vanishing H-flux we deduce from (4.6) that at most two of the three isometries can be gauged. The gauging of only a single isometry is well known and will be reviewed in Section 4.1. The situation of gauging two isometries will be discussed in Section 4.2.

4.1. One T-duality

We begin by considering one T-duality transformation for the three-torus with non-vanishing H-flux. In the present formalism, this has been analyzed in detail in [60] and so we will be brief here.

Gauged action and original model

For simplicity, let us chose the isometry direction along which we perform the T-duality to correspond to the Killing vector $k_1 = \partial_1$. From the H-flux (4.2) we deduce the following one-form

$$v = h\alpha X^2 dX^3 - h(1-\alpha)X^3 dX^2, \tag{4.7}$$

with $\alpha \in \mathbb{R}$. The gauged action is obtained from the general expression shown in Eq. (2.6) and reads (with the dilaton term omitted)

$$\begin{aligned}\widehat{\mathcal{S}} = &-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\Big[\tfrac{1}{2}R_1^2(dX^1+A)\wedge\star(dX^1+A) + \sum_{\mathrm{i}=2}^{3}\tfrac{1}{2}R_{\mathrm{i}}^2 dX^{\mathrm{i}}\wedge\star dX^{\mathrm{i}}\Big] \\ &-\frac{i}{2\pi\alpha'}\int_{\Sigma} h\, dX^1\wedge dX^2\wedge dX^3 \\ &-\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}(v+d\chi)\wedge A.\end{aligned} \tag{4.8}$$

The ungauged version is recovered by using the equation of motion $dA = 0$ as well as Stoke's theorem for the last term, which agrees with the general form (3.4). Defining then $dY^1 = dX^1 + A$, $dY^2 = dX^2$ and $dY^3 = dX^3$, we arrive at the original action.

Dual model

In order to obtain the dual theory, we first recall the general formulas shown in Eq. (3.21). For a non-vanishing field strength and one Killing vector we have

$$\begin{aligned} \mathcal{G} &= R_1^2, & \xi &= d\chi + v, \\ \mathcal{D} &= 0, & k &= R_1^2\, dX^1, \end{aligned} \tag{4.9}$$

from which we determine, using (3.25), the metric and field strength of the enlarged target space as follows

$$\begin{aligned} \check{G} &= G - R_1^2\, dX^1 \wedge \star dX^1 + \frac{1}{R_1^2}\xi \wedge \star\xi \\ &= \frac{1}{R_1^2}\xi \wedge \star\xi + R_2^2\, dX^2 \wedge \star dX^2 + R_3^2\, dX^3 \wedge \star dX^3, \end{aligned} \tag{4.10}$$

$$\check{H} = H + d\Big[dX^1 \wedge \xi\Big] = 0. \tag{4.11}$$

Employing these expressions in the action (3.24), we have obtained the dual model (up to a transformation of the dilaton). Note that the one-form ξ satisfies

$$d\xi = h\, dX^2 \wedge dX^3. \tag{4.12}$$

Hence, as expected, (4.10) and (4.11), together with (4.12), describe a twisted three-torus with vanishing field strength $\check{H} = 0$ [2,3].

4.2. Two T-dualities

Next, we turn to the case of two collective T-dualities for a three-torus with non-vanishing H-flux, and T-dualize along the directions of the Killing vectors $k_1 = \partial_1$ and $k_2 = \partial_2$.

Gauged action and original model

In this setting, the one-forms v_1 and v_2 corresponding to k_1 and k_2 are shown in (4.5). However, due to the first condition in (2.10), here reading $\mathcal{L}_{k_{[\underline{\alpha}}} v_{\underline{\beta}]} = 0$, we find a restriction on the constants α_m and β_m in (4.5). In particular, for the one-forms v_α we obtain

$$\begin{aligned} v_1 &= h\alpha X^2 dX^3 - h(1-\alpha) X^3 dX^2, \\ v_2 &= h(1+\alpha) X^3 dX^1 + h\alpha X^1 dX^3, \end{aligned} \tag{4.13}$$

with $\alpha \in \mathbb{R}$. Given these expressions, we can write down the gauged action following from (2.6) as

$$\begin{aligned} \widehat{\mathcal{S}} = &-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\Big[\sum_{\mathsf{i}=1}^{2}\tfrac{1}{2}R_{\mathsf{i}}^2\big(dX^{\mathsf{i}} + A^{\mathsf{i}}\big)\wedge\star\big(dX^{\mathsf{i}} + A^{\mathsf{i}}\big) + \tfrac{1}{2}R_3^2 dX^3 \wedge \star dX^3\Big] \\ &- \frac{i}{2\pi\alpha'}\int_{\Sigma} h\, dX^1 \wedge dX^2 \wedge dX^3 \\ &- \frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\Big[\sum_{\mathsf{i}=1}^{2}(v_{\mathsf{i}} + d\chi_{\mathsf{i}}) \wedge A^{\mathsf{i}} + hX^3 A^1 \wedge A^2\Big]. \end{aligned} \tag{4.14}$$

The original ungauged model is again obtained via the procedure discussed in Section 3.1, which in the present case is similar to the example of one T-duality.

Dual model

In order to determine the dual model, let us recall Eq. (3.21) and evaluate the there-mentioned quantities. We find

$$\begin{aligned} \mathcal{G}_{\alpha\beta} &= \begin{pmatrix} R_1^2 & 0 \\ 0 & R_2^2 \end{pmatrix}, & \xi_\alpha &= \begin{pmatrix} d\chi_1 + v_1 \\ d\chi_2 + v_2 \end{pmatrix}, \\ \mathcal{D}_{\alpha\beta} &= \begin{pmatrix} 0 & +hX^3 \\ -hX^3 & 0 \end{pmatrix}, & k_\alpha &= \begin{pmatrix} R_1^2 dX^1 \\ R_2^2 dX^2 \end{pmatrix}, \end{aligned} \tag{4.15}$$

and the matrix $\mathcal{M}_{\alpha\beta}$ defined in (3.23) takes the following form

$$\mathcal{M}_{\alpha\beta} = \begin{pmatrix} R_1^2 + \left[\frac{hX^3}{R_2}\right]^2 & 0 \\ 0 & R_2^2 + \left[\frac{hX^3}{R_1}\right]^2 \end{pmatrix}. \tag{4.16}$$

The general formula for the metric of the enlarged target-space was given in Eq. (3.25), which in the basis $\{dX^i, \xi_\alpha\}$ becomes

$$\check{G}_{IJ} = \frac{1}{\rho} \left(\begin{array}{ccc|cc} [R_1 hX^3]^2 & 0 & 0 & 0 & -R_1^2 hX^3 \\ 0 & [R_2 hX^3]^2 & 0 & +R_2^2 hX^3 & 0 \\ 0 & 0 & \rho R_3^2 & 0 & 0 \\ \hline 0 & +R_2^2 hX^3 & 0 & R_2^2 & 0 \\ -R_1^2 hX^3 & 0 & 0 & 0 & R_1^2 \end{array} \right), \tag{4.17}$$

where for notational convenience we have defined the quantity

$$\rho = R_1^2 R_2^2 + \left[hX^3\right]^2. \tag{4.18}$$

Next, recall that the matrix (4.17) has eigenvectors with vanishing eigenvalue; the eigenvectors can therefore be used to perform a change of coordinates. Let us consider $\check{\mathsf{G}}_{AB} = (\mathcal{T}^T \check{G} \mathcal{T})_{AB}$, where the matrix $\mathcal{T}$ is given by

$$\mathcal{T}^I{}_A = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & -hX^3 & 0 & 1 & 0 \\ +hX^3 & 0 & 0 & 0 & 1 \end{array} \right). \tag{4.19}$$

Explicitly evaluating the change of basis we find

$$\check{\mathsf{G}}_{AB} = (\mathcal{T}^T \check{G} \mathcal{T})_{AB} = \frac{1}{\rho} \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho R_3^2 & 0 & 0 \\ \hline 0 & 0 & 0 & R_2^2 & 0 \\ 0 & 0 & 0 & 0 & R_1^2 \end{array} \right). \tag{4.20}$$

A similar analysis can be carried out for the field strength: from (3.25) we determine an expression for $\check{H}_{IJK}$ and we perform the above change of coordinates, that is

$$\check{\mathsf{H}}_{ABC} = \check{H}_{IJK}\mathcal{T}^{I}{}_{A}\mathcal{T}^{J}{}_{B}\mathcal{T}^{K}{}_{C}. \tag{4.21}$$

We then find that the only non-vanishing resulting component is

$$\check{\mathsf{H}}_{3\xi_1\xi_2} = -h\frac{R_1^2 R_2^2 - \left[hX^3\right]^2}{\rho^2}. \tag{4.22}$$

Finally, we have to determine how the basis one-forms $\{dX^i\}$ and $\{\xi_\alpha\}$ transform under the change of basis given by (4.19). A short computation leads to

$$\mathsf{e} = \mathcal{T}^{-1}\begin{pmatrix} dX^1 \\ dX^2 \\ dX^3 \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} dX^1 \\ dX^2 \\ dX^3 \\ d\left(\chi_1 + h\alpha X^2 X^3\right) \\ d\left(\chi_2 + h\alpha X^1 X^3\right) \end{pmatrix}, \tag{4.23}$$

where the free parameter $\alpha \in \mathbb{R}$ was defined in (4.13). For the dual the model we therefore have the following metric and field strength

$$\begin{aligned} \check{\mathsf{G}} &= \frac{1}{\rho}\Big[R_1^2\, d\tilde{X}^1 \wedge \star d\tilde{X}^1 + R_2^2\, d\tilde{X}^2 \wedge \star d\tilde{X}^2\Big] + R_3^2\, dX^3 \wedge \star dX^3, \\ \check{\mathsf{H}} &= -h\frac{R_1^2 R_2^2 - \left[hX^3\right]^2}{\rho^2}\, d\tilde{X}^1 \wedge d\tilde{X}^2 \wedge dX^3, \end{aligned} \tag{4.24}$$

with new local coordinates $\tilde{X}^1 = \chi_1 + h\alpha X^2 X^3$ and $\tilde{X}^2 = \chi_2 + h\alpha X^1 X^3$. We also remind the reader that the quantity ρ was defined in Eq. (4.18), and we observe that the metric and field strength shown in (4.24) describe the well-known torus-example of a Q-flux background [4,6].

4.3. *Three T-dualities*

We finally consider three collective T-dualities for the three-torus. As explained below Eq. (4.6), in this case the H-flux has to vanish and so the one-forms v_α can be chosen to be zero. The gauged action (2.6) becomes

$$\check{\mathcal{S}} = -\frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\sum_{\mathsf{i}=1}^{3}\Big[\tfrac{1}{2}R_{\mathsf{i}}^2\left(dX^{\mathsf{i}} + A^{\mathsf{i}}\right)\wedge\star\left(dX^{\mathsf{i}} + A^{\mathsf{i}}\right) + i\,d\chi_{\mathsf{i}}\wedge A^{\mathsf{i}}\Big], \tag{4.25}$$

and the ungauged action is recovered from (4.25) by noting that the equations of motion for χ_α read $dA^\alpha = 0$. Applying then Stoke's theorem we observe that the last term in (4.25) vanishes. For the first terms we define new one-forms $dY^I = dX^I + A^I$, and therefore recover the original model.

Dual model

To obtain the dual model, we recall our discussion from Section 3.2. For the present setting, the quantities defined in Eq. (3.21) take the following form

$$\begin{aligned} \mathcal{G}_{\alpha\beta} &= R_\alpha^2\,\delta_{\alpha\beta}, & \xi_\alpha &= d\chi_\alpha, \\ \mathcal{D}_{\alpha\beta} &= 0, & k_\alpha &= R_\alpha^2\,\delta_{\alpha i}\, dX^i. \end{aligned} \tag{4.26}$$

Using these expressions, we can determine the metric of the enlarged target-space from (3.25) as

$$\check{G} = G + \begin{pmatrix} k \\ d\chi \end{pmatrix}^T \begin{pmatrix} -\mathcal{G}^{-1} & 0 \\ 0 & +\mathcal{G}^{-1} \end{pmatrix} \wedge \star \begin{pmatrix} k \\ d\chi \end{pmatrix} = \sum_{\mathsf{i}=1}^{3} \frac{1}{R_{\mathsf{i}}^2} d\chi_{\mathsf{i}} \wedge \star d\chi_{\mathsf{i}}, \tag{4.27}$$

and for the field strength we find

$$\check{H} = \tfrac{1}{2} d\left[\begin{pmatrix} k \\ d\chi \end{pmatrix}^T \begin{pmatrix} 0 & +\mathcal{G}^{-1} \\ -\mathcal{G}^{-1} & 0 \end{pmatrix} \wedge \begin{pmatrix} k \\ d\chi \end{pmatrix} \right] = \sum_{\mathsf{i}=1}^{3} d\Big[dX^{\mathsf{i}} \wedge d\chi_{\mathsf{i}} \Big] = 0. \tag{4.28}$$

Using these two results in the action (3.24), we see that it reduces to the dual theory specified by

$$\check{G}^{\alpha\beta} = \begin{pmatrix} \frac{1}{R_1^2} & 0 & 0 \\ 0 & \frac{1}{R_2^2} & 0 \\ 0 & 0 & \frac{1}{R_3^2} \end{pmatrix}, \qquad \check{H} = 0, \tag{4.29}$$

where for the metric tensor the basis $\{d\chi_\alpha\}$ with $\alpha = 1, 2, 3$ has been employed. Hence, as expected, we find that a collective T-duality along all three directions of a three-torus (without H-flux) inverts the radii.

4.4. Summary

To close our discussion of collective T-duality transformations for the three-torus with H-flux, let us briefly summarize our results. First, we have seen that the procedure of performing *collective* T-duality transformations introduced in Section 3 leads to the known results in the case of the torus. Our discussion in this section therefore serves as a check of that formalism. Second, the examples we have studied can be summarized as follows:

- In the case of vanishing field strength $H = 0$, a T-duality transformation along any of the Killing vectors in (4.3) inverts the corresponding component in the metric. For three collective T-dualities we have discussed this situation in Section 4.3.
- For non-vanishing field strengths $H \neq 0$, one T-duality leads to the so-called twisted torus. In the present formalism, this has been discussed in detail in [60], whose main results we reviewed in Section 4.1.
- The case of two *collective* T-dualities for $H \neq 0$ has been discussed in Section 4.2. As expected, we arrive at a Q-flux background.
- Finally, due to the requirement (4.6), we have seen that for a non-vanishing H-flux three collective T-dualities cannot be performed within the formalism presented in Section 3.

Let us also mention that in Appendix A, an analysis similar to the three-torus has been performed for the twisted three-torus with H-flux. In this case, different variants of a *twisted T-fold* are obtained.

5. Examples II: three-sphere

In this section, we study collective T-duality transformations for the three-sphere with H-flux. Some of the results obtained below have partially appeared already in the literature; but here we discuss them in a unified manner similar to the example of the three-torus. Furthermore, we note that in contrast to the three-torus with $H \neq 0$, the three-sphere with appropriately adjusted H-flux solves the string equations of motion.

Setup

Let us begin by specifying the setting we will be working in. For the three-sphere, we choose the round metric in terms of Hopf coordinates which takes the following form

$$ds^2 = R^2\Big[\sin^2\eta\,(d\zeta_1)^2 + \cos^2\eta\,(d\zeta_2)^2 + (d\eta)^2\Big], \tag{5.1}$$

where $\zeta_{1,2} = 0\ldots 2\pi$ and $\eta = 0\ldots\pi/2$, and where R denotes the radius of the three-sphere. We also consider a non-trivial field strength for the Kalb–Ramond field B,

$$H = \frac{h}{2\pi^2}\sin\eta\cos\eta\, d\zeta_1\wedge d\zeta_2\wedge d\eta, \tag{5.2}$$

for which the quantization condition shown in Eq. (2.2) implies that $h\in\mathbb{Z}$. Let us mention that this model solves the string equations of motion for a constant dilaton ϕ_0, hence it is a proper string theory model, if the field strength H and the radius R of the three-sphere are related as

$$R^2 = \frac{h}{4\pi^2}. \tag{5.3}$$

Killing vectors

The isometry group of the three-sphere S^3 is $O(4)$, and so there are six linearly independent Killing vectors. Employing the basis of vector fields $\{\partial_{\zeta_1}, \partial_{\zeta_2}, \partial_\eta\}$, the Killing vectors for the metric (5.1) can be expressed in the following way

$$\begin{aligned}
k_1 &= \frac{1}{2}\begin{pmatrix} +1 \\ -1 \\ 0\end{pmatrix}, & \tilde{k}_1 &= \frac{1}{2}\begin{pmatrix} +1 \\ +1 \\ 0\end{pmatrix},\\
k_2 &= \frac{1}{2}\begin{pmatrix} -\sin(\zeta_1-\zeta_2)\cot\eta \\ -\sin(\zeta_1-\zeta_2)\tan\eta \\ \cos(\zeta_1-\zeta_2)\end{pmatrix}, & \tilde{k}_2 &= \frac{1}{2}\begin{pmatrix} +\sin(\zeta_1+\zeta_2)\cot\eta \\ -\sin(\zeta_1+\zeta_2)\tan\eta \\ -\cos(\zeta_1+\zeta_2)\end{pmatrix},\\
k_3 &= \frac{1}{2}\begin{pmatrix} -\cos(\zeta_1-\zeta_2)\cot\eta \\ -\cos(\zeta_1-\zeta_2)\tan\eta \\ -\sin(\zeta_1-\zeta_2)\end{pmatrix}, & \tilde{k}_3 &= \frac{1}{2}\begin{pmatrix} +\cos(\zeta_1+\zeta_2)\cot\eta \\ -\cos(\zeta_1+\zeta_2)\tan\eta \\ +\sin(\zeta_1+\zeta_2)\end{pmatrix}.
\end{aligned} \tag{5.4}$$

Next, we note that $\mathfrak{so}(4)\cong\mathfrak{su}(2)\times\mathfrak{su}(2)$, which implies that the above Killing vectors satisfy the following algebra (with $\alpha,\beta,\gamma\in\{1,2,3\}$ and $\epsilon_{\alpha\beta\gamma}$ the Levi-Civita symbol)

$$[k_\alpha, k_\beta]_{\mathrm{L}} = \epsilon_{\alpha\beta}{}^{\gamma}k_\gamma, \qquad [k_\alpha, \tilde{k}_\beta]_{\mathrm{L}} = 0, \qquad [\tilde{k}_\alpha, \tilde{k}_\beta]_{\mathrm{L}} = \epsilon_{\alpha\beta}{}^{\gamma}\tilde{k}_\gamma. \tag{5.5}$$

Furthermore, the Killing vectors shown in (5.4) have constant non-vanishing norm, corresponding to the fact that they are dual to the invariant one-forms on the three-sphere

$$|k_\alpha|^2 = |\tilde{k}_\alpha|^2 = \frac{R^2}{4}. \tag{5.6}$$

Constraints on gauging the sigma model

After having introduced our notation, let us now investigate under which conditions the corresponding non-linear sigma model can be gauged. These constraints are governed by (2.10), however, in order to obtain the dual model we also have to check the condition (3.22). We consider three different cases:

- First, gauging a single isometry of the three-sphere has been discussed for instance in [75], and in the present formalism in [60]. In this case, the constraint (2.10) is always satisfied, and so we can allow for a non-trivial field strength $H \neq 0$. Also, since all vectors in (5.4) have constant norm, the condition (3.22) is satisfied.
- Second, for the case of two Killing vectors we have to choose one vector from $\{k_\alpha\}$ and one from $\{\tilde{k}_\alpha\}$ in order to obtain a closed algebra. Because these Killing vectors commute, the second constraint in (2.10) is always satisfied.
  Without loss of generality, let us then take $\mathsf{k}_1 = k_1$ and $\mathsf{k}_2 = \tilde{k}_1$ and determine the metric $\mathcal{G}_{\alpha\beta}$ defined in (3.21). We find that
  $$\mathcal{G}_{\alpha\beta} = \frac{R^2}{4}\begin{pmatrix} 1 & -\cos(2\eta) \\ -\cos(2\eta) & 1 \end{pmatrix} \quad \longrightarrow \quad \det\mathcal{G} = \frac{R^4}{16}\sin^2(2\eta), \tag{5.7}$$
  and thus (3.22) is not met at the two points $\eta = 0$ and $\eta = \pi/2$.
- Third, the most interesting case is to gauge three isometries. Due to the requirement of a closed algebra of Killing vectors, we choose the three vectors $\{k_\alpha\}$. For those we compute
  $$\mathcal{G}_{\alpha\beta} = \frac{R^2}{4}\delta_{\alpha\beta}, \tag{5.8}$$
  and so the constraint (3.22) is satisfied. However, the conditions (2.10) require a vanishing field strength $H = 0$.

5.1. One T-duality

We start with one T-duality for the three-sphere with H-flux. In the present formalism, this situation has been analyzed in detail in [60], which we review briefly. For the Killing vector, we choose $k = k_1$ from (5.4), that is

$$k = \frac{1}{2}\begin{pmatrix} +1 \\ -1 \\ 0 \end{pmatrix} \tag{5.9}$$

in the basis $\{\partial_{\zeta_1}, \partial_{\zeta_2}, \partial_\eta\}$. The corresponding one-form v is determined as

$$v = -\frac{h}{8\pi^2}\Big[\alpha_1\zeta_1 d\cos^2\eta - \beta_1\cos^2\eta d\zeta_1 + \alpha_2\zeta_2 d\cos^2\eta - \beta_2\cos^2\eta d\zeta_2\Big], \tag{5.10}$$

with $\alpha_m + \beta_m = 1$. The gauged action for this setting can be determined from the general form (2.6) using the metric G in (5.1) together with the one-form v in (5.10). The original model is recovered similarly to the example of the torus, as the components of k in (5.9) are constant. Alternatively, following the discussion in Section 3.1, we note that the relation (3.11) can be satisfied by choosing for e any vector from $\{k_1, \tilde{k}_\alpha\}$ shown in (5.4), which then obeys $[k, e]_{\mathrm{L}} = 0$.

Dual model

To obtain the dual model, we start by determining the quantities in (3.21) for the present setting:

$$\begin{aligned} \mathcal{G} &= \frac{R^2}{4}, & \xi &= d\chi + v, \\ \mathcal{D} &= 0, & \mathsf{k} &= \frac{R^2}{2}\Big(\sin^2\eta d\zeta_1 - \cos^2\eta d\zeta_2\Big); \end{aligned} \tag{5.11}$$

the matrix $\mathcal{M}$ is given by $\mathcal{M} = \mathcal{G}$. The metric and field strength of the enlarged target space appearing in the action (3.24) are determined by the general expressions (3.25), which in the present case become

$$\begin{aligned}\check{G} &= R^2\left[d\eta\wedge\star d\eta + \frac{1}{4}\sin^2(2\eta)\,(d\zeta_1+d\zeta_2)\wedge\star(d\zeta_1+d\zeta_2)\right] + \frac{4}{R^2}\xi\wedge\star\xi,\\ \check{H} &= 2\sin(2\eta)(d\zeta_1+d\zeta_2)\wedge d\eta\wedge\xi.\end{aligned}\tag{5.12}$$

If we now make the redefinitions $\tilde{\eta} = 2\eta$ and $\tilde{\zeta} = \zeta_1 + \zeta_2$, we can express the above metric and field strength as

$$\begin{aligned}\check{\mathsf{G}} &= \frac{R^2}{4}\left[d\tilde{\eta}\wedge\star d\tilde{\eta} + \sin^2\tilde{\eta}\,d\tilde{\zeta}\wedge\star d\tilde{\zeta}\right] + \frac{4}{R^2}\xi\wedge\star\xi,\\ \check{\mathsf{H}} &= \sin\tilde{\eta}\,d\tilde{\zeta}\wedge d\tilde{\eta}\wedge\xi.\end{aligned}\tag{5.13}$$

Noting then furthermore that

$$d\xi = -\frac{h}{16\pi^2}\sin\tilde{\eta}\,d\tilde{\eta}\wedge d\tilde{\zeta},\tag{5.14}$$

we conclude that the metric $\check{\mathsf{G}}$ in (5.13) corresponds to a circle of radius $\frac{2}{R}$ which is fibered over a round two-sphere of radius $\frac{R}{2}$, with the twisting characterized by (5.14) [50,75] (see also [76, 77] for related work). Note that the dual model solves again the string equations of motion for a constant dilaton.

5.2. Two T-dualities

Next, we turn to the three-sphere with non-trivial H-flux (5.2) and perform two duality transformations along the Killing vectors

$$\mathsf{k}_1 = k_1 = \frac{1}{2}\begin{pmatrix}+1\\-1\\0\end{pmatrix}, \qquad \mathsf{k}_2 = \tilde{k}_1 = \frac{1}{2}\begin{pmatrix}+1\\+1\\0\end{pmatrix},\tag{5.15}$$

which are written in the basis $\{\partial_{\zeta_1}, \partial_{\zeta_2}, \partial_\eta\}$. The corresponding one-forms are again specified by the second equation in (2.4), and take the form

$$\begin{aligned}\mathsf{v}_1 &= \frac{h}{8\pi^2}\Big[-\alpha_1\zeta_1 d\cos^2\eta + \beta_1\cos^2\eta d\zeta_1 - \alpha_2\zeta_2 d\cos^2\eta + \beta_2\cos^2\eta d\zeta_2\Big],\\ \mathsf{v}_2 &= \frac{h}{8\pi^2}\Big[+\alpha_3\zeta_1 d\cos^2\eta - \beta_3\cos^2\eta d\zeta_1 - \alpha_4\zeta_2 d\cos^2\eta + \beta_4\cos^2\eta d\zeta_2\Big].\end{aligned}\tag{5.16}$$

However, in order to satisfy the constraint (2.10), the constants α_m and β_m have to be restricted as $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 4$. The original model can be recovered along the lines discussed in Section 3.1 by choosing vector fields $\{e_a\} = \{k_1, \tilde{k}_1\}$, which satisfy the relation (3.11).

Dual model

In order to determine the dual model, we first compute the quantities shown in (3.21) for the present example. From the above data we obtain

$$\mathcal{G}_{\alpha\beta} = \frac{R^2}{4}\begin{pmatrix} 1 & -\cos(2\eta) \\ -\cos(2\eta) & 1 \end{pmatrix}, \qquad \xi_\alpha = d\chi_\alpha + \mathsf{v}_\alpha,$$
$$\mathcal{D}_{12} = -\frac{h}{8\pi^2}\cos^2\eta, \qquad k_\alpha = \frac{R^2}{2}\begin{pmatrix} \sin^2\eta d\zeta_1 - \cos^2\eta d\zeta_2 \\ \sin^2\eta d\zeta_1 + \cos^2\eta d\zeta_2 \end{pmatrix}. \tag{5.17}$$

The general form of the dual world-sheet action is again (3.24), where the corresponding metric and field strength are determined by (3.25). Employing (5.17), we find a rather complicated expression for the enlarged metric, which we do not present here. However, after a change of basis characterized by

$$\mathcal{T}^I{}_A = \left(\begin{array}{ccc|cc} +\frac{1}{2} & +\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & -\mathcal{D}_{12} & 0 & +\frac{1}{2} & +\frac{1}{2} \\ +\mathcal{D}_{12} & 0 & 0 & +\frac{1}{2} & -\frac{1}{2} \end{array}\right), \tag{5.18}$$

we obtain for the metric tensor in the new basis

$$\check{\mathsf{G}}_{AB} = (\mathcal{T}^T \check{G} \mathcal{T})_{AB} = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R^2 & 0 & 0 \\ \hline 0 & 0 & 0 & \check{\mathsf{G}}_{11} & 0 \\ 0 & 0 & 0 & 0 & \check{\mathsf{G}}_{22} \end{array}\right), \tag{5.19}$$

where the two components $\check{\mathsf{G}}_{11}$ and $\check{\mathsf{G}}_{22}$ are given by

$$\check{\mathsf{G}}_{11} = \frac{1}{R^2}\left[\sin^2\eta + \left(\frac{h}{4\pi^2R^2}\right)^2\cos^2\eta\right]^{-1},$$
$$\check{\mathsf{G}}_{22} = \frac{1}{R^2}\left[\cos^2\eta + \left(\frac{h}{4\pi^2R^2}\right)^2\frac{\cos^4\eta}{\sin^2\eta}\right]^{-1}. \tag{5.20}$$

The one forms in the transformed basis take the following general form

$$\mathsf{e}^A = \mathcal{T}^{-1}\begin{pmatrix} d\zeta_1 \\ d\zeta_2 \\ d\eta \\ \xi_1 \\ \xi_2 \end{pmatrix}, \tag{5.21}$$

and we note that e^{ξ_1} and e^{ξ_2} are exact, so we can introduce new coordinates $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ via $\mathsf{e}^{\xi_1} = d\tilde{\zeta}_1$ and $\mathsf{e}^{\xi_2} = d\tilde{\zeta}_2$. A similar analysis can be performed for the dual field strength: using the expression shown in (3.25) and performing the change of basis characterized by (5.18), we find that the only non-vanishing component of $\check{\mathsf{H}}_{ABC}$ reads

$$\check{\mathsf{H}}_{\eta\xi_1\xi_2} = -8h\pi^2\left(h^2 - 16\pi^2R^4\right)\frac{\sin\eta\cos\eta}{\left[16\pi^2R^4\sin^2\eta + h^2\cos^2\eta\right]^2}. \tag{5.22}$$

Summary and discussion

The expressions for the components of the dual metric and field strength were given in Eqs. (5.19) and (5.22), which we summarize as

$$\begin{aligned}\check{\mathsf{G}} &= R^2 d\eta \wedge \star d\eta + \frac{1}{R^2}\frac{d\tilde{\zeta}_1 \wedge \star d\tilde{\zeta}_1}{\sin^2\eta + \left[\frac{h}{4\pi^2 R^2}\right]^2 \cos^2\eta} + \frac{1}{R^2}\frac{d\tilde{\zeta}_2 \wedge \star d\tilde{\zeta}_2}{\cos^2\eta + \left(\frac{h}{4\pi^2 R^2}\right)^2 \frac{\cos^4\eta}{\sin^2\eta}},\\ \check{\mathsf{H}} &= -8h\pi^2\big(h^2 - 16\pi^4 R^4\big)\frac{\sin\eta\cos\eta}{\big[16\pi^2 R^4 \sin^2\eta + h^2\cos^2\eta\big]^2}\, d\eta \wedge d\tilde{\zeta}_1 \wedge d\tilde{\zeta}_2. \end{aligned} \tag{5.23}$$

These formulas are rather complicated, and it appears to be difficult to extract properties of the dual space. However, if we use the condition (5.3) for solving the string equations of motion of the original model, the above formulas simplify considerably. In particular, we find

$$\begin{aligned}\bar{\mathsf{G}} &= R^2 d\eta \wedge \star d\eta + \frac{1}{R^2}\Big[d\tilde{\zeta}_1 \wedge \star d\tilde{\zeta}_1 + \tan^2\eta\, d\tilde{\zeta}_2 \wedge \star d\tilde{\zeta}_2\Big],\\ \bar{\mathsf{H}} &= 0,\end{aligned} \tag{5.24}$$

which describes a non-compact but geometric background. This is in contrast to the example of the three-torus with H-flux discussed in Section 4.2, where after two T-dualities a non-geometric Q-flux background was obtained.

Let us also note that the dual configuration (5.24) solves again the string equations of motion if we transform the dilaton via the standard relation of the Buscher rules [69–71] as

$$\bar{\phi} = -\log\big(R^2\cos\eta\big) + \phi_0. \tag{5.25}$$

Note furthermore, this backgrounds is related to Witten's black hole [78], that is the group manifold $SL(2,\mathbb{R})/U(1)$, via analytic continuation.

5.3. Three T-dualities

We finally consider the situation of gauging three (non-abelian) isometries of the three-sphere. As explained in the beginning of this section, in this case the constraints in (2.10) require a vanishing field strength $H = 0$. Thus, we have

$$H = 0 \quad \longrightarrow \quad v_\alpha = 0. \tag{5.26}$$

For the Killing vectors, we can choose either of the sets $\{k_\alpha\}$ or $\{\tilde{k}_\alpha\}$; for definiteness we consider the first in the following.

Gauged action and original model

The gauged action can again be inferred from the general form shown in Eq. (2.6). Using coordinates $\{X^1, X^2, X^3\} = \{\zeta_1, \zeta_2, \eta\}$, we find

$$\begin{aligned}\widehat{\mathcal{S}} = &-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} \tfrac{1}{2} G_{ij}(dX^i + k^i_\alpha A^\alpha) \wedge \star(dX^j + k^j_\beta A^\beta)\\ &-\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\Big[d\chi_\alpha \wedge A^\alpha + \tfrac{1}{2} f_{\alpha\beta}{}^\gamma \chi_\gamma A^\alpha \wedge A^\beta\Big],\end{aligned} \tag{5.27}$$

where now the gauge fields are non-abelian. To recover the original ungauged model, we use the equations of motion (3.2) for A^α and rewrite the action as in Section 3.1. In particular, from (3.4) we obtain

$$\widehat{\mathcal{S}} = -\frac{1}{4\pi\alpha'} \int_{\partial\Sigma} G_{ij} DX^i \wedge \star DX^j, \tag{5.28}$$

where $DX^i = dX^i + k^i_\alpha A^\alpha$. However, we note that $d(DX^i) \neq 0$, and so we cannot make the replacements $DX^i \to dY^i$ as before. The way to proceed has been described in Section 3.1. We first need to find a set of vector fields $\{e_a\}$ which commute with $\{k_\alpha\}$ and thus satisfy Eq. (3.11). For the three-sphere we have an obvious candidate, namely $\{\tilde{k}_\alpha\}$,

$$e_a{}^i = \tilde{k}_\alpha{}^i = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ +\sin(\zeta_1+\zeta_2)\cot\eta & -\sin(\zeta_1+\zeta_2)\tan\eta & -\cos(\zeta_1+\zeta_2) \\ +\cos(\zeta_1+\zeta_2)\cot\eta & -\cos(\zeta_1+\zeta_2)\tan\eta & +\sin(\zeta_1+\zeta_2) \end{pmatrix}. \tag{5.29}$$

The metric (5.1) can then be transformed via

$$G_{ab} = e_a{}^i G_{ij} (e^T)^j{}_b = \frac{R^2}{4} \delta_{ab}, \tag{5.30}$$

and for DX^a in the new basis we compute

$$d(DX^a) = -\frac{1}{2} \epsilon^a{}_{bc} DX^b \wedge DX^c. \tag{5.31}$$

Hence, the one-forms $\{DX^a\}$ behave like a non-holonomic basis of the co-tangent space. Since the corresponding metric (5.30) is constant, we can define new vielbeins $E^a = DX^a$, and express them in a local basis dY^i as

$$E^a = e^a{}_i dY^i, \tag{5.32}$$

where we also performed the obvious relabeling $X^i \to Y^i$ in the matrix $e^a{}_i$. Using this form, we then arrive at the original ungauged action

$$\mathcal{S} = -\frac{1}{4\pi\alpha'} \int_{\partial\Sigma} R^2 \Big[\sin^2\eta\, d\zeta_1 \wedge \star d\zeta_1 + \cos^2\eta\, d\zeta_2 \wedge \star d\zeta_2 + d\eta \wedge \star d\eta \Big]. \tag{5.33}$$

Dual model

In order to determine the dual model, we fist specify the quantities in Eq. (3.21) as follows

$$\begin{aligned} \mathcal{G}_{\alpha\beta} &= \frac{R^2}{4} \delta_{\alpha\beta}, & \xi_\alpha &= d\chi_\alpha, \\ \mathcal{D}_{\alpha\beta} &= \epsilon_{\alpha\beta}{}^\gamma \chi_\gamma, & k_\alpha &= k^i_\alpha G_{ij} dX^j, \end{aligned} \tag{5.34}$$

where we did not spell out the expression for the one-forms k_α corresponding to the Killing vectors. Using then the general formulas shown in (3.25), the metric and field strength of the enlarged target-space can be determined. These expressions become quite involved, and so we only display the quantities after a change of basis given by the null-eigenvectors (3.26) has been performed and after a field redefinition. In a basis $\{d\tilde{\chi}_1, d\tilde{\chi}_2, d\tilde{\chi}_3\}$ we obtain

$$\bar{\mathsf{G}}_{\alpha\beta} = \frac{4}{R^2}\frac{1}{\frac{R^4}{16}+\tilde{\chi}_1^2+\tilde{\chi}_2^2+\tilde{\chi}_3^2}\begin{pmatrix} \frac{R^4}{16}+\tilde{\chi}_1^2 & \tilde{\chi}_1\tilde{\chi}_2 & \tilde{\chi}_1\tilde{\chi}_3 \\ \tilde{\chi}_1\tilde{\chi}_2 & \frac{R^4}{16}+\tilde{\chi}_2^2 & \tilde{\chi}_2\tilde{\chi}_3 \\ \tilde{\chi}_1\tilde{\chi}_3 & \tilde{\chi}_2\tilde{\chi}_3 & \frac{R^4}{16}+\tilde{\chi}_3^2 \end{pmatrix},$$

$$\bar{\mathsf{H}}_{123} = 16\frac{3\frac{R^4}{16}+\tilde{\chi}_1^2+\tilde{\chi}_2^2+\tilde{\chi}_3^2}{\frac{R^4}{16}+\tilde{\chi}_1^2+\tilde{\chi}_2^2+\tilde{\chi}_3^2}. \tag{5.35}$$

Performing now a further change to spherical coordinates $\{\rho, \phi_1, \phi_2\}$ with $\rho \geq 0$ and $\phi_{1,2} = 0, \ldots, 2\pi$, we find

$$\bar{\mathsf{G}} = \frac{4}{R^2}\, d\rho \wedge \star d\rho + \frac{R^2}{4}\frac{\rho^2}{\rho^2+\frac{R^4}{16}}\Big[d\phi_1 \wedge \star d\phi_1 + \sin^2(\phi_1)\, d\phi_2 \wedge \star d\phi_2\Big],$$

$$\bar{\mathsf{H}} = \frac{\rho^2}{\left(\rho^2+\frac{R^4}{16}\right)^2}\left[\rho^2 + 3\frac{R^4}{16}\right]\sin(\phi_1)\, d\rho \wedge d\phi_1 \wedge d\phi_2. \tag{5.36}$$

This configuration can be interpreted as a two-sphere (parametrized by ϕ_1 and ϕ_2) whose radius depends on the ray-variable ρ. (The same result has been obtained in [50] and [55], and related expressions can be found in [57].) Note that the volume of the two-sphere as well as the H-flux vanish at $\rho = 0$, but stay finite in the limit $\rho \to \infty$.

5.4. Summary

In this section we have considered collective T-duality transformations for the three-sphere with H-flux. One of the features of this background is that it solves the string equations of motion if the flux is adjusted properly, cf. (5.3). The main purpose of studying this example was to investigate whether results similar to the three-torus with H-flux can be obtained.

- After a single T-duality for the three-sphere with H-flux, we arrived at the background of a circle fibered over a two-sphere. This is a well-defined geometric background with geometric flux, which agrees with the result for the torus obtained in Section 4.1.
- After two collective T-dualities for the three-sphere we obtained at a rather complicated-looking background, shown in Eq. (5.23). However, when imposing the condition (5.3) for the original model to be conformal, the background simplified considerably. In particular, despite being non-compact, the dual background is *geometric*. This is in contrast to our discussion in Section 4.2, where two T-dualities for the three-torus lead to a *non-geometric* background.
- Finally, for three collective T-duality transformations we found that the H-flux has to vanish. The corresponding dual background shown in Eq. (5.36) is again geometric but non-compact.

6. Summary and conclusions

In this paper, we have studied T-duality transformations along one, two, and three directions of isometry for the three-sphere with H-flux. The question we wanted to answer was, whether after two T-dualities a non-geometric Q-flux background similarly to the example of the three-torus appears.

In order to perform the duality transformations, in Section 3 we have developed a novel formalism for collective, and in general non-abelian, T-duality. Our approach is different compared to the known literature, as we do not rely on a gauge fixing procedure nor on the specific structure of Wess–Zumino–Witten models. Furthermore, we derived a constraint, shown in Eq. (2.10), which restricts the allowed transformations in the case of non-vanishing H-flux. For the three-torus and three-sphere this implied that for $H \neq 0$ at most two T-dualities can be performed.

In Section 4 we illustrated our formalism with the example of the three-torus and reproduced the known results; this analysis served as a check of our procedure. In addition, in Appendix A we studied collective T-duality transformations for the twisted torus with H-flux, for which we found a new *twisted T-fold* background.

In Section 5 we investigated collective T-duality transformations for the three-sphere with H-flux. In contrast to the torus, this background solves the string equations of motion if the flux is properly adjusted. For one T-duality, we reproduced the known result, namely the dual background is a circle fibered over a two-sphere. In view of the duality chain (1.1), this configuration would correspond to a geometric-flux background. After applying two collective T-dualities, we obtained a rather complicated background, which resembled the form of the torus T-fold. However, if the radius of the three-sphere is appropriately related to the H-flux, making the original model conformal, the dual background simplified considerably. In particular, one obtains a two-sphere fibered over a line segment, which is a geometric but non-compact space. Finally, as mentioned above, for three T-dualities the restrictions (2.10) require a vanishing H-flux. We therefore chose $H = 0$ and obtained after a non-abelian T-duality transformation a two-sphere fibered over a ray.

Let us compare our results for two collective T-duality transformations on the three-torus and on the three-sphere with H-flux. For the torus we reviewed that one obtains a non-geometric Q-flux background, or more generally a T-fold. Note however, the torus with $H \neq 0$ does not solve the string equations of motion and therefore is, strictly speaking, not a proper string background. For the three-sphere, without requiring the model to be conformal, we found a background of a form similar to the torus T-fold. But, after requiring the original model to solve the string equations of motion, the dual background simplified. In particular, the dual space is geometric but non-compact.

Our findings in this paper therefore challenge the simple picture of T-duality transformations shown in (1.1). Namely, applying two T-duality transformations to a geometric background with H-flux does not necessarily lead to a non-geometric Q-flux background. However, we also want to emphasize that the two examples studied in this paper have drawbacks: the torus example does not solve the string equations of motion, and the three-sphere leads to a non-compact background. We therefore cannot draw general conclusions about the origin of non-geometry, but have to consider further examples in the future.

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Appendix A. Examples III: twisted three-torus

As a generalization of the three-torus with H-flux, in this appendix we discuss the twisted three-torus found in Section 4.1 together with a non-vanishing H-flux.

Setup

The components of the metric tensor of the twisted three-torus in a coordinate basis $\{dX^1, dX^2, dX^3\}$ are chosen as

$$G_{ij} = \begin{pmatrix} R_1^2 & 0 & R_1^2 f X^2 \\ 0 & R_2^2 & 0 \\ R_1^2 f X^2 & 0 & R_3^2 + R_1^2 \left[f X^2\right]^2 \end{pmatrix}, \tag{A.1}$$

where f denotes the geometric flux, and we allow for a non-vanishing field strength of the Kalb–Ramond field

$$H = h\, dX^1 \wedge dX^2 \wedge dX^3, \qquad h \in \ell_s^{-1}\mathbb{Z}. \tag{A.2}$$

The Killing vectors for the above metric in the basis $\{\partial_1, \partial_2, \partial_3\}$ are given by

$$k_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad k_2 = \begin{pmatrix} -f X^3 \\ 1 \\ 0 \end{pmatrix}, \qquad k_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{A.3}$$

which satisfy a non-abelian isometry algebra with commutation relations

$$\left[k_1, k_2\right]_{\mathrm{L}} = 0, \qquad \left[k_2, k_3\right]_{\mathrm{L}} = f k_1, \qquad \left[k_3, k_1\right]_{\mathrm{L}} = 0. \tag{A.4}$$

Furthermore, the topology of the twisted torus is specified by the identifications

$$\begin{aligned} &1) \quad X^1 \to X^1 + \ell_s, \\ &2) \quad X^2 \to X^2 + \ell_s, \\ &3) \quad X^3 \to X^3 + \ell_s, \qquad X^1 \to X^1 + \ell_s f X^2. \end{aligned} \tag{A.5}$$

One T-duality

As it is well known [75,79,80], a single T-duality along the Killing vector k_1 results in a twisted torus with the replacements

$$f \longleftrightarrow h, \qquad R_1 \longrightarrow \frac{1}{R_1}. \tag{A.6}$$

However, a T-duality along the Killing vectors k_2 or k_3 leads to a twisted T-fold. More concretely, after performing a T-duality transformation along the Killing vector k_2, we find for the dual metric and H-field the expressions

$$\begin{aligned} \check{G} &= \frac{1}{1 + \left[\frac{R_1}{R_2} f X^3\right]^2} \left(R_1^2 dX^1 \wedge \star dX^1 + \frac{1}{R_2^2} \xi \wedge \star \xi \right) + R_3^2 dX^3 \wedge \star dX^3, \\ \check{H} &= -f \frac{R_1^2}{R_2^2} \frac{1 - \left[\frac{R_1}{R_2} f X^3\right]^2}{\left(1 + \left[\frac{R_1}{R_2} f X^3\right]^2\right)^2} dX^1 \wedge \xi \wedge dX^3, \end{aligned} \tag{A.7}$$

where the one form ξ is not closed,

$$d\xi = -h\, dX^1 \wedge dX^3. \tag{A.8}$$

The result for a T-duality along k_3 leads to the same expression but with the replacements $X^2 \leftrightarrow -X^3$ and $R_2 \leftrightarrow R_3$.

Two T-dualities

When performing two collective dualities for the twisted torus, there are two combinations of Killing vectors which lead to a closed isometry algebra, namely $\{k_1, k_2\}$ and $\{k_1, k_3\}$. Both choices result in a twisted T-fold:

- For a T-duality along Killing vectors $\{k_1, k_2\}$, the dual metric and H-flux are given by (A.7) and (A.8), with the replacements $h \leftrightarrow f$ and $R_1 \to 1/R_1$.
- For a T-duality along Killing vectors $\{k_1, k_3\}$, the expressions are similar but now again with the additional changes $X^2 \leftrightarrow -X^3$ and $R_2 \leftrightarrow R_3$.

Three T-dualities

The case of three collective T-dualities for the twisted torus is interesting since here the isometry algebra is non-abelian. However, due to the constraints (2.10), the H-flux has to vanish. After applying the same formalism as above and performing the field redefinitions

$$\tilde{\chi}_1 = \chi_1, \qquad \tilde{\chi}_2 = \chi_2 + f\chi_1 X^3, \qquad \tilde{\chi}_3 = \chi_3 - f\chi_1 X^2, \tag{A.9}$$

we arrive at the following dual T-fold background

$$\begin{aligned}
\check{\mathsf{G}} &= \frac{1}{R_1^2}\, d\tilde{\chi}_1 \wedge \star d\tilde{\chi}_1 + \frac{1}{1+\big[\frac{f}{R_2R_3}\tilde{\chi}_1\big]^2}\left(\frac{1}{R_2^2}\, d\tilde{\chi}_2 \wedge \star d\tilde{\chi}_2 + \frac{1}{R_3^2}\, d\tilde{\chi}_3 \wedge \star d\tilde{\chi}_3\right),\\
\check{\mathsf{H}} &= -\frac{f}{R_1^2R_2^2}\, \frac{1-\big[\frac{f}{R_2R_3}\tilde{\chi}_1\big]^2}{\Big(1+\big[\frac{f}{R_2R_3}\tilde{\chi}_1\big]^2\Big)^2}\, d\tilde{\chi}^1 \wedge d\tilde{\chi}_2 \wedge d\tilde{\chi}_3.
\end{aligned} \tag{A.10}$$

Let us finally recall our discussion in Section 3.1 about recovering the original model from the gauged action. We found that in the non-abelian case a change of basis characterized by a matrix ${e_a}^i$ has to be performed. In the present case, this matrix takes the form

$${e_a}^i = \begin{pmatrix} 1 & 0 & -fX^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.11}$$

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