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Topology 44 (2005) 131–149
TOPOLOGY
www.elsevier.com/locate/top

Signatures of foliated surface bundles and the symplectomorphism groups of surfaces☆

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Received 27 November 2003; accepted 18 March 2004

Abstract

For any closed oriented surface Σ_g of genus $g \geqslant 3$, we prove the existence of *foliated* Σ_g-bundles over surfaces such that the signatures of the total spaces are non-zero. We can arrange that the total holonomy of the horizontal foliations preserve a prescribed symplectic form ω on the fiber. We relate the cohomology class represented by the transverse symplectic form to a *crossed* homomorphism $\widetilde{\text{Flux}}: \text{Symp}\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R})$ which is an extension of the flux homomorphism $\text{Flux} : \text{Symp}_0\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R})$ from the identity component $\text{Symp}_0\,\Sigma_g$ to the whole group $\text{Symp}\,\Sigma_g$ of symplectomorphisms of Σ_g with respect to the symplectic form ω.

MSC: primary 57R1757R30; 57R50; secondary 37E3057M99; 58H10;

Keywords: Surface bundle; Signature; Foliated bundle; Mapping class group; Symplectomorphism; Flux homomorphism; Calabi homomorphism

1. Introduction and statement of the main results

Let Σ_g be a closed oriented surface of genus g. It is a classical result of Kodaira [19] and Atiyah [1] that, for any $g \geqslant 3$, there exist oriented Σ_g-bundles over closed oriented surfaces such that the signatures of the total spaces are non-zero. In this paper, we prove the existence of such bundles which, in addition to having non-zero signature, are flat, or foliated. This means that there exist

☆ Support from the *Deutscher Akademischer Austauschdienst* and the *Deutsche Forschungsgemeinschaft* is gratefully acknowledged. D. Kotschick is a member of the *European Differential Geometry Endeavour* (EDGE), Research Training Network HPRN-CT-2000-00101, supported by The European Human Potential Programme. S. Morita is partially supported by JSPS Grant 13440017.
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doi:10.1016/j.top.2004.05.002

codimension two foliations complementary to the fibers, which is equivalent to the existence of lifts of the holonomy homomorphisms from the mapping class group to the diffeomorphism group of the fiber. We will further show that such lifts can be chosen to preserve a prescribed area form, or equivalently a symplectic form ω, on the fiber. More precisely, we prove the following result.

Theorem 1. *For any $g \geqslant 3$, there exist foliated oriented Σ_g-bundles $\pi : E \to B$ over closed oriented surfaces B such that the total holonomy group is contained in the symplectomorphism group* $\mathrm{Symp}\,\Sigma_g$ *with respect to a prescribed symplectic form ω on Σ_g, and sign $E \neq 0$.*

Recall that due to Thurston's results [36,37] Haefliger's classifying space $\mathrm{B}\bar{\Gamma}_2$ is known to be 3-connected, which implies that the tangent bundle along the fibers of any surface bundle over a surface is homotopic to the normal bundle of a codimension two foliation on the total space. However, it is impossible to determine whether this foliation can be made everywhere transverse to the fibers.

Theorem 1 raises the question how two naturally defined cohomology classes, the transverse symplectic class and the Euler class of the vertical bundle, compare on the total space of a foliated bundle with total holonomy in the symplectomorphism group of the fiber. Let $\pi : E \to B$ be a foliated oriented Σ_g-bundle as in Theorem 1, so that sign $E \neq 0$ and the image of the total holonomy homomorphism

$$\pi_1 B \to \mathrm{Diff}_+\,\Sigma_g$$

is contained in the symplectomorphism subgroup $\mathrm{Symp}\,\Sigma_g \subset \mathrm{Diff}_+\Sigma_g$ with respect to ω. Since the total holonomy preserves the symplectic form ω on Σ_g, the pullback of this form to the product $\Sigma_g \times \tilde{B}$ descends to $E = (\Sigma_g \times \tilde{B})/\pi_1 B$ as a globally defined closed 2-form $\tilde{\omega}$ of rank 2 which restricts to ω on the fiber. Hence we have the corresponding cohomology class

$$v = [\tilde{\omega}] \in H^2(E; \mathbb{R})$$

which we call the *transverse symplectic class*. At the universal space level, this cohomology class v can be considered as an element of $H^2(\mathrm{ESymp}^\delta \Sigma_g; \mathbb{R})$, where the discrete group $\mathrm{ESymp}^\delta \Sigma_g$ is defined as follows. Let $\mathcal{M}_g$ and $\mathcal{M}_{g,*}$ denote the mapping class group of Σ_g, respectively, the mapping class group relative to a base point. Then we have the universal extension $\pi_1 \Sigma_g \to \mathcal{M}_{g,*} \to \mathcal{M}_g$. If we pull back this extension by the natural projection $\mathrm{Symp}^\delta \Sigma_g \to \mathcal{M}_g$, where the symbol δ indicates the *discrete* topology, we obtain an extension

$$1 \to \pi_1 \Sigma_g \to \mathrm{ESymp}^\delta\, \Sigma_g \to \mathrm{Symp}^\delta\, \Sigma_g \to 1.$$

Thus, $\mathrm{ESymp}^\delta\, \Sigma_g$ is the universal model for the fundamental groups of the total spaces of foliated Σ_g-bundles with area-preserving holonomy. On the other hand, we have the Euler class $e \in H^2(E; \mathbb{Z})$ of the tangent bundle along the fibers of π. This bundle is the normal bundle of the horizontal foliation on E. The two cohomology classes v and e are proportional on the fiber, and if we normalize ω so that

$$\int_{\Sigma_g} \omega = 2g - 2$$

then $e + v$ restricts to 0 on the fiber. However, we cannot have the equality $v = -e$, because of the following reason. Clearly, we have $v^2 = 0$ (since $\tilde{\omega}^2$ vanishes identically), while $e^2 \neq 0$ since its fiber integral is nothing but the first Mumford–Morita–Miller class $e_1 \in H^2(B;\mathbb{Z})$ which represents the signature of the total space (see [1,26,29]). Therefore, we would like to identify the difference between v and $-e$. We shall do this by making use of certain basic facts in symplectic topology.

Let $\mathrm{Symp}_0\,\Sigma_g$ denote the identity component of $\mathrm{Symp}\,\Sigma_g$. Then there is a well-defined surjective homomorphism

$$\mathrm{Flux} : \mathrm{Symp}_0\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R}),$$

called the flux homomorphism. We refer to the book [25] by McDuff and Salamon for generalities of symplectic topology, including the flux homomorphism as well as the Calabi homomorphism used in the proof of Theorem 3 below.

Theorem 2. *For all $g \geqslant 2$, there is a unique cohomology class*

$$[\widetilde{\mathrm{Flux}}] \in H^1(\mathrm{Symp}^\delta\,\Sigma_g; H^1(\Sigma_g;\mathbb{R}))$$

which extends $[\mathrm{Flux}] \in H^1(\mathrm{Symp}_0^\delta\,\Sigma_g; H^1(\Sigma_g;\mathbb{R}))$ to the whole symplectomorphism group.

Furthermore, in the cohomology spectral sequence of the extension

$$1 \to \pi_1\Sigma_g \to \mathrm{ESymp}^\delta\,\Sigma_g \to \mathrm{Symp}^\delta\,\Sigma_g \to 1,$$

the class

$$e + v \in \mathrm{Ker}(H^2(\mathrm{ESymp}^\delta\,\Sigma_g;\mathbb{R}) \to H^2(\Sigma_g;\mathbb{R}))$$

projects to $[\widetilde{\mathrm{Flux}}]$.

Note that because $\mathrm{Symp}^\delta\,\Sigma_g$, acts non-trivially on $H^1(\Sigma_g;\mathbb{R})$, a representing cocycle for this cohomology class is not a homomorphism, but a *crossed* homomorphism

$$\widetilde{\mathrm{Flux}} : \mathrm{Symp}\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R})$$

with respect to this action. We will actually determine the group $H^1(\mathrm{Symp}^\delta\,\Sigma_g; H^1(\Sigma_g;\mathbb{R}))$ completely, see Proposition 9. Furthermore, in Section 6 we generalize Theorem 2 to a certain class of closed symplectic manifolds of higher dimensions; see also [18].

Theorem 1 can be reformulated and extended in the context of the Mumford–Morita–Miller classes (see [34,28,27]). Let $\mathscr{M}_g$ be the mapping class group of Σ_g as before and let $e_i = \pi_* e^{i+1} \in H^{2i}(\mathscr{M}_g;\mathbb{Q})$ be the ith Mumford–Morita–Miller class with *rational* coefficients defined by integration over the fiber in the universal Σ_g-bundle.

Let $\mathrm{Diff}_+\,\Sigma_g$ be the group of orientation-preserving diffeomorphisms of Σ_g. Then $\mathscr{M}_g$ can be considered as the group of path components of $\mathrm{Diff}_+\,\Sigma_g$ and we have an extension

$$1 \to \mathrm{Diff}_0\,\Sigma_g \to \mathrm{Diff}_+\,\Sigma_g \xrightarrow{p} \mathscr{M}_g \to 1,$$

where $\mathrm{Diff}_0\,\Sigma_g$ is the identity component of $\mathrm{Diff}_+\,\Sigma_g$ and p is the natural projection. It follows from the Bott vanishing theorem for the characteristic classes of the normal bundles of foliations that

$$p^*(e_i) = 0 \in H^{2i}(\mathrm{BDiff}_+^\delta\,\Sigma_g;\mathbb{Q})$$

for all $i \geqslant 3$, see [29] and also [32]. (Here, as before, δ indicates the *discrete* topology, so that the space $\mathrm{BDiff}_+^\delta\,\Sigma_g$ is the classifying space of *foliated* oriented Σ_g-bundles.) More precisely, the

Bott vanishing theorem applied to the horizontal foliation shows that $e^i = 0 \in H^{2i}(\mathrm{BEDiff}_+^\delta \Sigma_g; \mathbb{Q})$ for all $i \geqslant 4$, where $\mathrm{EDiff}_+^\delta \Sigma_g$ is defined as in the case of symplectomorphism groups considered above. On the other hand, Theorem 1 shows that $e^2 \in H^4(\mathrm{BEDiff}_+^\delta \Sigma_g; \mathbb{Q})$ and its fiber integral $e_1 \in H^2(\mathrm{BDiff}_+^\delta \Sigma_g; \mathbb{Q})$ are non-zero.

It remains to determine whether other polynomials in e_1 and e_2 are trivial in $H^*(\mathrm{BDiff}_+^\delta \Sigma_g; \mathbb{Q})$, or not. By extending Theorem 1, we can give a partial answer to this question. Namely, we show the non-triviality of any power $e_1^k \in H^{2k}(\mathrm{BDiff}_+^\delta \Sigma_g; \mathbb{Q})$ of the first characteristic class e_1. In fact, we can prove the following stronger non-vanishing result for the subgroup $\mathrm{Symp}\, \Sigma_g \subset \mathrm{Diff}_+ \Sigma_g$.

Theorem 3. *Let* $\mathrm{Symp}^\delta \Sigma_g$ *denote the group of symplectomorphisms of* (Σ_g, ω) *equipped with the discrete topology. Then, for any* $k \geqslant 1$, *the power* $e_1^k \in H^{2k}(\mathrm{BSymp}^\delta \Sigma_g; \mathbb{Q})$ *of the first Mumford–Morita–Miller class* e_1 *is non-trivial for all* $g \geqslant 3k$.

Thus, we are left with the following open problem.

Problem 4. Determine whether the second Mumford–Morita–Miller class e_2 is non-trivial in $H^4(\mathrm{BDiff}_+^\delta \Sigma_g; \mathbb{R})$ (or in $H^4(\mathrm{BSymp}^\delta \Sigma_g; \mathbb{R})$).

More generally, one can ask about polynomials in e_1 and e_2.

2. Proof of Theorem 1

In this section we prove Theorem 1 by constructing foliated surface bundles with non-zero signatures. In fact, we prove more than was stated in Theorem 1, in that we show that *any* surface bundle over a surface can be made flat by fiber summing with a trivial bundle.

First we treat the case where there is no constraint on the total holonomy group in $\mathrm{Diff}_+ \Sigma_g$. Let $\pi : E \to \Sigma_h$ be any oriented Σ_g-bundle over a closed oriented surface of genus h, for example one with $\mathrm{sign}\, E \neq 0$. Such bundles are classified by their monodromy homomorphisms

$$\rho : \pi_1 \Sigma_h \to \mathcal{M}_g.$$

Choose a standard system $\alpha_1, \ldots, \alpha_h, \beta_1, \ldots, \beta_h$ of generators for $\pi_1 \Sigma_h$ with a unique relation

$$[\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h] = 1$$

and set

$$\tilde{\alpha}_i = \text{any lift of } \rho(\alpha_i) \in \mathcal{M}_g \text{ to } \mathrm{Diff}_+ \Sigma_g,$$

$$\tilde{\beta}_i = \text{any lift of } \rho(\beta_i) \in \mathcal{M}_g \text{ to } \mathrm{Diff}_+ \Sigma_g.$$

Then clearly we have

$$\xi = [\tilde{\alpha}_1, \tilde{\beta}_1] \cdots [\tilde{\alpha}_h, \tilde{\beta}_h] \in \mathrm{Diff}_0 \Sigma_g.$$

According to a special case of a deep theorem of Thurston [37], the group $\mathrm{Diff}_0 \Sigma_g$ is perfect (and simple). Hence the above element ξ can be written as a product of commutators of elements of $\mathrm{Diff}_0\,\Sigma_g$:

$$\xi = [\varphi_1, \psi_1] \cdots [\varphi_{h'}, \psi_{h'}] \quad (\varphi_i, \psi_i \in \mathrm{Diff}_0\,\Sigma_g).$$

By considering the surface $\Sigma_{h+h'}$ of genus $h + h'$ as the connected sum $\Sigma_h \sharp \Sigma_{h'}$, we can define a homomorphism

$$\tilde{\rho} : \pi_1 \Sigma_{h+h'} \to \mathrm{Diff}_+\,\Sigma_g$$

by using $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ on $\Sigma_h \setminus D^2$ and the elements $\varphi_i, \psi_i \in \mathrm{Diff}_0\,\Sigma_g$ on $\Sigma_{h'} \setminus D^2$. Let

$$\tilde{\pi} : \tilde{E} \to \Sigma_{h+h'}$$

be the corresponding foliated Σ_g-bundle. Topologically, if we ignore the horizontal foliation, this new bundle is nothing but the fiber sum of the original bundle and the product Σ_g-bundle $\Sigma_{h'} \times \Sigma_g$. Hence, by Novikov additivity we have

$$\mathrm{sign}\,\tilde{E} = \mathrm{sign}\ E \neq 0.$$

This proves Theorem 1 in the case when we do not impose any constraint on the holonomy of the horizontal foliation.

Next we prove that, in the above construction, we can replace the group $\mathrm{Diff}_+\,\Sigma_g$ by the subgroup $\mathrm{Symp}\,\Sigma_g$ with respect to a symplectic or area form ω on Σ_g. It is elementary to see that any Dehn twist on Σ_g can be represented by an area-preserving diffeomorphism. Since the mapping class group is generated by Dehn twists, it follows that the natural map $\mathrm{Symp}\,\Sigma_g \to \mathscr{M}_g$ is surjective. In fact, Moser's celebrated result [33] on isotopy of volume-preserving diffeomorphisms implies the stronger assertion that the inclusion

$$\mathrm{Symp}\,\Sigma_g \subset \mathrm{Diff}_+\,\Sigma_g$$

is a weak homotopy equivalence. It follows that $\mathrm{Symp}\,\Sigma_g \cap \mathrm{Diff}_+\,\Sigma_g = \mathrm{Symp}_0\,\Sigma_g$, and we have an extension

$$1 \to \mathrm{Symp}_0\,\Sigma_g \to \mathrm{Symp}\,\Sigma_g \to \mathscr{M}_g \to 1.$$

Remark 5. Earle and Eells [9] proved that $\mathrm{Diff}_0\,\Sigma_g$ is contractible for any $g \geqslant 2$. Hence $\mathrm{Symp}_0\,\Sigma_g$ is also contractible by Moser's result mentioned above, and we have isomorphisms

$$H^*(\mathrm{BSymp}\,\Sigma_g) \cong H^*(\mathrm{BDiff}_+\,\Sigma_g) \cong H^*(\mathscr{M}_g).$$

Thus there is no difference between the characteristic classes of smooth surface bundles and those of symplectic surface bundles, and they are all detected by the cohomology of the mapping class group. However, if we endow the groups $\mathrm{Diff}_+\,\Sigma_g$ and $\mathrm{Symp}\,\Sigma_g$ with the *discrete* topology, then the situation is completely different. This is the main concern of the present paper.

Now going back to the construction above, we replace $\mathrm{Diff}_+\,\Sigma_g$ by $\mathrm{Symp}\,\Sigma_g$ and set

$$\tilde{\alpha}_i = \text{any lift of } \rho(\alpha_i) \in \mathscr{M}_g \text{ to } \mathrm{Symp}\,\Sigma_g,$$

$$\tilde{\beta}_i = \text{any lift of } \rho(\beta_i) \in \mathscr{M}_g \text{ to } \mathrm{Symp}\,\Sigma_g.$$

Then the element

$$\xi = [\tilde{\alpha}_1, \tilde{\beta}_1] \cdots [\tilde{\alpha}_h, \tilde{\beta}_h]$$

belongs to $\mathrm{Symp}_0\,\Sigma_g$, not just to $\mathrm{Diff}_0\,\Sigma_g$. But now, the group $\mathrm{Symp}_0\,\Sigma_g$ is not perfect. In fact, it is known that there is a surjective homomorphism

$$\mathrm{Flux} : \mathrm{Symp}_0\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R})$$

called the flux homomorphism, whose kernel is the subgroup $\mathrm{Ham}\,\Sigma_g$ consisting of Hamiltonian symplectomorphisms of Σ_g. Fortunately, $\mathrm{Ham}\,\Sigma_g$ is known to be perfect by a general result of Thurston [35] on the group of volume-preserving diffeomorphisms of closed manifolds (which was generalized to the case of closed symplectic manifolds by a theorem of Banyaga [4]. See also the books [5,25] for these well-known results.) Thus, we have an extension

$$1 \to \mathrm{Ham}\,\Sigma_g \to \mathrm{Symp}_0\,\Sigma_g \stackrel{\mathrm{Flux}}{\to} H^1(\Sigma_g;\mathbb{R}) \to 1.$$

In our situation, if $\mathrm{Flux}(\xi)=0$, then ξ belongs to the perfect group $\mathrm{Ham}\,\Sigma_g$ and we are done. In general, we cannot expect this and we have to kill $\mathrm{Flux}(\xi) \in H^1(\Sigma_g;\mathbb{R})$ in some way. Since $\mathrm{Ham}\,\Sigma_g$ is perfect, it is easy to see that the flux homomorphism gives an isomorphism

$$H_1(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{Z}) \cong H^1(\Sigma_g;\mathbb{R}).$$

The natural action of $\mathrm{Symp}\,\Sigma_g$ on $H_1(\mathrm{Symp}_0^\delta\,\Sigma_g;\mathbb{Z})$ by outer conjugation factors through that of the mapping class group $\mathcal{M}_g$ because any inner automorphism of a group acts trivially on its integral first homology, i.e. its abelianization.

Lemma 6. *The flux homomorphism* $\mathrm{Flux} : \mathrm{Symp}_0\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R})$ *is equivariant with respect to the natural actions of* $\mathcal{M}_g$. *In other words, for any two elements* $\varphi \in \mathrm{Symp}\,\Sigma_g$ *and* $\psi \in \mathrm{Symp}_0\,\Sigma_g$, *we have the identity*

$$\mathrm{Flux}(\varphi\,\psi\,\varphi^{-1}) = \bar{\varphi}(\mathrm{Flux}(\psi)),$$

where $\bar{\varphi} \in \mathcal{M}_g$ *denotes the mapping class of* φ *and* $\mathcal{M}_g$ *acts on* $H^1(\Sigma_g;\mathbb{R})$ *from the left by the rule* $\bar{\varphi}(w) = (\bar{\varphi}^{-1})^*(w)$ $(w \in H^1(\Sigma_g;\mathbb{R}))$.

Proof. Recall that the flux homomorphism (for the case Σ_g $(g \geqslant 2)$) can be defined as follows. For any element $\psi \in \mathrm{Symp}_0\,\Sigma_g$, choose an isotopy $\psi_t \in \mathrm{Symp}_0\,\Sigma_g$ such that $\psi_0 = \mathrm{Id}$ and $\psi_1 = \psi$. Then

$$\mathrm{Flux}(\psi) = \int_0^1 i_{\dot{\psi}_t}\omega\,\mathrm{d}t \in H^1(\Sigma_g;\mathbb{R}).$$

The assertion follows easily from this. □

As usual, let $H^1(\Sigma_g;\mathbb{R})_{\mathcal{M}_g}$ denote the group of co-invariants of $H^1(\Sigma_g;\mathbb{R})$ with respect to the action of $\mathcal{M}_g$. This is the quotient of $H^1(\Sigma_g;\mathbb{R})$ by the subgroup generated by the elements of the form $\varphi(w) - w\,(\varphi \in \mathcal{M}_g, w \in H^1(\Sigma_g;\mathbb{R}))$. Notice that we have to consider $H^1(\Sigma_g;\mathbb{R})$ as an abelian group, rather than as a vector space, so that the action of $\mathcal{M}_g$ on it is far from being irreducible. However, we have the following simple fact.

Lemma 7. *For any* $g \geqslant 1$, *we have* $H^1(\Sigma_g;\mathbb{R})_{\mathcal{M}_g} = 0$.

Proof. Let $u \in H_1(\Sigma_g; \mathbb{Z})$ be the homology class represented by any oriented non-separating simple closed curve on Σ_g. Then it is easy to see that there exist elements $\varphi \in \mathcal{M}_g$ and $v \in H_1(\Sigma_g; \mathbb{Z})$ such that $u = \varphi(v) - v$ (consider the Dehn twist along a non-separating simple closed curve which intersects u transversely and at only one point). The assertion follows easily from this fact. Moreover, it can be shown that any element in $H^1(\Sigma_g; \mathbb{R})$ can be represented as the sum of at most $2g$ elements of the form $\varphi(w) - w$. □

With the above preparation, we can now finish the proof of Theorem 1. By Lemma 7, there exist elements $\varphi_i \in \mathcal{M}_g$, $w_i \in H^1(\Sigma_g; \mathbb{R})$ $(1 \leqslant i \leqslant 2g)$ such that

$$\mathrm{Flux}(\xi) = \sum_{i=1}^{2g} (\varphi_i(w_i) - w_i). \tag{1}$$

On the other hand, since the flux homomorphism is surjective, for any i there exists an element $\psi_i \in \mathrm{Symp}_0\, \Sigma_g$ such that $\mathrm{Flux}(\psi_i) = w_i$. By Lemma 6

$$\mathrm{Flux}(\tilde{\varphi}_i \psi_i \tilde{\varphi}_i^{-1}) = \varphi_i(\mathrm{Flux}(\psi_i)) = \varphi_i(w_i),$$

where $\tilde{\varphi}_i \in \mathrm{Symp}\, \Sigma_g$ is any lift of φ_i. Since Flux is a homomorphism, we can conclude

$$\mathrm{Flux}([\tilde{\varphi}_i, \psi_i]) = \mathrm{Flux}(\tilde{\varphi}_i \psi_i \tilde{\varphi}_i^{-1}) + \mathrm{Flux}(\psi_i^{-1}) = \varphi_i(w_i) - w_i. \tag{2}$$

Now consider the element

$$\eta = [\tilde{\varphi}_1, \psi_1] \cdots [\tilde{\varphi}_{2g}, \psi_{2g}] \in \mathrm{Symp}_0\, \Sigma_g. \tag{3}$$

It follows from equalities (1) and (2) that

$$\mathrm{Flux}(\eta) = \mathrm{Flux}(\xi).$$

Hence, $\mathrm{Flux}(\xi\eta^{-1}) = 0$ so that we have $\xi\eta^{-1} \in \mathrm{Ham}\, \Sigma_g$. Since $\mathrm{Ham}\, \Sigma_g$ is perfect, $\xi\eta^{-1}$ can be represented as a product of commutators of elements of $\mathrm{Ham}\, \Sigma_g$. Let h' be the number of commutators needed for this. Then a similar argument as before yields a homomorphism

$$\pi_1 \Sigma_{h+2g+h'} \to \mathrm{Symp}\, \Sigma_g$$

such that the corresponding foliated Σ_g-bundle $\tilde{\pi} : \tilde{E} \to \Sigma_{h+2g+h'}$ has total holonomy group in $\mathrm{Symp}\, \Sigma_g$. Now the part of $\tilde{E}$ over Σ_h is the same as E and the part of $\tilde{E}$ over $\Sigma_{h'}$ is topologically trivial. The remaining part of $\tilde{E}$ over Σ_{2g} may be non-trivial topologically. However, its monodromy homomorphism to the mapping class group factors through a free group because the mapping class of ψ_i is trivial for any i, and so its signature vanishes because a free group has no second cohomology to accommodate the signature cocycle, or first Mumford–Morita–Miller class. Hence Novikov additivity implies that $\mathrm{sign}\, \tilde{E} = \mathrm{sign}\, E \neq 0$. This completes the proof of Theorem 1.

2.1. Interpretation of Theorem 1 in terms of group homology

Theorem 1 can be translated into algebraic terms in the context of group homology. The extension

$$1 \to \mathrm{Diff}_0\, \Sigma_g \to \mathrm{Diff}_+\, \Sigma_g \to \mathcal{M}_g \to 1$$

gives rise to the 5-term exact sequence

$$H_2(\mathrm{Diff}^\delta_+\,\Sigma_g) \to H_2(\mathscr{M}_g) \to H_1(\mathrm{Diff}^\delta_0\,\Sigma_g)_{\mathscr{M}_g} \to \; H_1(\mathrm{Diff}^\delta_+\,\Sigma_g) \to H_1(\mathscr{M}_g) \to 0$$

of integral homology groups of discrete groups. From Thurston's theorem [37] that $\mathrm{Diff}_0\,M$ is perfect for any closed manifold M, we see that $H_1(\mathrm{Diff}^\delta_0\,\Sigma_g)=0$. Therefore, the exact sequence implies two things. Firstly, for all $g \geqslant 3$, the group $\mathrm{Diff}_+\,\Sigma_g$ is perfect. Secondly, the map $H_2(\mathrm{Diff}^\delta_+\,\Sigma_g) \to H_2(\mathscr{M}_g)$ is surjective. We know from work of Harer that $H_2(\mathscr{M}_g) \cong \mathbb{Z}$ for any $g \geqslant 4$ and that the generator is detected by the first Mumford–Morita–Miller class $e_1 \in H^2(\mathscr{M}_g;\mathbb{Z})$, see [26,14,15,20]. This also holds for $g=3$, except that $H_2(\mathscr{M}_3)$ may have an additional torsion summand. Hence we conclude that the homomorphism

$$H_2(\mathrm{Diff}^\delta_+\,\Sigma_g) \to \mathbb{Z}$$

given by the cap product with e_1 is non-trivial for any $g \geqslant 3$. This is equivalent to the existence of foliated Σ_g-bundles with non-zero signatures.

Next consider the extension

$$1 \to \mathrm{Symp}_0\,\Sigma_g \to \mathrm{Symp}\,\Sigma_g \to \mathscr{M}_g \to 1$$

and the associated 5-term exact sequence

$$H_2(\mathrm{Symp}^\delta\,\Sigma_g) \to H_2(\mathscr{M}_g) \to H_1(\mathrm{Symp}^\delta_0\,\Sigma_g)_{\mathscr{M}_g} \to \; H_1(\mathrm{Symp}^\delta\,\Sigma_g) \to H_1(\mathscr{M}_g) \to 0.$$

As mentioned above, the flux homomorphism yields an isomorphism $H_1(\mathrm{Symp}^\delta_0\,\Sigma_g) \cong H^1(\Sigma_g;\mathbb{R})$. Hence Lemma 7 implies that $H_1(\mathrm{Symp}^\delta_0\,\Sigma_g)_{\mathscr{M}_g}$ vanishes. We can now conclude that the homomorphism

$$H_2(\mathrm{Symp}^\delta\,\Sigma_g) \to H_2(\mathscr{M}_g)$$

is surjective and that there is an isomorphism $H_1(\mathrm{Symp}^\delta\,\Sigma_g) \cong H_1(\mathscr{M}_g)$. The former fact is equivalent to the existence of foliated Σ_g-bundles with area-preserving total holonomy and with non-zero signatures as in Theorem 1. The latter fact implies that the natural projection $\mathrm{Symp}\,\Sigma_g \to \mathscr{M}_g$ induces an isomorphism on the first integral homology groups. In particular, the group $\mathrm{Symp}\,\Sigma_g$ is perfect for all $g \geqslant 3$.

3. The transverse symplectic class and the flux homomorphism

In this section, we prove Theorem 2. In particular, we show that the flux homomorphism

$$\mathrm{Flux}\ :\ \mathrm{Symp}_0\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R})$$

can be extended to a crossed homomorphism

$$\widetilde{\mathrm{Flux}}:\ \mathrm{Symp}\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R})$$

in an essentially unique way.

If we consider the flux homomorphism as an element of $H^1(\mathrm{Symp}^\delta_0\,\Sigma_g;H^1(\Sigma_g;\mathbb{R}))$, then Lemma 6 implies that it is invariant under the canonical actions of $\mathscr{M}_g$. In other words, we can write

$$\mathrm{Flux} \in H^1(\mathrm{Symp}^\delta_0\,\Sigma_g;H^1(\Sigma_g;\mathbb{R}))^{\mathscr{M}_g}. \tag{4}$$

Now, we consider the cohomology class $e+v\in H^2(\mathrm{ESymp}^\delta\Sigma_g;\mathbb{R})$ mentioned in Section 1. As we noted there, this class restricts to 0 on the fiber of the extension $\pi_1\Sigma_g\to\mathrm{ESymp}^\delta\Sigma_g\to\mathrm{Symp}^\delta\Sigma_g$. Hence, in the spectral sequence $\{E_i^{p,q}\}$ for its real cohomology, we have the natural projection

$$p\colon \mathrm{Ker}(H^2(\mathrm{ESymp}^\delta\Sigma_g;\mathbb{R})\to H^2(\Sigma_g;\mathbb{R}))\ni e+v\to p(e+v)\in E_\infty^{1,1}\subset H^1(\mathrm{Symp}^\delta\Sigma_g;H^1(\Sigma_g;\mathbb{R})).$$

To prove Theorem 2, we first show the following: if we pull back $p(e+v)$ to

$$H^1(\mathrm{Symp}_0^\delta\Sigma_g;H^1(\Sigma_g;\mathbb{R}))\cong\mathrm{Hom}(\mathrm{Symp}_0\Sigma_g,H^1(\Sigma_g;\mathbb{R}))$$

then we have the equality

$$p(e+v)=\mathrm{Flux}\ :\ \mathrm{Symp}_0\Sigma_g\to H^1(\Sigma_g;\mathbb{R}).\tag{5}$$

To see this, it suffices to prove the following:

Lemma 8. *Let $I=[0,1]$. For any $\varphi\in\mathrm{Symp}_0\Sigma_g$ let $\pi:M_\varphi\to S^1$ be the foliated Σ_g-bundle over S^1 with monodromy φ. It is the quotient space of $\Sigma_g\times I$ by the equivalence relation $(p,0)\sim(\varphi(p),1)$. By assumption, there is an isotopy $\varphi_t\in\mathrm{Symp}_0\Sigma_g$ such that $\varphi_0=\mathrm{Id}$ and $\varphi_1=\varphi$. Let $f:M_\varphi\to\Sigma_g\times S^1$ be the induced diffeomorphism given by the correspondence*

$$M_\varphi\ni(p,t)\mapsto(\varphi_t^{-1}(p),t)\in\Sigma_g\times S^1.$$

Then the transverse symplectic class $v\in H^2(M_\varphi;\mathbb{R})$ is equal to

$$(2g-2)\mu+\mathrm{Flux}(\varphi)\otimes\nu\in H^2(\Sigma_g\times S^1;\mathbb{R})\cong H^2(\Sigma_g;\mathbb{R})\oplus(H^1(\Sigma_g;\mathbb{R})\otimes H^1(S^1;\mathbb{R}))$$

under the above isomorphism, where $\mu\in H^2(\Sigma_g;\mathbb{R})$ and $\nu\in H^1(S^1;\mathbb{R})$ denote the fundamental cohomology classes of Σ_g and S^1, respectively.

Proof. The foliation on M_φ is induced from the trivial foliation $\{\Sigma_g\times\{t\}\}$ on $\Sigma_g\times I$. Hence the transverse symplectic class v is represented by the form $p^*\omega$ on $\Sigma_g\times I$, where $p:\Sigma_g\times I\to\Sigma_g$ denotes the projection to the first factor. It is clear that the $H^2(\Sigma_g;\mathbb{R})$-component of v is equal to $(2g-2)\mu$ so that we only need to prove that for any closed oriented curve $\gamma\subset\Sigma_g$, the value of v on the cycle $f^{-1}(\gamma\times S^1)\subset M_\varphi$ is equal to $\mathrm{Flux}(\varphi)([\gamma])$ where $[\gamma]\in H_1(\Sigma_g;\mathbb{Z})$ denotes the homology class of γ. Now on $\Sigma_g\times I$, the above cycle is expressed as the image of the map

$$\gamma\times I\ni(q,t)\mapsto(\varphi_t(q),t)\in\Sigma_g\times I$$

because $f^{-1}(q,t)=(\varphi_t(q),t)\,((q,t)\in\Sigma_g\times S^1)$. Hence the required value is equal to the symplectic area of the image of the mapping

$$\gamma\times I\ni(q,t)\mapsto\varphi_t(q)\in\Sigma_g.$$

But this is exactly equal to the value of $\mathrm{Flux}(\varphi)$ on the homology class represented by the cycle $\gamma\subset\Sigma_g$. This completes the proof. □

Now we can finish the proof of Theorem 2 as follows. The extension

$$1\to\mathrm{Symp}_0\Sigma_g\to\mathrm{Symp}\,\Sigma_g\to\mathscr{M}_g\to 1$$

gives rise to the exact sequence

$$0 \to H^1(\mathcal{M}_g; H^1(\Sigma_g;\mathbb{R})) \to H^1(\mathrm{Symp}^\delta \Sigma_g; H^1(\Sigma_g;\mathbb{R}))$$
$$\to H^1(\mathrm{Symp}^\delta_0 \Sigma_g; H^1(\Sigma_g;\mathbb{R}))^{\mathcal{M}_g} \to H^2(\mathcal{M}_g; H^1(\Sigma_g;\mathbb{R})) \to \qquad (6)$$

Eqs. (4) and (5) show that the element $p(e+v) \in H^1(\mathrm{Symp}^\delta \Sigma_g; H^1(\Sigma_g;\mathbb{R}))$ is mapped to $\mathrm{Flux} \in H^1(\mathrm{Symp}^\delta_0 \Sigma_g; H^1(\Sigma_g;\mathbb{R}))^{\mathcal{M}_g}$ in the above sequence (6). In other words, the flux homomorphism can be lifted to a crossed homomorphism

$$\widetilde{\mathrm{Flux}}: \mathrm{Symp}\,\Sigma_g \to H^1(\Sigma_g;\mathbb{R}).$$

On the other hand, it was proved in [30] that $H^1(\mathcal{M}_g; H^1(\Sigma_g;\mathbb{Z}))=0$ for any $g \geqslant 1$. It follows that $H^1(\mathcal{M}_g; H^1(\Sigma_g;\mathbb{R}))=0$ for any $g \geqslant 1$. The exact sequence (6) now shows that the above lift is essentially unique. This completes the proof of Theorem 2.

The cohomology group $H^1(\mathrm{Symp}^\delta \Sigma_g; H^1(\Sigma_g;\mathbb{R}))$ can be completely determined as follows.

Proposition 9. *For any $g \geqslant 2$, there exists an isomorphism*

$$H^1(\mathrm{Symp}^\delta \Sigma_g; H^1(\Sigma_g;\mathbb{R})) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{R},\mathbb{R}),$$

where the right-hand side denotes the $\mathbb{Q}$-vector space consisting of all $\mathbb{Q}$-linear mappings $\mathbb{R} \to \mathbb{R}$. Under this isomorphism, the element

$$[\widetilde{\mathrm{Flux}}] \in H^1(\mathrm{Symp}^\delta \Sigma_g; H^1(\Sigma_g;\mathbb{R}))$$

corresponds to $\mathrm{Id} \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{R},\mathbb{R})$.

Proof. Consider the exact sequence (6). As mentioned above, we know that

$$H^1(\mathcal{M}_g; H^1(\Sigma_g;\mathbb{R}))=0 \quad (g \geqslant 1).$$

On the other hand, we have the vanishing result

$$H^2(\mathcal{M}_g; H^1(\Sigma_g;\mathbb{Q}))=0 \quad (g \geqslant 1,\ g \neq 4,5).$$

This is a special case of a general result of Looijenga [22] (for a stable range $g \geqslant 6$), while the case $g \geqslant 9$ was already mentioned in [31]. (See also Proposition 21 of [13]. The proof there should be modified to use Harer's result [16] on the *third* homology group of the moduli spaces as well as results of Igusa [17] and Looijenga [21] for low genera $g=2,3$, instead of Harer's earlier result [14] on the second homology. This correction forces us to exclude $g=4$ or 5 for now.)

Thus we have an isomorphism

$$H^1(\mathrm{Symp}^\delta \Sigma_g; H^1(\Sigma_g;\mathbb{R})) \cong H^1(\mathrm{Symp}^\delta_0 \Sigma_g; H^1(\Sigma_g;\mathbb{R}))^{\mathcal{M}_g}, \qquad (7)$$

except possibly for $g=4$, 5 for the moment.

Now the flux homomorphism gives rise to an isomorphism

$$\mathrm{Flux}\ : H_1(\mathrm{Symp}^\delta_0 \Sigma_g;\mathbb{Z}) \cong H^1(\Sigma_g;\mathbb{R}).$$

Hence we can write

$$H^1(\mathrm{Symp}^\delta_0 \Sigma_g; H^1(\Sigma_g;\mathbb{R})) \cong \mathrm{Hom}(H^1(\Sigma_g;\mathbb{R}), H^1(\Sigma_g;\mathbb{R}))$$

and under this isomorphism, the flux homomorphism clearly corresponds to the identity. On the other hand, an analysis of the action of $\mathscr{M}_g$ on the right-hand side yields an isomorphism

$$\mathrm{Hom}(H^1(\Sigma_g;\mathbb{R}), H^1(\Sigma_g;\mathbb{R}))^{\mathscr{M}_g} \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{R},\mathbb{R}). \tag{8}$$

More precisely, if we choose a Hamel basis $\{a_\lambda\}_\lambda$ for $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$, then we have an isomorphism

$$H^1(\Sigma_g;\mathbb{R}) \cong \bigoplus_\lambda H^1(\Sigma_g; a_\lambda\mathbb{Q}).$$

It is easy to see that any endomorphism $f : H^1(\Sigma_g;\mathbb{R}) \to H^1(\Sigma_g;\mathbb{R})$, which is equivariant with respect to the natural action of $\mathscr{M}_g$ must send any summand $H^1(\Sigma_g; a_\lambda\mathbb{Q})$ to a direct sum of finitely many such summands by some scalar multiplication in each factor. The isomorphism (8) follows from this.

We can eliminate the possible exceptions for (7) by a stabilization argument using the simple behavior of the flux homomorphisms under the inclusions

$$\mathrm{Symp}_0^c\,\Sigma_g^0 \subset \mathrm{Symp}_0\,\Sigma_g, \qquad \mathrm{Symp}_0^c\,\Sigma_g^0 \subset \mathrm{Symp}_0^c\,\Sigma_{g+1}^0.$$

Here $\Sigma_g^0 = \Sigma_g \setminus D^2$ and $\mathrm{Symp}_0^c\,\Sigma_g^0$ denotes the group of symplectomorphisms of Σ_g^0 with compact supports (see the next section for the flux homomorphism for the group $\mathrm{Symp}_0^c\,\Sigma_g^0$).

Thus isomorphism (7) holds for any $g \geqslant 2$ and the required result follows. □

Remark 10. Kawazumi kindly pointed out the following simple argument which avoids the use of the vanishing result for $H^2(\mathscr{M}_g; H^1(\Sigma_g;\mathbb{Q}))$. Any element in

$$H^1(\mathrm{Symp}_0^\delta\,\Sigma_g; H^1(\Sigma_g;\mathbb{R}))^{\mathscr{M}_g} \cong \mathrm{Hom}(H^1(\Sigma_g;\mathbb{R}), H^1(\Sigma_g;\mathbb{R}))^{\mathscr{M}_g}$$

is obtained from the flux homomorphism (= the identity) by applying some endomorphism to the coefficients $H^1(\Sigma_g;\mathbb{R})$ which is equivariant under the action of $\mathscr{M}_g$. Since we already proved that the identity can be lifted, any other element can also be lifted simply by applying an $\mathscr{M}_g$-equivariant change of coefficients.

4. The symplectomorphism groups of open surfaces

In this section, we prepare a few facts concerning the symplectomorphism groups of *open* surfaces. These will be used in the proof of Theorem 3 given in the next section.

Let $D^2 \subset \Sigma_g$ be a closed embedded disk and Σ_g^0 the open surface $\Sigma_g \setminus D^2$. We consider the group $\mathrm{Symp}^c\,\Sigma_g^0$ of symplectomorphisms of Σ_g^0 with *compact supports*. Let $\mathrm{Symp}_0^c\,\Sigma_g^0$ denote the identity component of $\mathrm{Symp}^c\,\Sigma_g^0$. In this context, we again have a flux homomorphism

$$\mathrm{Flux} : \mathrm{Symp}_0^c\,\Sigma_g^0 \to H_c^1(\Sigma_g^0;\mathbb{R}),$$

where $H_c^1(\Sigma_g^0;\mathbb{R})$ denotes the first real cohomology group of Σ_g^0 with compact supports. It is easy to see that the inclusion $\Sigma_g^0 \subset \Sigma_g$ induces an isomorphism $H_c^1(\Sigma_g^0;\mathbb{R}) \cong H^1(\Sigma_g;\mathbb{R})$, and that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Symp}_0^c\,\Sigma_g^0 & \xrightarrow{\mathrm{Flux}} & H_c^1(\Sigma_g^0;\mathbb{R}) \\ \downarrow & & \downarrow \cong \\ \mathrm{Symp}_0\,\Sigma_g & \xrightarrow{\mathrm{Flux}} & H^1(\Sigma_g;\mathbb{R}). \end{array} \tag{9}$$

It is known that the kernel of the flux homomorphism is equal to the subgroup $\mathrm{Ham}^c\,\Sigma_g^0$ consisting of Hamiltonian symplectomorphisms with compact supports. Thus we have an extension

$$1 \to \mathrm{Ham}^c\,\Sigma_g^0 \to \mathrm{Symp}_0^c\,\Sigma_g^0 \overset{\mathrm{Flux}}{\to} H_c^1\,(\Sigma_g^0;\mathbb{R}) \to 1.$$

In contrast to the case of closed surfaces, the group $\mathrm{Ham}^c\,\Sigma_g^0$ is not perfect. In fact, there is a surjective homomorphism

$$\mathrm{Cal} : \mathrm{Ham}^c\,\Sigma_g^0 \to \mathbb{R},$$

called the (second) Calabi homomorphism, see [8]. The kernel of this homomorphism is known to be simple, and hence perfect, by a result of Banyaga [4,5].

The flux and the Calabi homomorphisms can be defined for any non-compact symplectic manifold. Here we only consider the case of *exact* symplectic manifolds, assuming the existence of a 1-form λ such that $\omega = -d\lambda$. The open surface Σ_g^0 which we are concerned with is an exact symplectic manifold.

For any exact symplectic manifold (M,ω) of dimension $2n$, the flux and the Calabi homomorphisms can be expressed as

$$\mathrm{Flux}(\varphi) = [\lambda - \varphi^*\lambda] \in H_c^1(M;\mathbb{R}) \quad (\varphi \in \mathrm{Symp}_0^c\,M)$$

and

$$\mathrm{Cal}(\varphi) = -\frac{1}{n+1}\int_M \varphi^*\lambda \wedge \lambda \wedge \omega^{n-1} \quad (\varphi \in \mathrm{Ham}^c M), \tag{10}$$

respectively (see [25, Lemmas 10.14 and 10.27]). The formula (10) above can be used for any $\varphi \in \mathrm{Symp}_0^c\,M$, not necessarily in $\mathrm{Ham}^c\,M$. It defines a *map*

$$\mathrm{Cal} : \mathrm{Symp}_0^c\,M \to \mathbb{R} \tag{11}$$

and a straightforward calculation shows that

$$\mathrm{Cal}(\varphi\psi) = \mathrm{Cal}(\varphi) + \mathrm{Cal}(\psi) + \frac{1}{n+1}\int_M \mathrm{Flux}(\varphi) \wedge \mathrm{Flux}(\psi) \wedge \omega^{n-1}$$

for any two elements $\varphi, \psi \in \mathrm{Symp}_0^c\,M$. Hence the map (11) above is a homomorphism if and only if the pairing

$$H_c^1(M;\mathbb{R}) \otimes H_c^1(M;\mathbb{R}) \ni ([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta \wedge \omega^{n-1} \tag{12}$$

is trivial. This is the case if $\dim M = 2n \geqslant 4$, because then the integrand is exact (and compactly supported). However, in our case of an open surface $M = \Sigma_g^0$, the pairing (12) is non-trivial, and even non-degenerate. Define the Heisenberg group $\mathscr{H}$ to be the central extension

$$0 \to \mathbb{R} \to \mathscr{H} \to H_c^1(\Sigma_g^0;\mathbb{R}) \to 1$$

corresponding to the cup product pairing $H_c^1(\Sigma_g^0;\mathbb{R}) \otimes H_c^1(\Sigma_g^0;\mathbb{R}) \to \mathbb{R}$. Now we obtain the following fact:

Proposition 11. *For any* $g \geqslant 2$, *the mapping* Cal + Flux *defines a surjective homomorphism*

$$\mathrm{Cal} + \mathrm{Flux} \ : \ \mathrm{Symp}_0^c\, \Sigma_g^0 \to \mathscr{H}.$$

Corollary 12. *The flux homomorphism induces an isomorphism*

$$\mathrm{Flux}\ : H_1(\mathrm{Symp}_0^c\, \Sigma_g^0; \mathbb{Z}) \cong H_c^1(\Sigma_g^0; \mathbb{R}).$$

Furthermore for any real number $r \in \mathbb{R}$, *there exist two elements* $\varphi, \psi \in \mathrm{Symp}_0^c\, \Sigma_g^0$ *such that the commutator* $\eta = [\varphi, \psi] \in \mathrm{Ham}^c\, \Sigma_g^0$ *satisfies* $\mathrm{Cal}(\eta) = r$.

Proof. The first statement follows from Proposition 11 together with the fact that $\mathrm{Ker}\,\mathrm{Cal} \subset \mathrm{Ham}^c\, \Sigma_g^0$ is perfect. The second statement follows easily from the above argument. □

5. Proof of Theorem 3

In this section we prove Theorem 3, which shows the non-triviality of any power $e_1^k \in H^{2k}(\mathrm{BSymp}^\delta\, \Sigma_g; \mathbb{R})$ of the first Mumford–Morita–Miller class whenever $g \geqslant 3k$.

We first treat the case where the total holonomy group is in $\mathrm{Diff}_+\, \Sigma_g$, rather than in $\mathrm{Symp}\, \Sigma_g$. As in the previous section, fix a closed embedded disk $D^2 \subset \Sigma_g$. We denote by $\mathrm{Diff}(\Sigma_g, D^2)$ the group of diffeomorphisms of Σ_g which are the identity on some open neighborhoods of D^2. This is the same as the group $\mathrm{Diff}^c\, \Sigma_g^0$ of diffeomorphisms with compact supports of the open surface Σ_g^0.

Let $\pi : E \to \Sigma_h$ be any Σ_g-bundle over Σ_h, for example one with $\mathrm{sign}\, E \neq 0$. Then we can apply the same argument as in Section 2 to this bundle replacing the group $\mathrm{Diff}_+\, \Sigma_g$ by $\mathrm{Diff}(\Sigma_g, D^2)$. Fortunately Thurston's theorem (see [37,5]) is also valid for this relative case, giving that the identity component $\mathrm{Diff}_0(\Sigma_g, D^2)$ is simple and hence perfect. Thus for some h', we obtain a homomorphism

$$\pi_1 \Sigma_{h+h'} \to \mathrm{Diff}(\Sigma_g, D^2)$$

such that the signature of the total space of the associated foliated Σ_g-bundle over $\Sigma_{h+h'}$ is equal to $\mathrm{sign}\, E \neq 0$. This implies the non-triviality

$$e_1 \neq 0 \in H^2(\mathrm{BDiff}^\delta(\Sigma_g, D^2); \mathbb{Q}).$$

To prove the non-triviality of higher powers e_1^k, consider a genus kg surface $\Sigma_{kg,1} = \Sigma_{kg} \setminus \mathrm{Int}\, D^2$ with one boundary component as the boundary connected sum

$$\Sigma_{kg,1} = \Sigma_{g,1} \natural \cdots \natural \Sigma_{g,1}$$

of k copies of $\Sigma_{g,1} = \Sigma_g \setminus \mathrm{Int}\, D^2$. This induces a homomorphism

$$f_k : \ \mathrm{Diff}(\Sigma_g, D^2) \times \cdots \times \mathrm{Diff}(\Sigma_g, D^2) \to \mathrm{Diff}(\Sigma_{kg}, D^2) \tag{13}$$

from the direct product of k copies of the group $\mathrm{Diff}(\Sigma_g, D^2)$ to $\mathrm{Diff}(\Sigma_{kg}, D^2)$. It can be shown that

$$f_k^*(e_1) = e_1 \times 1 \times \cdots \times 1 + \cdots + 1 \times \cdots \times 1 \times e_1,$$

see [27,29] or [32]. It follows that

$$f_k^*(e_1^k) = e_1 \times \cdots \times e_1 + \text{other terms},$$

where the other terms belong to various summands of

$$H^*(\mathrm{BDiff}^\delta(\Sigma_g, D^2); \mathbb{Q}) \otimes \cdots \otimes H^*(\mathrm{BDiff}^\delta(\Sigma_g, D^2); \mathbb{Q})$$

other than

$$H^2(\mathrm{BDiff}^\delta(\Sigma_g, D^2); \mathbb{Q}) \otimes \cdots \otimes H^2(\mathrm{BDiff}^\delta(\Sigma_g, D^2); \mathbb{Q}).$$

Since $e_1 \times \cdots \times e_1 \neq 0$, we can now conclude that $f_k^*(e_1^k) \neq 0$. This proves the non-triviality

$$e_1^k \neq 0 \in H^{2k}(\mathrm{BDiff}_+ \Sigma_g; \mathbb{Q}) \quad \text{for any } g \geqslant 3k.$$

Next we consider the case where the total holonomy is contained in $\mathrm{Symp}\, \Sigma_g$. We apply the same argument as in Section 2, but replacing the group $\mathrm{Symp}\, \Sigma_g$ by $\mathrm{Symp}^c\, \Sigma_g^0$. At the final stage, we must use the second statement of Corollary 12 to kill the value of the Calabi homomorphism. To summarize, we kill the value of the flux homomorphism by adding $2g$ commutators in $\mathrm{Symp}^c\, \Sigma_g^0$ as in Section 2 and then kill the value of the Calabi homomorphism by adding one commutator in $\mathrm{Symp}_0^c\, \Sigma_g^0$. Then we can use the perfection of the subgroup $\mathrm{Ker\ Cal} \subset \mathrm{Symp}_0^c\, \Sigma_g^0$ to show the non-triviality

$$e_1 \neq 0 \in H^2(\mathrm{BSymp}^{c,\delta}\, \Sigma_g^0; \mathbb{Q}).$$

Finally we consider the homomorphism

$$h_k : \mathrm{Symp}^c\, \Sigma_g^0 \times \cdots \times \mathrm{Symp}^c\, \Sigma_g^0 \to \mathrm{Symp}^c\, \Sigma_{kg}^0$$

which is defined similarly to the f_k in (13) and apply the same argument as above to show the non-triviality of the power e_1^k.

This completes the proof of Theorem 3.

6. Further results

6.1. Symplectic pairs

A symplectic pair on a smooth manifold is a pair of closed two-forms ω_1, ω_2 of constant and complementary ranks, for which ω_1 restricts as a symplectic form to the leaves of the kernel foliation of ω_2, and vice versa. This definition is analogous to that of contact pairs and of contact-symplectic pairs discussed by Bande [2,3].

Manifolds with symplectic pairs are always symplectic, but they satisfy much stronger topological restrictions than general symplectic manifolds. For example, a four-manifold with a symplectic pair admits symplectic structures for both choices of orientation, because $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$ are symplectic forms inducing opposite orientations.

Theorem 1 implies:

Corollary 13. *There exist smooth closed oriented four-manifolds of non-zero signature which admit symplectic pairs.*

The signature vanishes for all other four-manifolds which we know to admit symplectic pairs.

6.2. Perfect versus uniformly perfect groups

Combining our discussion in 2.1 with the main result of [10], we obtain the following.

Corollary 14. *Let $G = \mathrm{Diff}_+ \Sigma_g$ or $\mathrm{Symp}\, \Sigma_g$. For all $g \geqslant 3$ the group G is perfect but not uniformly perfect.*

Moreover, if $\varphi \in G$ represents the Dehn twist along any homotopically non-trivial simple closed curve on Σ_g, then the commutator length of φ^k in G grows linearly with k, for all $g \geqslant 2$.

Proof. We saw in 2.1 that $H_1(G^\delta) = H_1(\mathcal{M}_g)$ for all $g \geqslant 2$. For $g \geqslant 3$, the mapping class group is known to be perfect, see for example [15].

The projection $G \to \mathcal{M}_g$ is surjective, and the commutator length of φ^k is bounded below by that of its image in $\mathcal{M}_g$. Thus the main result of [10] gives the conclusion, compare also [7]. □

For a perfect group G not being uniformly perfect is equivalent to the statement that the comparison map $c : H_b^2(G^\delta) \to H^2(G^\delta)$ from the second bounded cohomology to the usual cohomology with real coefficients is not injective. If we denote the kernel of c by $K(G^\delta)$, it is easy to see that $K(\mathcal{M}_g)$ injects into $K(G^\delta)$ for $G = \mathrm{Diff}_+ \Sigma_g$ or $\mathrm{Symp}\, \Sigma_g$. The result of [10] to the effect that $K(\mathcal{M}_g)$ is non-zero has been generalized by Bestvina and Fujiwara [6] to show that it is infinite-dimensional. Thus, $K(G^\delta)$ is also infinite-dimensional.

Note that because the Mumford–Morita–Miller class $e_1 \in H^2(\mathcal{M}_g)$ is a bounded class, i.e. in the image of c, the same is true for $e_1 \in H^2(G^\delta)$ and its powers e_1^k. Thus Theorem 3 shows in particular that the comparison map c is non-trivial on $H_b^{2k}(G^\delta)$ for $g \geqslant 3k \geqslant 3$ and $G = \mathrm{Diff}_+ \Sigma_g$ or $\mathrm{Symp}\, \Sigma_g$.

The groups $\mathrm{Diff}_0 \Sigma_g$, $\mathrm{Diff}_0(\Sigma_g, D^2)$, $\mathrm{Ham}\, \Sigma_g$ and $\mathrm{Ker\, Cal} \subset \mathrm{Ham}^c \Sigma_g^0$ are perfect by the results of Thurston [35,37] and Banyaga [4], compare also [5]. In parallel to our work on this paper, Gambaudo and Ghys [12] have proved that $\mathrm{Ham}\, \Sigma_g$ is not uniformly perfect. Their arguments also apply to the group $\mathrm{Ker\ Cal} \subset \mathrm{Ham}^c \Sigma_g^0$, although they do not state this in [12]. The result of Gambaudo–Ghys implies that in our proof of Theorem 1 one cannot control the base genus of the trivial fibration which we fiber sum to a given surface bundle to obtain a flat bundle with total holonomy in $\mathrm{Symp}\, \Sigma_g$.

Note that Entov and Polterovich [11] recently proved that $\mathrm{Ham}\, M$ is not uniformly perfect if M belongs to a certain subclass of the spherically monotone symplectic manifolds which includes S^2 and many high-dimensional manifolds, but not the surfaces of positive genus.

Whether or not $\mathrm{Diff}_0 \Sigma_g$ and $\mathrm{Diff}_0(\Sigma_g, D^2)$ are uniformly perfect remains a very interesting open question.

6.3. The crossed flux homomorphism in higher dimensions

Let (M, ω) be any closed symplectic manifold and $\mathcal{M}_\omega$ its symplectic mapping class group defined to be the quotient of $\mathrm{Symp}(M, \omega)$ by its identity component $\mathrm{Symp}_0(M, \omega)$, so that we have an extension

$$1 \to \mathrm{Symp}_0(M, \omega) \to \mathrm{Symp}(M, \omega) \to \mathcal{M}_\omega \to 1. \tag{14}$$

In view of Theorem 2, it appears to be an interesting problem to determine whether the flux homomorphism

$$\mathrm{Flux} : \mathrm{Symp}_0(M,\omega) \to H^1(M;\mathbb{R})/\Gamma_\omega \tag{15}$$

can be extended to a crossed homomorphism on the whole group $\mathrm{Symp}(M,\omega)$ or not. Here Γ_ω denotes the flux subgroup corresponding to the flux of non-trivial loops in $\mathrm{Symp}_0(M,\omega)$. (This has to be divided out to make the flux well-defined, see [25].) Extension (14) yields an exact sequence

$$0 \to H^1(\mathcal{M}_\omega; H^1(M;\mathbb{R})/\Gamma_\omega) \to H^1(\mathrm{Symp}(M,\omega); H^1(M;\mathbb{R})/\Gamma_\omega)$$
$$\to H^1(\mathrm{Symp}_0(M,\omega); H^1(M;\mathbb{R})/\Gamma_\omega)^{\mathcal{M}_\omega} \xrightarrow{\delta} H^2(\mathcal{M}_\omega; H^1(M;\mathbb{R})/\Gamma_\omega) \to$$

It is easy to generalize Lemma 6, which treats the case of surfaces, to the case of closed symplectic manifolds. Hence we can write

$$\mathrm{Flux} \in H^1(\mathrm{Symp}_0(M,\omega); H^1(M;\mathbb{R})/\Gamma_\omega)^{\mathcal{M}_\omega}$$

and we may ask whether the element $\delta(\mathrm{Flux})$ is trivial or not. This is equivalent to asking whether the extension

$$1 \to H^1(M;\mathbb{R})/\Gamma_\omega \to \mathrm{Symp}(M,\omega)/\mathrm{Ham}(M,\omega) \to \mathcal{M}_\omega \to 1 \tag{16}$$

splits or not. If this is the case, then the flux extends and the group $H^1(\mathcal{M}_\omega; H^1(M;\mathbb{R})/\Gamma_\omega)$ measures the differences between the possible extensions.

As a partial answer to this problem, we have the following. Assume that the cohomology class $[\omega] \in H^2(M;\mathbb{R})$ is a multiple of the first Chern class $c_1(M) \in H^2(M;\mathbb{Z})$. (This is a variant of the monotonicity assumption.) Then it was proved by McDuff [24] and by Lupton–Oprea [23] that the flux subgroup Γ_ω is trivial. We can extend this result as follows, thereby also reproving the triviality of Γ_ω from our point of view.

Proposition 15. *Let (M,ω) be a closed symplectic manifold and assume that the cohomology class $[\omega] \in H^2(M;\mathbb{R})$ is a multiple of the first Chern class $c_1(M) \in H^2(M;\mathbb{Z})$. Then the flux subgroup Γ_ω is trivial and the flux homomorphism* $\mathrm{Flux} : \mathrm{Symp}_0(M,\omega) \to H^1(M;\mathbb{R})$ *can be extended to a crossed homomorphism*

$$\widetilde{\mathrm{Flux}} : \mathrm{Symp}(M,\omega) \to H^1(M;\mathbb{R}).$$

Proof. We modify the argument in the proof of Theorem 2, given in Section 3, as follows. Observe first that the Euler class $e \in H^2(\mathrm{ESymp}^\delta\,\Sigma_g;\mathbb{Z})$ considered there is nothing but the first Chern class of the tangent bundle along the fibers of the universal surface bundle over $\mathrm{BSymp}^\delta\,\Sigma_g$. Let $\mathrm{BSymp}^\delta(M,\omega)$ be the classifying space of the discrete group $\mathrm{Symp}^\delta(M,\omega)$ and let

$$\pi : \mathrm{ESymp}^\delta(M,\omega) \to \mathrm{BSymp}^\delta(M,\omega)$$

be the universal foliated M-bundle over $\mathrm{BSymp}^\delta(M,\omega)$ with total holonomy group in $\mathrm{Symp}(M,\omega)$. Then we have the first Chern class

$$c_1(\xi) \in H^2(\mathrm{ESymp}^\delta(M,\omega);\mathbb{Z}),$$

where ξ denotes the tangent bundle along the fibers of π. By assumption, there exists a non-zero real number r such that $[\omega] = rc_1(M)$. Now consider the cohomology class

$$u = v - rc_1(\xi) \in H^2(\mathrm{ESymp}^\delta(M,\omega); \mathbb{R}),$$

where v denotes the transverse symplectic class represented by the global 2-form $\tilde{\omega}$ on $\mathrm{ESymp}^\delta(M,\omega)$ which restricts to ω on each fiber. The restriction of u to the fiber vanishes so that, in the spectral sequence $\{E_r^{p,q}\}$ for the real cohomology, we have

$$p(u) \in E_\infty^{1,1} \subset H^1(\mathrm{BSymp}^\delta(M,\omega); H^1(M;\mathbb{R})).$$

Now we consider the composition of homomorphisms

$$H^1(\mathrm{BSymp}^\delta(M,\omega); H^1(M;\mathbb{R})) \to H^1(\mathrm{BSymp}_0^\delta(M,\omega); H^1(M;\mathbb{R}))$$
$$\to H^1(\mathrm{BSymp}_0^\delta(M,\omega); H^1(M;\mathbb{R})/\Gamma_\omega) \cong \mathrm{Hom}(\mathrm{Symp}_0(M,\omega), H^1(M;\mathbb{R})/\Gamma_\omega),$$

where the first homomorphism is induced by the restriction to the subgroup $\mathrm{Symp}_0(M,\omega) \subset \mathrm{Symp}(M,\omega)$ while the second one is induced by the natural projection $H^1(M;\mathbb{R}) \to H^1(M;\mathbb{R})/\Gamma_\omega$. Let

$$\overline{p(u)} \in \mathrm{Hom}(\mathrm{Symp}_0(M,\omega), H^1(M;\mathbb{R})/\Gamma_\omega)$$

be the image of $p(u)$ under the above composition. Then we have the equality

$$\overline{p(u)} = \mathrm{Flux} : \mathrm{Symp}_0(M,\omega) \to H^1(M;\mathbb{R})/\Gamma_\omega. \tag{17}$$

This can be shown by suitably adapting Lemma 8 to the case of a general closed symplectic manifold M instead of Σ_g. Thus we see that the flux homomorphism can be extended to a homomorphism from $\mathrm{Symp}_0(M,\omega)$ to the whole of $H^1(M;\mathbb{R})$ (rather than its quotient by Γ_ω). On the other hand, Banyaga's result [4] that Ker Flux=Ham(M,ω) is perfect (and simple) implies that the abelianization of $\mathrm{Symp}_0(M,\omega)$ is equal to $H^1(M;\mathbb{R})/\Gamma_\omega$. We can now conclude that the flux subgroup Γ_ω is trivial and further that the flux homomorphism can be extended canonically to a crossed homomorphism on the whole group $\mathrm{Symp}(M,\omega)$. This completes the proof. □

Example 16. The above proof does not apply to the torus T^2 with the standard symplectic form ω_0 because the first Chern class is trivial in this case. In fact, the flux subgroup is isomorphic to $H^1(T^2;\mathbb{Z})$, which is non-trivial. However, the flux homomorphism does extend canonically to a crossed homomorphism

$$\widetilde{\mathrm{Flux}} : \mathrm{Symp}(T^2,\omega_0) \to H^1(T^2;\mathbb{R})/H^1(T^2;\mathbb{Z}).$$

This is because the mapping class group $\mathscr{M}_1 \cong \mathrm{SL}(2,\mathbb{Z})$ acts on T^2 linearly by symplectomorphisms and hence the extension (16) splits canonically.

Remark 17. In forthcoming joint work with Kędra [18] we further generalize Theorem 2 and Proposition 15 to other situations.

Remark 18. After this paper was written, the problem of exhibiting explicit crossed homomorphisms representing the extended flux was raised by R. Kasagawa, by D. McDuff, and by an anonymous referee. We shall return to this problem elsewhere.

Acknowledgements

S. Morita would like to thank the Mathematisches Institut der Universität München, where the present work was done, for its hospitality. Thanks are also due to N. Kawazumi, S. Matsumoto, K. Ono and T. Tsuboi for enlightening discussions and helpful information.

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